

Pencil and Paper Exercises Week 4 – Solutions

1. Prove that the result of updating an S5 epistemic model with an S5 action model is an S5 model.

Answer:

Let $\mathbf{M} = (W, V, R)$ be an S5 epistemic model, and let $\mathbf{A} = (A, P, S)$ be an S5 action model. We have to show that $\mathbf{M} \otimes \mathbf{A}$, the result of updating \mathbf{M} with \mathbf{A} , is S5.

Let b be an arbitrary agent. We have to show that the accessibility relation for b in $\mathbf{M} \otimes \mathbf{A}$ is an equivalence relation, i.e., we have to check that the relation is reflexive, symmetric and transitive.

Reflexivity. Let (w, s) be a world in $\mathbf{M} \otimes \mathbf{A}$. We have to show that $(w, s) \xrightarrow{b} (w, s)$. From the fact that \mathbf{M} is S5, it follows that $w \xrightarrow{b} w$. From the fact that \mathbf{A} is S5, it follows that $s \xrightarrow{b} s$. By the definition of update product, $(w, s) \xrightarrow{b} (w, s)$.

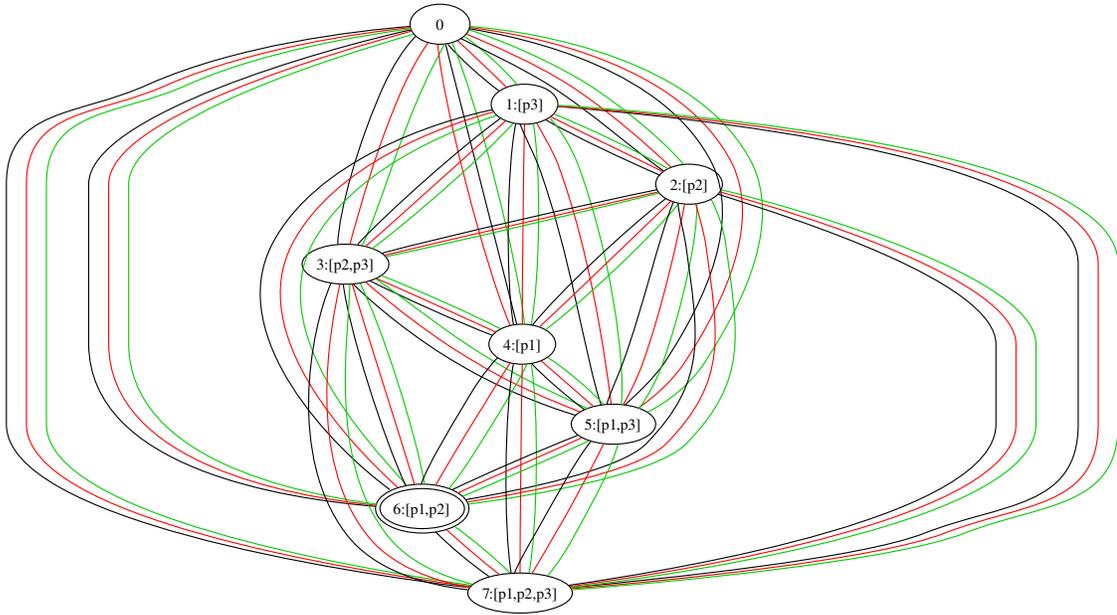
Symmetry. Let $(w, s) \xrightarrow{b} (w', s')$. By definition of update product, it follows that $w \xrightarrow{b} w'$ in \mathbf{M} and $s \xrightarrow{b} s'$ in \mathbf{A} . By the fact that \mathbf{M} and \mathbf{A} are both S5, it follows from $w \xrightarrow{b} w'$ that $w' \xrightarrow{b} w$, for \xrightarrow{b} is symmetric on \mathbf{M} , and from $s \xrightarrow{b} s'$ that $s' \xrightarrow{b} s$, for \xrightarrow{b} is symmetric on \mathbf{A} . By definition of update product, $(w', s') \xrightarrow{b} (w, s)$.

Transitivity. Let $(w_1, s_1) \xrightarrow{b} (w_2, s_2)$ and $(w_2, s_2) \xrightarrow{b} (w_3, s_3)$. We have to show $(w_1, s_1) \xrightarrow{b} (w_3, s_3)$. By the definition of update product, $w_1 \xrightarrow{b} w_2$, $w_2 \xrightarrow{b} w_3$ in \mathbf{M} , and $s_1 \xrightarrow{b} s_2$, $s_2 \xrightarrow{b} s_3$ in \mathbf{A} . By transitivity of \xrightarrow{b} on \mathbf{M} , it follows from $w_1 \xrightarrow{b} w_2$, $w_2 \xrightarrow{b} w_3$ that $w_1 \xrightarrow{b} w_3$, and by transitivity of \xrightarrow{b} on \mathbf{A} , it follows from $s_1 \xrightarrow{b} s_2$, $s_2 \xrightarrow{b} s_3$ that $s_1 \xrightarrow{b} s_3$. From $w_1 \xrightarrow{b} w_3$ and $s_1 \xrightarrow{b} s_3$, by the definition of update product, $(w_1, s_1) \xrightarrow{b} (w_3, s_3)$.

2. In the following exercises we look at the muddy children problem from an update perspective. Here is the story again, now in a more detailed version. Three children are standing in a circle. Each child is facing the others. The children *have their eyes closed*. Assume that it is common knowledge among the children that they all have their eyes closed. Also assume that in fact children a and b have mud on their foreheads, but child c is clean. Let p_1 express that a is muddy, p_2 that b is muddy, and p_3 that c is muddy. Give the epistemic model for this initial situation.

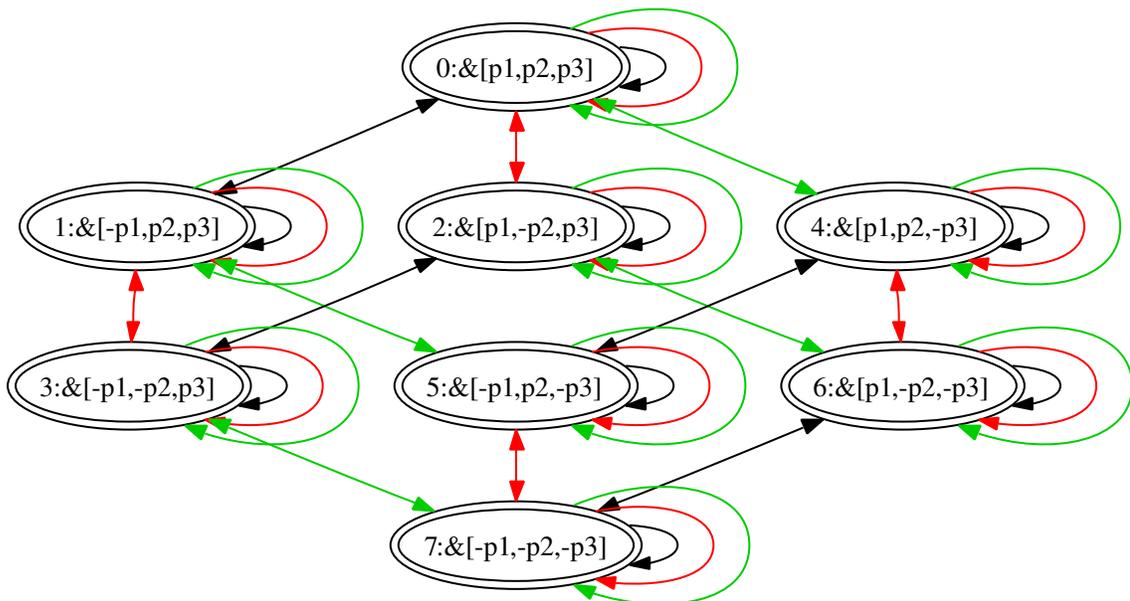
Answer:¹

¹In the pictures of epistemic models, we omit reflexive arrows and do not indicate direction on the arrows. In the pictures of action models, reflexive arrows are always included and directions are always indicated.



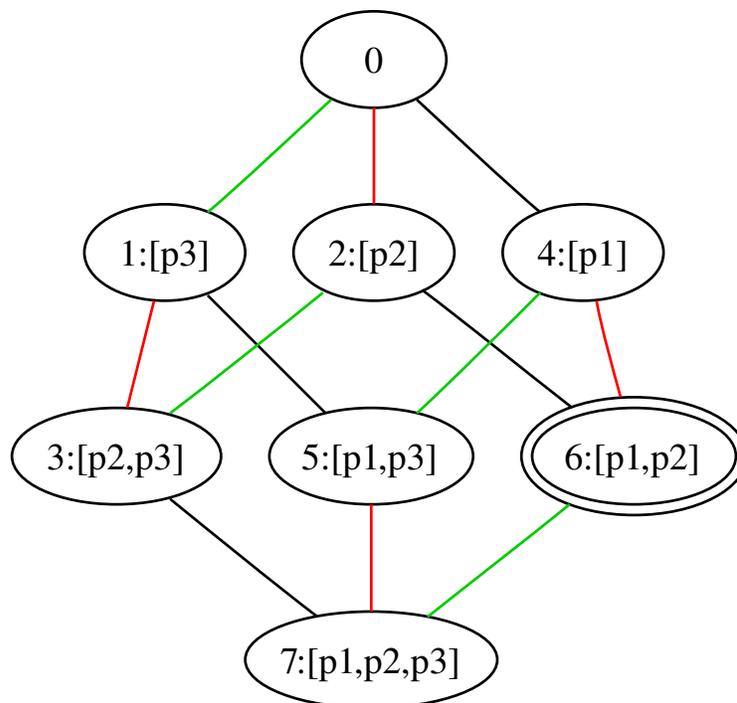
3. The children open their eyes. Two of them see mud on the forehead of one of the others. The third one (c) sees two muddy faces. Define an appropriate action model to capture what goes on in this scenario, specifically in the action of the children opening their eyes.

Answer:



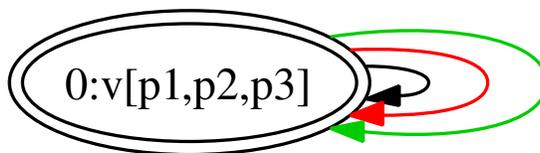
4. Give the epistemic model that results from updating the epistemic model from Exercise 2 with the action model for Exercise 3.

Answer:



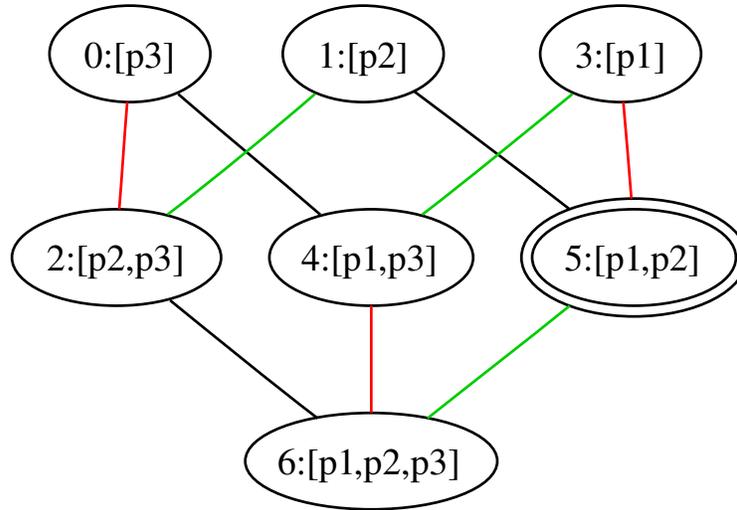
5. Father says: “At least one of you is muddy”. Give the corresponding action model.

Answer:



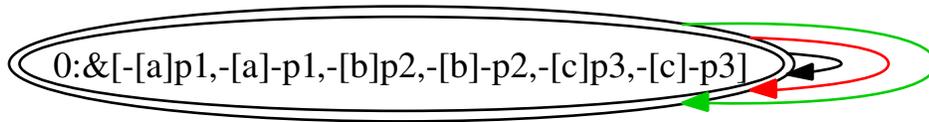
6. Give the epistemic model that results from updating the model from Exercise 4 with the action model from Exercise 5.

Answer:



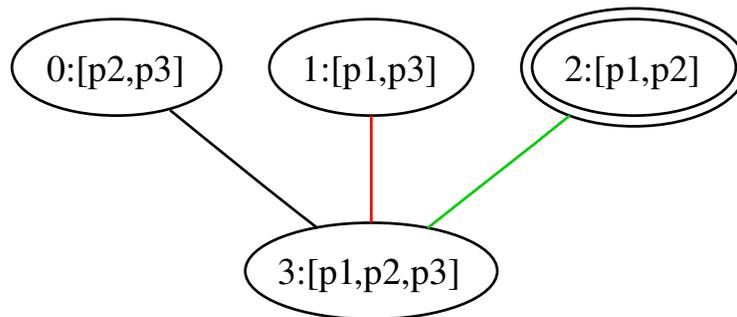
7. Children a , b and c make public announcements to the effect that they do not know whether they are muddy or not. Give a single action model to capture this.

Answer:



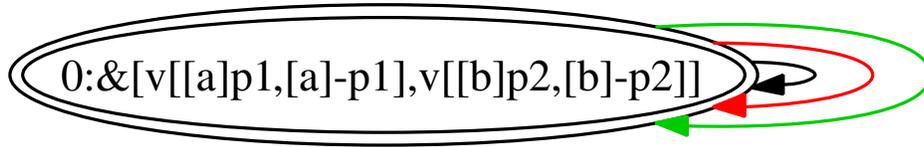
8. Give the epistemic model that results from updating the epistemic model from Exercise 6 with the action model from Exercise 7.

Answer:



9. Now children a and b publicly announce that they know whether they are muddy or not. Give an appropriate action model for this.

Answer:



10. Give the epistemic model that results from updating the epistemic model from Exercise 8 with the action model from Exercise 9.

Answer:

