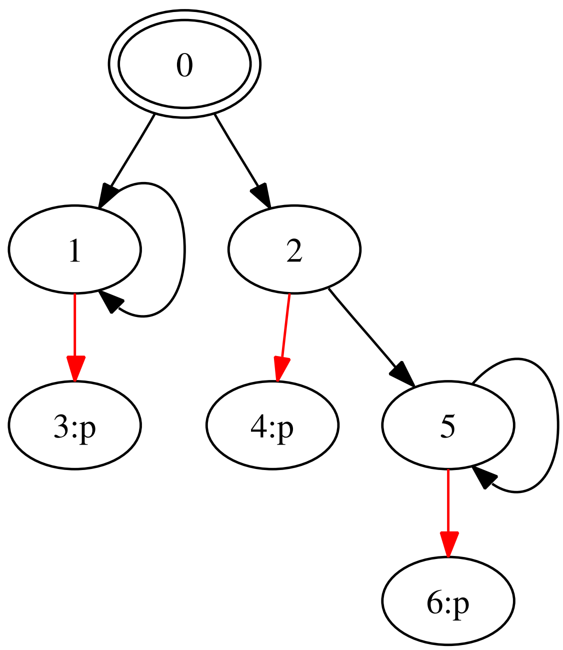
Pencil and Paper Exercises Week 3 — Solutions

1. Find the largest bisimulation on the following model, and also give the corresponding partition.



Answer: the largest bisimulation is the bisimulation Z given by the following list of pairs:

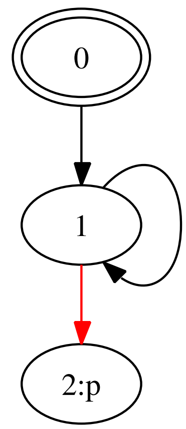
$$\{(0,0),(1,1),(1,2),(1,5),(2,1),(2,2),(2,5),(5,1),(5,2),(5,5),$$
$$(3,3),(3,4),(3,6),(4,3),(4,4),(4,6),(6,3),(6,4),(6,6)\}$$

The partition that corresponds with this equivalence relation:

$$\{\{0\},\{1,2,5\},\{3,4,6\}\}.$$

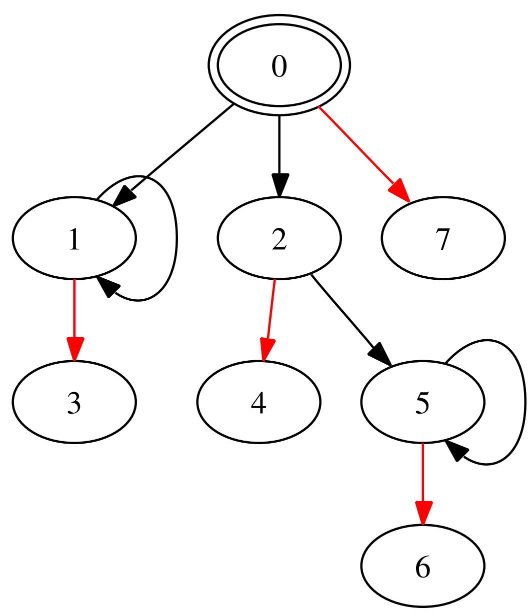
2. Give the bisimulation minimal version of the model from the previous exercise, and show the stages in which this bisimulation minimal model gets computed by the partition refinement algorithm.

Answer: the bisimulation minimal model looks like this:



This is computed by the partition refinement algorithm in two steps. Initial blocks are $\{0, 1, 2, 5\}$ and $\{3, 4, 6\}$, for these are the blocks of worlds with the same valuation. Next, the first block gets split, for $1, 2, 5$ have a red arrow to the other block, while 0 has not. The other block does not get split in this round. This gives partition $\{\{0\}, \{1, 2, 5\}, \{3, 4, 6\}\}$. In the next round no further splitting takes place, so this partition is a fixpoint.

3. Find the largest bisimulation on the following model, and also give the corresponding partition:



Answer: The largest bisimulation is:

$$\{(0, 0), (0, 1), (0, 2), (0, 5), (1, 0), (1, 1), (1, 2), (1, 5), (2, 0), (2, 1), (2, 2), (2, 5),$$

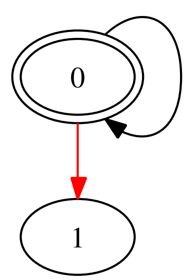
$$(5,0),(5,1),(5,2),(5,5),(3,3),(3,4),(3,6),(3,7),(4,3),(4,4),(4,6),(4,7),$$
$$(6,3),(6,4),(6,6),(6,7),(7,3),(7,4),(7,6),(7,7)\}$$

The corresponding partition:

$$\{\{0,1,2,5\},\{3,4,6,7\}\}.$$

4. Give the bisimulation minimal version of the model from the previous exercise, and show the stages in which this bisimulation minimal model gets computed by the partition refinement algorithm.

   Answer: The bisimulation minimal model looks like this:



   This is found by the partition refinement algorithm in two steps.

   First, all the worlds are put together in a single block (for they all have the same valuation). Next, the block is split because worlds 0, 1, 2, and 5 have a black arrow pointing to the block, but worlds 3, 4, 6 and 7 have not. In the next round, no further splitting takes place, so the partition $\{\{0,1,2,5\},\{3,4,6,7\}\}$ is a fixpoint.

5. The generated submodel of a Kripke model $\mathbf{M}, s$ (where s is a designated state, called the *root*) is the model $\mathbf{M}', s$ that results from restricting the state set of $\mathbf{M}$ to

$$\{t \mid (s,t) \in (\bigcup_{b \in B} R_b)^*\},$$

   where $\{R_b \mid b \in B\}$ are the accessibility relations of $\mathbf{M}$.

   Show that $\mathbf{M}, s \leftrightsquigarrow \mathbf{M}', s$.

   (Recall from the lectures that $\leftrightsquigarrow$ denotes modal equivalence.)

   Answer: We will show that the relation Z that links those s in $\mathbf{M}$ that belong to the set of worlds of $\mathbf{M}'$ to the corresponding s in $\mathbf{M}'$ is a bisimulation. The result then follows from Theorem 1 in the Lecture Notes on Bisimulation.

   The *Invariance* property follows at once from the fact that the states linked by Z are the same.

For the *Zig* property, assume $Z : \mathbf{M}, w \underline{\leftrightarrow} \mathbf{M}', w$ and assume $w \xrightarrow{a} w'$ in $\mathbf{M}$. Then $(s, w) \in (\bigcup_{b \in B} R_b)^*\}$, so $(s, w') \in (\bigcup_{b \in B} R_b)^*\}$. Thus, s' is in the state set of $\mathbf{M}$, and $w \xrightarrow{a} w'$ in $\mathbf{M}'$, and Z links w' in $\mathbf{M}$ to w' in $\mathbf{M}'$.

The *Zag* property follows immediately from the way $\mathbf{M}'$ is defined.

6. Prove that if Z and Z' are bisimulations on a model $\mathbf{M}$, then $Z \cup Z'$ is also a bisimulation on $\mathbf{M}$.

   Answer: We show that $Z \cup Z'$ satisfies the invariance, zig and zag conditions.

   Assume $s(Z \cup Z')t$.

   Invariance: Either sZt or $sZ't$. In both cases s and t have the same valuation, either by the invariance condition of Z or by that of Z'.

   Zig: suppose $s \xrightarrow{a} s'$. Case 1. sZt. Then by the zig condition of Z there is a t' with $t \xrightarrow{a} t'$ and $s'Zt'$. But then also $s'(Z \cup Z')t'$, and the zig condition for $Z \cup Z'$ is satisfied. Case 2. $sZ't$. Then by the zig condition of Z' there is a t' with $t \xrightarrow{a} t'$ and $s'Z't'$. But then also $s'(Z \cup Z')t'$, and the zig condition for $Z \cup Z'$ is satisfied.

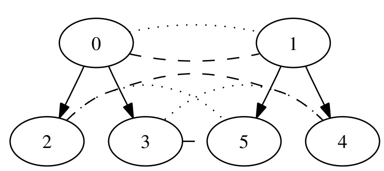
   Zag: similar reasoning.

7. Prove that every bisimulation Z on a model is contained in a largest bisimulation.

   Answer. Let $\{Z_k \mid k \in K\}$ be the set of all bisimulations on a model $\mathbf{M}$. Then it follows from the previous exercise that $Z = \bigcup_{k \in K} Z_k$ is a bisimulation on $\mathbf{M}$. By definition, Z is the largest bisimulation on $\mathbf{M}$.

8. Does it hold that if Z and Z' are bisimulations on a model $\mathbf{M}$, then $Z \cap Z'$ is a bisimulation on $\mathbf{M}$? Give a proof or a counterexample.

   Answer: no, this does not hold. Consider the following model.



The dotted lines picture the relation Z, the dashed lines the relation Z' (reflexive arrows are not shown). Surely, these are bisimulations. But the only pairs in the intersection of Z and Z' are $(0, 0)(0, 1), (1, 1)$, and $\{(0, 0)(0, 1), (1, 1)\}$ is not a bisimulation for it does not satisfy the zig and zag requirements.