Knowledge Updates and Common Knowledge

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Abstract

Further examples of knowledge updating, with special emphasis on common knowledge.

Module Declaration

```
module LAI15 where

import List
import Char
import LAI9
import LAI10
import LAI11
import LAI12
import LAI13
import LAI14
```

Card Showing

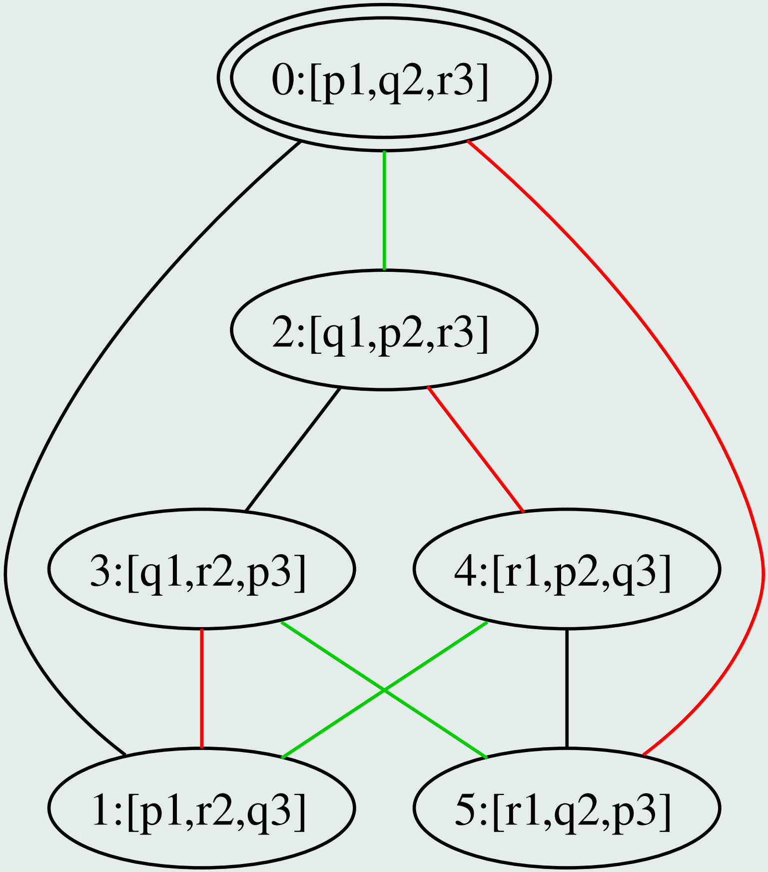
Three players a, b, c, in that order.

Three cards p, q, r.

Formula p_i expresses that the i-th player gets card p, and similarly for the other cards.

If the actual deal is p_1, q_2, r_3 and it is common knowledge that each player has inspected his own card, then the following is an appropriate model.

```
hexa :: EpistM State
hexa = Mo
       [0..5] [a..c]
       [(0,[P 1, Q 2, R 3]),(1,[P 1, R 2, Q 3]),
        (2,[Q 1, P 2, R 3]),(3,[Q 1, R 2, P 3]),
        (4,[R 1, P 2, Q 3]),(5,[R 1, Q 2, P 3])]
       ([(ag,x,x) | ag <- [a..c], x  <- [0..5]]
         ++ [(a,0,1),(a,1,0),(a,2,3),
             (a,3,2),(a,4,5),(a,5,4)]
         ++ [(b,0,5),(b,5,0),(b,1,3),
             (b,3,1),(b,2,4),(b,4,2)]
         ++ [(c,0,2),(c,2,0),(c,1,4),
             (c,4,1),(c,3,5),(c,5,3)])
       [0]
```


0:[p1,q2,r3]
2:[q1,p2,r3]
3:[q1,r2,p3]
4:[r1,p2,q3]
1:[p1,r2,q3]
5:[r1,q2,p3]

Useful Abbreviations for Propositions

```
q1,q2,q3,r1,r2,r3 :: Form
q1 = Prop (Q 1); q2 = Prop (Q 2); q3 = Prop (Q 3)
r1 = Prop (R 1); r2 = Prop (R 2); r3 = Prop (R 3)
```

General Knowledge versus Common Knowledge

General Knowledge versus Common Knowledge

```
LAI15> isTrue hexa (EK [a,b,c] (Neg (Conj [q1,r2,p3])))
True
LAI15> isTrue hexa (CK [a,b,c] (Neg (Conj [q1,r2,p3])))
False
```

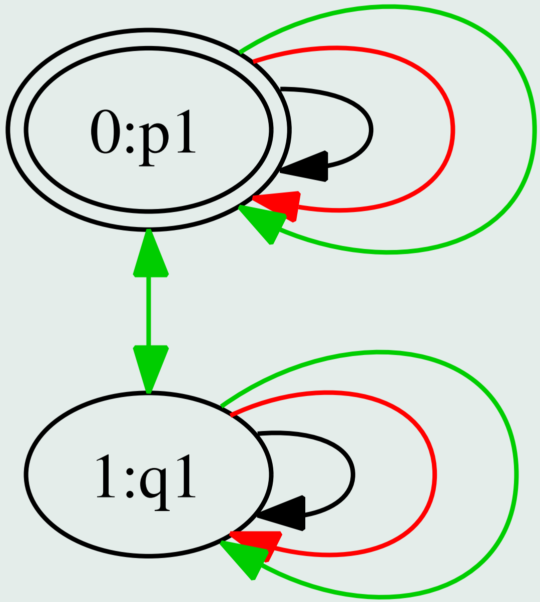
Player a shows her card . . .

Action: a shows p to b with c looking on (c sees that a card is shown, but does not see that it is p).

First attempt:

```
showABp :: FAM State
showABp = \ ags ->
 Am [0,1] ags pre susp [0]
  where
    pre = [(0,p1),(1,q1)]
    susp = [(a,0,0),(a,1,1),(b,0,0),(b,1,1),
            (c,0,0),(c,0,1),(c,1,0),(c,1,1)]
```

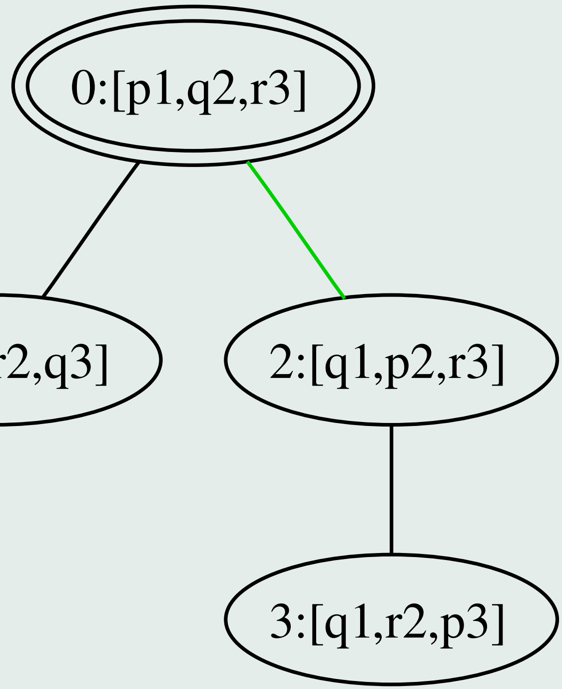
As a graph:



The result of updating with this:

```
LAI15> displayS5 (upd hexa showABp)
[0,1,2,3]
[(0,[p1,q2,r3]),(1,[p1,r2,q3]),
 (2,[q1,p2,r3]),(3,[q1,r2,p3])]
(a,[[0,1],[2,3]])
(b,[[0],[1],[2],[3]])
(c,[[0,2],[1],[3]])
[0]
```

Viewed as a graph:



Too crude . . .

This is not quite good enough.

What did we do wrong?

Too crude ...

This is not quite good enough.

What did we do wrong?

We have 'hard-coded' into the update that c holds card r.

How can we repair this?

Reveal actions — ‘Revelations’

What a does is reveal her card situation.

- If she holds p, she reveals p_1,
- if she holds q, she reveals q_1,
- if she holds r, she reveals r_1.

What we need is revelation of a choice from a number of alternatives.

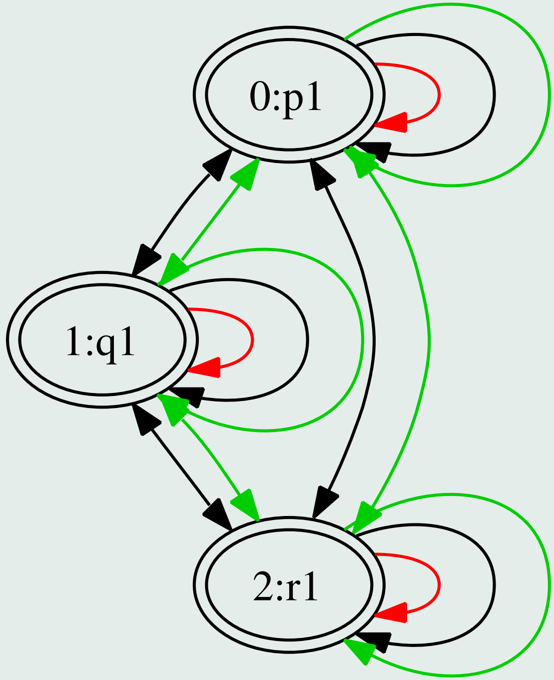
Action Model for Revelation

```
reveal :: [Agent] -> [Form] -> FAM State
reveal gr forms ags =
   Am
   states ags pre
   ([ (ag,s,s) | s <- states, ag <- gr ]
     ++
    [ (ag,s,s') | s <- states, s' <- states,
                  ag <- others ])
   states
     where pre    = zip [0..] forms
           states = map fst pre
           others = ags \\ gr
```

The Reveal Action of Player a

```
revealBpqr :: FAM State
revealBpqr = reveal [b] [p1,q1,r1]
```

Viewed as a graph:

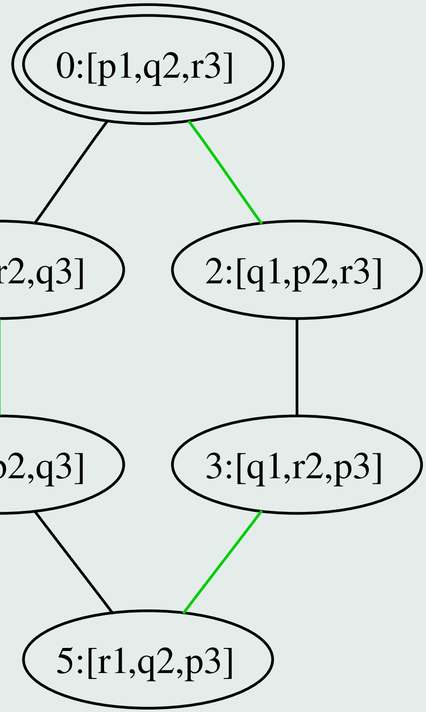


Player a shows her card

```
hexa2 = upd hexa revealBpqr
```

```
LAI15> displayS5 hexa2
[0,1,2,3,4,5]
[(0,[p1,q2,r3]),(1,[p1,r2,q3]),(2,[q1,p2,r3]),
 (3,[q1,r2,p3]),(4,[r1,p2,q3]),(5,[r1,q2,p3])]
(a,[[0,1],[2,3],[4,5]])
(b,[[0],[1],[2],[3],[4],[5]])
(c,[[0,2],[1,4],[3,5]])
[0]
```

Result viewed as a graph:



Again: General Knowledge versus Common Knowledge

Again: General Knowledge versus Common Knowledge

```
LAI15> isTrue hexa2 (EK [a,c] (Neg r1))
True
LAI15> isTrue hexa2 (CK [a,c] (Neg r1))
False
```

Multimodal Logic with Public Announcements

Let a set of agents B and a set of basic propositions P be given. Assume that in the following definition of a multimodal language with public announcements, b ranges over B and p ranges over P.

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid [b]\varphi \mid [\varphi_1!]\varphi_2$$

$[\varphi_1!]\varphi_2$ expresses that φ_2 holds after successful public announcement of φ_1.

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$[\varphi_1!]\varphi_2$ expresses that φ_2 holds after successful public announcement of φ_1.

The formal truth definition of $[\varphi_1!]\varphi_2$ runs as follows:

$$\mathbf{M}, w \models [\varphi_1!]\varphi_2 \text{ iff } (\mathbf{M}, w \models \varphi_1 \text{ implies } \mathbf{M} \mid \varphi_1, w \models \varphi_2).$$

Useful Abbreviations

- $\top \stackrel{\text{def}}{=} \neg\bot$,
- $\varphi_1 \vee \varphi_2 \stackrel{\text{def}}{=} \neg(\neg\varphi_1 \wedge \neg\varphi_2)$,
- $\varphi_1 \Rightarrow \varphi_2 \stackrel{\text{def}}{=} \neg(\varphi_1 \wedge \neg\varphi_2)$,
- $\varphi_1 \Leftrightarrow \varphi_2 \stackrel{\text{def}}{=} (\varphi_1 \Rightarrow \varphi_2) \wedge (\varphi_2 \Rightarrow \varphi_1)$,
- $\langle b\rangle\varphi \stackrel{\text{def}}{=} \neg[b]\neg\varphi$.
- $\langle\varphi_1!\rangle\varphi_2 \stackrel{\text{def}}{=} \neg[\varphi_1!]\neg\varphi_2$.

Remarkable Equivalences

$$
\begin{array}{lll}
[\varphi!]\bot & \text{iff} & \neg\varphi, \\
[\varphi!]p & \text{iff} & \varphi \Rightarrow p, \\
[\varphi!]\neg\psi & \text{iff} & \varphi \Rightarrow \neg[\varphi!]\psi, \\
[\varphi!](\psi_1 \wedge \psi_2) & \text{iff} & [\varphi!]\psi_1 \wedge [\varphi!]\psi_2, \\
[\varphi!][b]\psi & \text{iff} & \varphi \Rightarrow [b][\varphi!]\psi
\end{array}
$$

$[\varphi!]\bot$ **iff** $\neg\varphi$

$\mathbf{M}, w \models [\varphi!]\bot$

iff ($\mathbf{M}, w \models \varphi$ implies $\mathbf{M} \mid \varphi, w \models \bot$)

iff ($\mathbf{M}, w \models \varphi$ implies $\mathbf{M}, w \models \bot$)

iff $\mathbf{M}, w \models \neg\varphi$.

$[\varphi!]p$ **iff** $\varphi \Rightarrow p$

$\mathbf{M}, w \models [\varphi!]p$

iff ($\mathbf{M}, w \models \varphi$ implies $\mathbf{M} \mid \varphi, w \models p$)

iff ($\mathbf{M}, w \models \varphi$ implies $\mathbf{M}, w \models p$)

iff $\mathbf{M}, w \models \varphi \Rightarrow p$.

$[\varphi!]\neg\psi$ **iff** $\varphi \Rightarrow \neg[\varphi!]\psi$

$\mathbf{M}, w \models [\varphi!]\neg\psi$

iff $(\mathbf{M}, w \models \varphi$ implies $\mathbf{M} \mid \varphi, w \models \neg\psi)$

iff $(\mathbf{M}, w \models \varphi$ implies $\mathbf{M} \mid \varphi, w \not\models \psi)$

iff $(\mathbf{M}, w \models \varphi$ implies $\mathbf{M}, w \not\models [\varphi!]\psi)$

iff $(\mathbf{M}, w \models \varphi$ implies $\mathbf{M}, w \models \neg[\varphi!]\psi)$

iff $\mathbf{M}, w \models \varphi \Rightarrow \neg[\varphi!]\psi$.

$[\varphi!](\psi_1 \land \psi_2)$ **iff** $[\varphi!]\psi_1 \land [\varphi!]\psi_2$

$\mathbf{M}, w \models [\varphi!](\psi_1 \land \psi_2)$

iff ($\mathbf{M}, w \models \varphi$ implies $\mathbf{M} \mid \varphi, w \models \psi_1 \land \psi_2$)

iff ($\mathbf{M}, w \models \varphi$ implies ($\mathbf{M} \mid \varphi, w \models \psi_1$ and $\mathbf{M} \mid \varphi, w \models \psi_2$))

iff ($\mathbf{M}, w \models \varphi$ implies $\mathbf{M} \mid \varphi, w \models \psi_1$) and ($\mathbf{M}, w \models \varphi$ implies $\mathbf{M} \mid \varphi, w \models \psi_2$)

iff $\mathbf{M}, w \models [\varphi!]\psi_1$ and $\mathbf{M}, w \models [\varphi!]\psi_2$

iff $\mathbf{M}, w \models ([\varphi!]\psi_1 \land [\varphi!]\psi_2)$.

$[\varphi!][b]\psi$ **iff** $\varphi \Rightarrow [b][\varphi!]\psi$

$\mathbf{M}, w \models [\varphi!][b]\psi$

iff ($\mathbf{M}, w \models \varphi$ implies $\mathbf{M}, w \models [b]\psi$)

iff ($\mathbf{M}, w \models \varphi$ implies for all w' with $w \xrightarrow{b} w'$ and $\mathbf{M}, w \models \varphi$: $\mathbf{M} \mid \varphi, w' \models \psi$)

iff ($\mathbf{M}, w \models \varphi$ implies for all w' with $w \xrightarrow{b} w'$: ($\mathbf{M}, w' \models \varphi$ implies $\mathbf{M} \mid \varphi, w' \models \psi$))

iff ($\mathbf{M}, w \models \varphi$ implies for all w' with $w \xrightarrow{b} w'$: $\mathbf{M}, w' \models [\varphi!]\psi$)

iff ($\mathbf{M}, w \models \varphi$ implies $\mathbf{M}, w' \models [b][\varphi!]\psi$)

iff $\mathbf{M}, w \models \varphi \Rightarrow [b][\varphi!]\psi$.

Reduction of MM Logic With PA to MM Logic Without PA

The multimodal language with public announcements can be reduced to the multimodal language without public announcements.

Reduction of MM Logic With PA to MM Logic Without PA

The multimodal language with public announcements can be reduced to the multimodal language without public announcements.

For every formula from the multimodal language with public announcements there is an equivalent formula in the multimodal language without public announcements.

Axiomatisation of Multimodal S5 with Public Announcement

- Knowledge Distribution: $[b](\varphi \Rightarrow \psi) \Rightarrow ([b]\varphi \Rightarrow [b]\psi)$,
- Truth axiom: $[b]\varphi \Rightarrow \varphi$.
- Positive introspection: $[b]\varphi \Rightarrow [b][b]\varphi$
- Negative introspection: $\neg[b]\varphi \Rightarrow [b]\neg[b]\varphi$.
- Reduction axioms:
  - $[\varphi!]\bot \Leftrightarrow \neg\varphi$
  - $[\varphi!]p \Leftrightarrow (\varphi \Rightarrow p)$
  - $[\varphi!]\neg\psi \Leftrightarrow (\varphi \Rightarrow \neg[\varphi!]\psi)$
  - $[\varphi!](\psi_1 \wedge \psi_2) \Leftrightarrow ([\varphi!]\psi_1 \wedge [\varphi!]\psi_2)$
  - $[\varphi!][b]\psi \Leftrightarrow (\varphi \Rightarrow [b][\varphi!]\psi)$.

Plus:

- Axioms of Propositional Logic.
- Modus Ponens

$$\frac{\varphi \quad \varphi \Rightarrow \psi}{\psi}$$

- Knowledge Generalisation

$$\frac{\varphi}{[b]\varphi}$$

Translation Instruction

$$
\begin{aligned}
t(\bot) &= \bot \\
t(p) &= p \\
t(\neg\varphi) &= \neg t(\varphi) \\
t(\varphi_1 \wedge \varphi_2) &= t(\varphi) \wedge t(\varphi_2) \\
t([b]\varphi) &= [b]t(\varphi) \\
t([\varphi!]\bot) &= \neg t(\varphi) \\
t([\varphi!]p) &= t(\varphi) \Rightarrow p \\
t([\varphi!](\neg\psi)) &= t(\varphi) \Rightarrow \neg t([\varphi!]\psi) \\
t([\varphi!](\psi_1 \wedge \psi_2)) &= t([\varphi!]\psi_1) \wedge t([\varphi!]\psi_2) \\
t([\varphi!][b]\psi) &= t(\varphi) \Rightarrow [b]t([\varphi!]\psi) \\
t([\varphi_1!][\varphi_2!]\psi) &= t([\varphi_1!]t([\varphi_2!]\psi))
\end{aligned}
$$

Question about Common Knowledge

Does the modal invariance theorem (Theorem 1 from the lecture notes on bisimulation) still hold if we extend the modal language with a common knowledge operator C_B?

Give a proof if your answer is 'yes', and a counterexample otherwise.

Answer

The modal invariance theorem still holds for the modal language extended with a common knowledge operator.

Answer

The modal invariance theorem still holds for the modal language extended with a common knowledge operator.

We can show this by proving that every bisimulation $Z : \mathbf{M}, s \underline{\leftrightarrow} \mathbf{N}, t$ satisfies the following extended zig and zag conditions.

Let R be the common knowledge relation $(\bigcup_{b \in B} R_b)^*$, where B is the set of all agents.

Extended Zig. If sRs' then there is a t' with tRt' and $Z : \mathbf{M}, s' \underline{\leftrightarrow} \mathbf{N}, t'$.

Extended Zag. If tRt' then there is an s' with sRs' and $Z : \mathbf{M}, s' \underline{\leftrightarrow} \mathbf{N}, t'$.

Proof of Extended Zig. Let sRs'. Then by the definition of R there is sequence

$$s \xrightarrow{b_1} s_1 \xrightarrow{b_2} s_2 \ldots \xrightarrow{b_n} s_n \xrightarrow{b_{n+1}} s',$$

with every $b_i \in B$.

By $n+1$ applications of Zig for Z there are $t_1, \ldots, t_n, t'$ with

$$t \xrightarrow{b_1} t_1 \xrightarrow{b_2} t_2 \ldots \xrightarrow{b_n} t_n \xrightarrow{b_{n+1}} t',$$

with $Z : s_i \underline{\leftrightarrow} t_i$, and $Z : s' \underline{\leftrightarrow} t'$. This means tRt' and $Z : s \underline{\leftrightarrow} t'$, so this proves Extended Zig.

Proof of Extended Zag: by a similar argument.

Extended Zig and Zag can be used to prove invariance for formulas of the form $C_B\psi$.

Multimodal Logic with Common Knowledge

Axiomatisation: previous part of the course. See [2], page 46:

- Axiomatisation of multimodal logic, plus:
- $E_B\varphi \Leftrightarrow \bigwedge_{b\in B}[b]\varphi$.
- $C_B\varphi \Rightarrow \varphi$
- $C_B\varphi \Rightarrow E_B C_B\varphi$
- $C_B(\varphi \Rightarrow \psi) \Rightarrow (C_B\varphi \Rightarrow C_B\psi)$.
- $C_B(\varphi \Rightarrow E_B\psi) \Rightarrow (\varphi \Rightarrow C_B\psi)$.
- Common knowledge generalisation:

$$\frac{\varphi}{C_B\varphi}$$

Multimodal Logic with Public Announcement and Common Knowledge

What happens if we add common knowledge to multimodal logic with public announcement?

Sound and complete axiomatisation is possible but difficult.

Multimodal Logic with Public Announcement and Common Knowledge

What happens if we add common knowledge to multimodal logic with public announcement?

Sound and complete axiomatisation is possible but difficult.

There is no reduction axiom for $[\varphi!]C_B\psi$.

The reason for this is that multimodal logic is not strong enough to express what such a reduction axiom would have to say.

Relativized Common Knowledge

Intuitively, what we want to say about the interaction of public announcement and common knowledge is this:

$[\varphi!]C_B\psi$ expresses that every B-path that consists entirely of φ worlds ends in a $[\varphi!]\psi$ world.

Relativized Common Knowledge

Intuitively, what we want to say about the interaction of public announcement and common knowledge is this:

$[\varphi!]C_B\psi$ expresses that every B-path that consists entirely of φ worlds ends in a $[\varphi!]\psi$ world.

Suppose we want to say this, what do we need?

Answer [1]: relativized common knowledge: $C_B(\varphi, \psi)$

Intended meaning:

"if φ is announced it will become common knowledge among B that ψ."

Truth Definition of Relativized Common Knowledge Operator

$\mathbf{M}, w \models C_B(\varphi, \psi)$ iff $\mathbf{M}, v \models \psi$ for all v such that $(w, v) \in (R_B \cap [\![\varphi]\!]^2)^*$.

Truth Definition of Relativized Common Knowledge Operator

$\mathbf{M}, w \models C_B(\varphi, \psi)$ iff $\mathbf{M}, v \models \psi$ for all v such that $(w, v) \in (R_B \cap [\![\varphi]\!]^2)^*$.

Ordinary common knowledge can be defined in terms of relativized common knowledge:

$$C_B\varphi \stackrel{\text{def}}{=} C_B(\top, \varphi).$$

Axiomatisation of Multimodal Logic with Relativized Common Knowledge

- All (instantiations of) propositional tautologies
- Knowledge distribution

$$[b](\varphi \Rightarrow \psi) \Rightarrow ([b]\varphi \Rightarrow [b]\psi).$$

- Relativized Common Knowledge Distribution

$$C_B(\varphi, \psi \Rightarrow \chi) \Rightarrow (C_B(\varphi, \psi) \Rightarrow C_B(\varphi, \chi)).$$

- Mix

$$C_B(\varphi, \psi) \Leftrightarrow (\varphi \Rightarrow (\psi \wedge E_B(\varphi \Rightarrow C_B(\varphi, \psi)))).$$

- Induction

$$((\varphi \Rightarrow \psi) \wedge C_B(\varphi, \psi \Rightarrow E_B(\varphi \Rightarrow \psi))) \Rightarrow C_B(\varphi, \psi).$$

Rules of Inference

- Modus Ponens

$$\frac{\varphi \quad \varphi \Rightarrow \psi}{\psi}$$

- Knowledge Generalisation

$$\frac{\varphi}{[b]\varphi}$$

- Relativized Common Knowledge Generalisation

$$\frac{\varphi}{C_B(\psi, \varphi)}$$

Adding Public Announcement

Reduction Axiom:

$$[\varphi!]C_B(\psi, \chi) \Leftrightarrow C_B(\varphi \wedge [\varphi!]\psi, [\varphi!]\chi)$$

Motivation for this is that the following are equivalent:

> It holds in $\mathbf{M}$ that after public announcement of φ it is the case (in the new model) that every ψ-and-B path ends in a χ world
>
> It holds in $\mathbf{M}$ the every $\varphi \wedge [\varphi!]\psi$-and-$B$ path ends in a $[\varphi!]\chi$ world.

References

[1] J. van Benthem, J. van Eijck, and B. Kooi. Logics of communication and change. Under submission, 2005. Available from www.cwi.nl/~jve/papers/05/lcc/.

[2] J.J.Ch. Meyer and W. van der Hoek. Epistemic Logic for AI and Computer Science. Cambridge University Press, 1995.