1. Let $X_1, \dots, X_n$ be some random variables on the same probability space. Consider the $n \times n$ covariance matrix A with entries $a_{ij} = \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])]$. Show that any covariance matrix is PSD.

   Solution: Let us define $Y_i = X_i - \mathbb{E}[X_i]$, so that Y_i has mean 0. Then $a_{ij} = \mathbb{E}[Y_i Y_j]$. For any $w \in \mathbb{R}^n$ we have

   $$w^T A w = \sum_{ij} w_i \mathbb{E}[Y_i Y_j] w_j = \mathbb{E}[\sum_{ij} w_i Y_i w_j Y_j] = \mathbb{E}[(\sum_i w_i Y_i)^2] \geq 0$$

   and hence A is PSD.

2. Show that given any PSD matrix A, one can construct random variables $X_1, \dots, X_n$ (in fact jointly Gaussian random variables) with covariance matrix A.

   [Hint: Do Cholesky decomposition of A and set $X_i = \langle g, u_i \rangle$.]

   Solution: Setting $X_i = \langle g, u_i \rangle$, we have

   $$\mathbb{E}[X_i X_j] = \mathbb{E}[\langle g, u_i \rangle \langle g, u_j \rangle] = \mathbb{E}[\sum_k g(k) u_i(k) \sum_{k'} g(k') u_j(k')] = \sum_k u_i(k) u_j(k) = u_i \cdot u_j$$

   where the second last step follows as $\mathbb{E}[g(k)^2] = 1$ and $\mathbb{E}[g(k)g(k')] = 0$ for $k \neq k'$, as the entries of g are iid $N(0,1)$.

3. Given any graph $G = (V, E)$, construct an explicit solution for the max-cut SDP such that the objective is at least $|E|/2$.

   Solution: Consider the all-orthogonal solution where vector i is assigned the vector $v_i = e_i$, i.e. unit vector in the i-th direction. Then for any two vertices i and j, we have $(1/4)\|v_i - v_j\|^2 = 1/2$, and thus the maxcut SDP objective is $|E|/2$.

4. Here we will show an improved bound of $2/\pi$ for the maximizing the quadratic form $x^T A y$ where A is a PSD matrix. Note that this is a generalization of the max-cut problem, where A corresponds to the Laplacian of G.

   We do this in the following steps

   - First, show using Cauchy-Schwarz that for any PSD matrix A one has

     $$x^T A y \leq (x^T A x)^{1/2} (y^T A y)^{1/2}$$

     and hence one can assume in the optimum solution that $x = y$.

     Solution: As A is PSD, $A = V^T V$ for some matrix V. So, $x^T A y = x^T V^T V y = (Vx)^T (Vy)$. By Cauchy Schwarz (i.e. $a^T b \leq \|a\|_2 \|b\|_2$ for any two vectors a and b) it follows that

     $$(Vx)^T (Vy) \leq \|Vx\|_2 \|Vy\|_2 = (x^T V^T V x)^{1/2} (y^T V^T V y)^{1/2} = (x^T A x)^{1/2} (y^T A y)^{1/2}.$$

   - Second, show the following identity. If b and c are two unit vectors, and g is a random gaussian then

     $$\frac{\pi}{2} \mathbb{E}[\text{sign}(g \cdot b) \cdot \text{sign}(g \cdot c)] = b \cdot c + \mathbb{E}\left([b \cdot g - \sqrt{\frac{\pi}{2}} \text{sign}(b \cdot g)][c \cdot g - \sqrt{\frac{\pi}{2}} \text{sign}(c \cdot g)] \right)$$

Hint: To compute $\mathbb{E}[(b \cdot G) \cdot \text{sign}(c \cdot G)]$, by rotation invariance assume that $b = (b_1, b_2, 0, \ldots, 0)$ and $c = (1, 0, ..., 0)$. This is $\mathbb{E}[(b_1 g_1 + b_2 g_2)(\text{sign}(g_1))] = \mathbb{E}[b_1 g_1 \text{sign}(g_1)]$. Show that this integral is $(\sqrt{2/\pi}) b_1$.

Solution: Let us expand

$$\mathbb{E}\left([b \cdot g - \sqrt{\frac{\pi}{2}}\text{sign}(b \cdot g)][c \cdot g - \sqrt{\frac{\pi}{2}}\text{sign}(c \cdot g)] \right)$$

as

$$\mathbb{E}[(b \cdot g)(c \cdot g)] - \mathbb{E}[(b \cdot g)\sqrt{\frac{\pi}{2}}\text{sign}(c \cdot g)] - \mathbb{E}[(c \cdot g)\sqrt{\frac{\pi}{2}}\text{sign}(b \cdot g)] + \mathbb{E}[\frac{\pi}{2}\text{sign}(b \cdot g)\text{sign}(c \cdot g)]$$

As we have seen before, the first term is simply $b \cdot c$. Moreover the last term is identical to the lhs of the identity we wish to show. So, it suffices to show that second term $\mathbb{E}[(b \cdot g)\sqrt{\frac{\pi}{2}}\text{sign}(c \cdot g)]$ is $b \cdot c$ (similarly for the third term by symmetry). By the hint above, this is the same as $\sqrt{\frac{\pi}{2}}\mathbb{E}[b_1 g_1 \text{sign}(g_1)]$ which we need to show is b_1 (which is $b \cdot c$). Now,

$$\begin{aligned}
\mathbb{E}[b_1 g_1 \text{sign}(g_1)] &= \int_{-\infty}^{\infty} b_1 |x| \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= 2b_1 \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x e^{-x^2/2} dx \\
&= 2b_1 \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-y} dy \qquad (\text{ setting } y = x^2/2) \\
&= (\frac{2}{\pi})^{1/2} b_1
\end{aligned}$$

- Finally, use these two facts (and that A is PSD) to show that the random hyperplane rounding gives at least a $2/\pi$ approximation.

  Solution: By part 1, we can consider maximizing $x^T A x$ and consider the vector relaxation $\sum_{ij} u_i a_{ij} u_j$. Let B denote the optimum value of this SDP and let u_i denote some optimum solution.

  Applying the hyperplane rounding gives an expected solution value of $\sum_{ij} a_{ij} \text{sign}(\langle u_i, g \rangle) \cdot \text{sign}(\langle u_i, g \rangle)$. Call this value A.

  Now by the identity in the second part (applying it to each pair u_i, u_j, multiplying it by a_{ij} and combining), we have

$$\frac{\pi}{2} A = B + \mathbb{E}\left(\sum_{ij} a_{ij} [u_i \cdot g - \sqrt{\frac{\pi}{2}}\text{sign}(u_i \cdot g)][u_j \cdot g - \sqrt{\frac{\pi}{2}}\text{sign}(u_j \cdot g)] \right).$$

  Now the key point is that for any value of g, the big term on the rhs is non-negative. This follows as A is PSD, and hence $w^T A w \geq 0$ for any fixed vector w. For any fixed g, consider the vector w with entry $w_i = u_i \cdot g - \sqrt{\frac{\pi}{2}}\text{sign}(u_i \cdot g)$.

  Thus $\frac{\pi}{2} A \geq B$ and we are done.