Incremental Dynamics

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Abstract. A new system of dynamic logic is introduced and motivated, with a novel approach to variable binding for incremental interpretation. The system is shown to be equivalent to first order logic and complete.

The new logic combines the dynamic binding idea from Dynamic Predicate Logic with De Bruijn style variable free indexing. Quantifiers bind the next available variable register; the indexing mechanism guarantees that active registers are never overwritten by new quantifier actions. Apart from its interest in its own right, the resulting system has certain advantages over Dynamic Predicate Logic or Discourse Representation Theory. It comes with a more well behaved (i.c., transitive) consequence relation, it gives a more explicit account of how anaphoric context grows as text gets processed, and it yields new insight into the dynamics of anaphoric linking in reasoning. Incremental dynamics also points to a new way of handling context dynamically in Montague grammar.

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1. Contexts as Updating Anaphoric Contexts

Predicate logics without variables have a long history. A key paper is (Quine, 1966). Based on this, (Kuhn, 1983) and (Purdy, 1991) have proposed variable free representations for natural language understanding. Based on an even older approach, Peirce's existential graphs, (Sanchez, 1991) has developed a variable free natural logic. There is also a long tradition of variable free notation in lambda calculus: combinatory logic (Barendregt, 1984) and De Bruijn indices (de Bruijn, 1980) come to mind here. We will take our cue from this tradition, represent a variable context as a stack of registers, and give the simplest representation on can imagine for quantification, as the action of pushing a new element on top of the context stack. The interpretation of quantification as context transition puts the approach of this paper in the tradition of dynamic first order logic (Barwise, 1987; Groenendijk and Stokhof, 1991). For another recent combination of dynamic quantification and variable free representation see (Clark and Kurtonina, 1999).

The resulting perspective, apart from being interesting in its own right, sheds new light on the account of pronominal reference and anaphoric linking in natural language understanding. Persistent problems of pronominal reference and anaphoric linking have inspired logicians interested in natural language semantics to turn away from classical first and higher order logics, and to look elsewhere for a principled account of the process of linking pronouns to their antecedents. This ‘dynamic turn’ in natural language understanding gave rise to Discourse Representation Theory or DRT (Kamp, 1981) and to File Change Semantics (Heim, 1982). These in turn have led to various attempts at rational reconstruction, with (Barwise, 1987) and (Groenendijk and Stokhof, 1991) as the most prominent examples. The gist of all of the resulting frameworks is that the static variable binding regime from standard predicate logic gets replaced by a dynamic regime, where meanings are viewed as relations between variable states in a model.

In the original version of the ‘dynamic shift’, the basic ingredients are contexts and constraints on contexts. A DRT-style representation for a piece of text (or: discourse) looks basically like a *context* consisting of a list (or set) of variables, plus a set of *constraints* on this context. The informal picture of how the information conveyed by a piece of text grows is that of ‘updating’ of representation structures:

context		new context
constraints on context	$\longrightarrow$ update $\longrightarrow$	new constraints on context

This picture can only be made to work if we make sure that the contexts are represented smartly. Contexts are essentially sets of variables: a context just is a list of dynamically bound variables. These variables represent the antecedents which are available in any extension of that context.

The rational reconstructions of dynamic discourse representation given by (Barwise, 1987) and (Groenendijk and Stokhof, 1991) essentially represent introduction of new antecedents by means of random assignment to a variable. The meaning of $\exists x$ becomes the relation between variable states f, g with the property that f and g differ at most in their x value: ${}_f[\![\exists x]\!]_g$ iff $f[x]g$.

This does indeed solve the problem of how to use dynamic scoping of variables to account for unbounded anaphoric linkings. However, it does not give a rational reconstruction of the fact that discourse representation is supposed to work incrementally. The problem with incrementality is the awkwardness of the constraint that embedded contexts (contexts occurring inside the constraints on a given context) and extensions of contexts (representing extensions of anaphoric possibilities) should always employ fresh variables. Such a constraint is needed, for if old variables get reused existing anaphoric possibilities may get blocked off by destructive value assignment.

What one would like is illustrated by the following example, where we assume an initial representation for the sentence ‘A man entered’, which gets updated by subsequent processing of ‘A woman entered’, and next of ‘He smiled at her.’

$$\begin{array}{|l|}\hline \mathrm{x} \\ \hline \mathrm{Mx} \\ \mathrm{Ex} \\ \hline \end{array} \rightarrow \text{‘A woman entered’} \rightarrow \begin{array}{|l|}\hline \mathrm{x\ y} \\ \hline \mathrm{Mx} \\ \mathrm{Ex} \\ \mathrm{Wy} \\ \mathrm{Ey} \\ \hline \end{array} \rightarrow \text{‘He smiled at her’} \rightarrow \begin{array}{|l|}\hline \mathrm{x\ y} \\ \hline \mathrm{Mx} \\ \mathrm{Ex} \\ \mathrm{Wy} \\ \mathrm{Ey} \\ \mathrm{Sxy} \\ \hline \end{array}$$

In a rational reconstruction of this, one would assume that the sentences to be added to the existing representation have a representation of their own, so one would get something like:

$$\begin{array}{|l|}\hline \mathrm{x} \\ \hline \mathrm{Mx} \\ \mathrm{Ex} \\ \\ \hline \end{array} \quad + \quad \begin{array}{|l|}\hline \mathrm{y} \\ \hline \mathrm{Wy} \\ \mathrm{Ey} \\ \\ \hline \end{array} \quad = \quad \begin{array}{|l|}\hline \mathrm{x\ y} \\ \hline \mathrm{Mx} \\ \mathrm{Ex} \\ \mathrm{Wy} \\ \mathrm{Ey} \\ \hline \end{array}$$

Problem: what happens if we get a variable clash:

$$\begin{array}{|l|}\hline \mathrm{x} \\ \hline \mathrm{Mx} \\ \mathrm{Ex} \\ \hline \end{array} \quad + \quad \begin{array}{|l|}\hline \mathrm{x} \\ \hline \mathrm{Wx} \\ \mathrm{Ex} \\ \hline \end{array} \quad = \quad ?$$

In Kamp’s original version of discourse representation theory, and also in the extended version presented in (Kamp and Reyle, 1993), this problem does not occur, for the algorithm presented there always parses new sentences in the context of an existing representation structure, and for any indefinite noun phrase it encounters, it simply gives the instruction *take a fresh variable*. In other words, Kamp never *merges* representation structures, for instead of first building a Discourse Representation Structure (or: DRS) for the new sentence and next linking that DRS to the 'background' DRS representing the context, Kamp always processes the new sentence in the context of an existing DRS.

Still, it makes sense to abstract from context in a more radical way than discourse representation theory (in Kamp's original formulation) does, and to ask for the meaning of the sentence as a 'context change potential'. The merge problem arises in carrying out a Montagovian or Fregean programme of natural language analysis in a setting that takes context and context change into account. A Montagovian or Fregean approach insists on a genuine analysis of sentence or discourse meaning as context change potential. Such an analysis calls for the construction of sentence or discourse representations in a bottom-up fashion. For such an analysis, one can still represent a context as a DRS. A new sentence S is processed by first translating S into a DRS K, and next merging K with the context DRS.

But *how* should the DRSs be merged? There are various approaches to the merge problem for DRSs; see (van Eijck and Kamp, 1996) for an overview. These strategies amount to various ways of avoiding destructive assignments to variables, i.e., to various ways of arriving at structures which can be interpreted monotonically in terms of an information ordering on the meanings of the representation structures.

In this paper we will argue that incremental use of an indexed stack of dynamic variables leads to a natural monotonic interpretation, and thus to a natural approach to the merge problem. We get 'fresh variables' for free if we replace the dynamic variable binding mechanism of dynamic predicate logic with an indexing mechanism.

2. Dynamic Predicate Logic Without Variables

The De Bruijn notation for lambda calculus consists of replacing variables by indices that indicate the distance to their binding lambda operator. The lambda term $\lambda x \lambda y.(\lambda z.(y(zx))(yx)$ is written in De Bruijn notation as $\lambda\lambda.(\lambda.(2\ (1\ 3))(1\ 2)$. This approach carries over to predicate logic in a straightforward fashion. Rather than carry out the program in any detail we refer to (BenShalom, 1994) (but note that the connection to De Bruijn is not mentioned there).

For convenient comparison with formalisms that use ordinary variable binding we will write the indices 1, 2, 3, ... as $u_1, u_2, u_3, \ldots$ The reader may think of the registers as a one-dimensional array u with cells $u[1]$, $u[2]$, $u[3]$, ... Since only one array is employed, it does not matter whether we refer to the cells as 1, 2, 3, ... , or as $u_1, u_2, u_3, \ldots$

Let $V := \{u_j \mid j \in \mathbb{N}^+\}$. Let $i : V \to \mathbb{N}^+$ be given by $i(u_j) = j$. Extend this function to a function $i : \mathcal{P}(V) \to \mathcal{P}(\mathbb{N}^+)$ by means of $i(X) := \{j \mid u_j \in X\}$ (a map from sets of registers to the sets of their indices). For any finite set X of positive natural numbers, let $\sup(X)$ give the maximum of X in case X is non-empty, 0 otherwise (the supremum function).

The language L of incremental dynamic predicate logic consists of the union $\bigcup_{n\in\mathbb{N}} L_n$, where each L_n gives the formulas that assume a *context* of size n. Each formula is a pair (n, ϕ), where n gives the size of the context. The languages L_n are defined by simultaneous recursion, as follows. (Note that the v_i in the definition below are meta-variables over registers; thus, if the value of v_j happens to be u_k, then $i(v_j) = k$. In other words, the index of v_j may well be different from j.)

Definition 1 (Terms and formulas of L)

$$
\begin{array}{rcll}
v & ::= & u_1 \mid u_2 \mid \cdots & \\
L & ::= & (n, \top) & \textit{if } n \in \mathbb{N} \\
 & \mid & (n, \exists; \phi) & \textit{provided } (n+1, \phi) \in L \\
 & \mid & (n, Pv_1 \cdots v_m; \phi) & \textit{provided } \sup\{i(v_1), \ldots, i(v_m)\} \le n \\
 & & & \quad \textit{and } (n, \phi) \in L \\
 & \mid & (n, v_1 \doteq v_2; \phi) & \textit{provided } \max(i(v_1), i(v_2)) \le n \\
 & & & \quad \textit{and } (n, \phi) \in L \\
 & \mid & (n, \neg\phi_1; \phi_2) & \textit{provided } (n, \phi_1) \in L \textit{ and } (n, \phi_2) \in L
\end{array}
$$

Note that we have built into the language that formulas are flat list structures. (In fact, $\top$ may be viewed as the nil formula list that goes with a flat list formula list constructor that prefixes formulas to formula lists.) Every formula has the form $(n, \phi_1; \ldots; \phi_k; \top)$, with $k \ge 0$. If $k > 0$ we will write $(n, \phi_1; \ldots; \phi_k; \top)$ as $(n, \phi_1; \ldots; \phi_k)$.

We will omit unnecessary parentheses, writing the formula

$$(3, \neg(Ru_2u_3))$$

as $3, \neg Ru_2u_3$, etcetera. Occasionally, we will write $\exists; \phi$ as $\exists\phi$. Also, we abbreviate $\neg\top$ as $\bot$, and $\neg(\phi_1; \cdots; \phi_n; \neg\phi_{n+1})$ as $(\phi_1; \cdots; \phi_n) \to \phi_{n+1}$.

The static interpretation is replaced by a dynamic one. Let $\mathcal{M} = (M, I)$ be a first order model, and σ an element of M^* (the set of all finite sequences of elements taken from M). We use $l(\sigma)$ for the length of $\sigma \in M^*$, and $\sigma[n]$ for the n-th element of σ. Then the term interpretation with respect to $\mathcal{M}$ and σ is given by (we use $\uparrow$ for 'undefined'):

$$[\![v]\!]^{\mathcal{M}}_{\sigma} := \begin{cases} \sigma[i(v)] & \text{if } i(v) \le l(\sigma), \\ \uparrow & \text{otherwise.} \end{cases}$$

In what follows, we will often extend a stack $\sigma \in M^*$ with a single value $a \in M$. Notation for this: $\sigma\hat{\ }a$. Concatenation of two stacks $\sigma, \tau \in M^*$, in that order, is written as $\sigma\hat{\ }\tau$. If $\sigma, \tau \in M^*$ we use $\sigma \sqsubseteq \tau$ for: there is a $\theta \in M^*$ with $\sigma\hat{\ }\theta = \tau$. It is easy to see that $\sqsubseteq$ is a partial order on M^* (reflexive, transitive and anti-symmetric).

As an alternative to the De Bruijn style binding regime, where the binding quantifier is found by counting from the inside out, it is also possible to count from the outside in. This is similar to the way lambdas are counted in Cartesian closed category models of the lambda calculus (see e.g. (Gunter, 1992, Ch. 3); also (Aczel, 1996)). Call this 'reverse De Bruijn style.'

The reason why it is more convenient to use reverse De Bruijn indexing rather than regular De Bruijn style is this. The key feature of dynamic anaphora logics is the ability of the existential quantifier to bind variables outside its proper scope. Consider the DPL text $\exists x; Px; \exists y; Qy; Rxy$. Here the x and y of Rxy are bound outside of the proper scope by $\exists x$ and $\exists y$ respectively, so variables can be viewed as anaphoric elements linked to a preceding existential quantifier that introduces a referent. Similarly, in DRT, the introduction of a reference marker acts as an existential quantifier with dynamic scope.

The regular De Bruijn analogue of the above DPL formula would be the following (we assume that the anaphoric context is empty):

$$(0, \exists; Pu_1; \exists; Qu_1; Ru_2u_1)$$

The register u_1 in Pu_1 and the register u_2 in Ru_2u_1 are bound by the same quantifier (the leftmost occurrence of $\exists$). This illustrates that anaphoric coreference (or: dynamic binding) is no longer encoded by use of the same index, but the antecedent of an index has to be worked out by taking the 'existential depth' of the intervening formula into account.

The awkwardness in antecedent recovery can be avoided by using reverse De Bruijn indexing. The reverse De Bruijn analogue of the example $\exists x; Px; \exists y; Qy; Rxy$ looks like this:

$$(0, \exists; Pu_1; \exists; Qu_2; Ru_1u_2)$$

All occurrences of u_1 are bound by the same quantifier (the leftmost occurrence of $\exists$), and similarly, all occurrences of u_2 are bound by the rightmost occurrence of $\exists$.

The semantic definition of satisfaction, for incremental dynamic logic under the reverse De Bruijn indexing scheme, runs as follows:

Definition 2 (Satisfaction for L)

$$
\begin{array}{rl}
{}_{\sigma}[\![n,\top]\!]^{\mathcal{M}}_{\tau} & \textit{iff } \sigma=\tau,\\
{}_{\sigma}[\![n,\exists;\phi]\!]^{\mathcal{M}}_{\tau} & \textit{iff } l(\sigma)=n \textit{ and there is an } a\in M\\
 & \quad \textit{with } {}_{\sigma\hat{\ }a}[\![n+1,\phi]\!]^{\mathcal{M}}_{\tau},\\
{}_{\sigma}[\![n,Pv_1\cdots v_m;\phi]\!]^{\mathcal{M}}_{\tau} & \textit{iff } l(\sigma)=n, \langle [\![v_1]\!]^{\mathcal{M}}_{\sigma}\dots,[\![v_m]\!]^{\mathcal{M}}_{\sigma}\rangle\in I(P),\\
 & \quad \textit{and } {}_{\sigma}[\![n,\phi]\!]^{\mathcal{M}}_{\tau},\\
{}_{\sigma}[\![n,v_1\doteq v_2;\phi]\!]^{\mathcal{M}}_{\tau} & \textit{iff } l(\sigma)=n, [\![v_1]\!]^{\mathcal{M}}_{\sigma}=[\![v_2]\!]^{\mathcal{M}}_{\sigma}, \textit{ and } {}_{\sigma}[\![n,\phi]\!]^{\mathcal{M}}_{\tau},\\
{}_{\sigma}[\![n,\neg\phi_1;\phi_2]\!]^{\mathcal{M}}_{\tau} & \textit{iff } l(\sigma)=n, \textit{ there is no } \theta\in M^* \textit{ with } {}_{\sigma}[\![n,\phi_1]\!]^{\mathcal{M}}_{\theta},\\
 & \quad \textit{and } {}_{\sigma}[\![n,\phi_2]\!]^{\mathcal{M}}_{\tau}.
\end{array}
$$

Note that the proviso $l(\sigma)=n$ in the semantic clauses guarantees that the term functions $[\![v_i]\!]^{\mathcal{M}}_{\sigma}$ in predicates $Pv_1\cdots v_m$ and identities $v_1\doteq v_2$ are well defined.

The definition of the semantics for L is in fact a straightforward adaptation of the dynamic semantics for predicate logic defined in (Groenendijk and Stokhof, 1991), which is in turn closely related to a proposal made in (Barwise, 1987). However, this semantics is *not* equivalent to the semantics given by Groenendijk and Stokhof, but has an important advantage over it. In Groenendijk and Stokhof's semantics for DPL, a repeated assignment to a single variable by means of a repeated use of the same existential quantifier-variable combination blocks off the individual introduced by the first use of the quantifier from further anaphoric reference. After $\exists xPx;\exists xQx$, the variable x will refer to the individual introduced by $\exists xQx$, and the individual introduced by $\exists xPx$ has become inaccessible.

In the so-called sequence semantics proposed in (Vermeulen, 1993) this problem is solved by making every variable refer to a stack, and interpreting an existential quantification for variable x as a push operation on the x stack. The quantification $\exists x$ now gets a counterpart xE, interpreted as a pop of the x stack. In our reverse De Bruijn semantics for L we use a single finite stack, and we do not allow pops. This ensures that existential quantification is non-destructive, in other words that our semantics is *incremental.* The push stack operations are replaced by a single push operation (the interpretation of the existential quantifier). Note that quantifications never can destroy previous dynamic assignments in the same formula, nor can they overwrite initially given values.

To reason about the semantics for L we need the notion of the 'existential depth' of a formula. Intuitively, the existential depth of (n,ϕ) calculates the number of positions by which the stack grows during the semantic processing of ϕ. E.g., the existential depth of $(n,\exists)$

is 1, for any n. If $(n, \phi) \in L$, the existential depth of ϕ is given by:

$$\begin{array}{lll} e(\top) & := & 0 \\ e(\exists; \phi) & := & 1 + e(\phi) \\ e(Pv_1 \cdots v_m; \phi) & := & e(\phi) \\ e(v_1 \doteq v_2; \phi) & := & e(\phi) \\ e(\neg\phi_1; \phi_2) & := & e(\phi_2) \end{array}$$

Note that the definition of the semantics for L ensures that registers $u_1, \ldots, u_k$ of an L-formula (k, ϕ) are not affected by the stack dynamics of the existential quantifier. The values of these positions are read from the input state; these are the anaphoric references picked up from the surrounding context. Positions higher up on the 'stack' get their value from an existential quantifier action inside ϕ. This is made formal in the following lemma (the proof is by induction on existential depth):

Lemma 3 (Incrementality) *If* ${}_\sigma[\![n, \phi]\!]_\tau^{\mathcal{M}}$ *then* $\sigma \sqsubseteq \tau$.

The language L is designed for the translation of open texts: texts that may contain occurrences of pronouns which take their reference from the surrounding context. One may think of such texts as annotated by means of anaphoric indices $i, j, \ldots$, where each index indicates that an anaphoric element is to be linked to some contextually given antecedent. Such a text presupposes that the context provides referents for such indices, in such a way that all these referents might be different. In other words, the context must be such that different indices correspond to different register cells. The essence of anaphoric linking is the process of picking up antecedents from context, and the framework of incremental dynamics defines context as the sort of thing that can provide such antecedents.

Here is a translation from L to standard DPL:

$$\begin{array}{rcl} (n, \top)^\bullet & := & \top \\ (n, \exists; \phi)^\bullet & := & \exists u_{n+1}; (n+1, \phi)^\bullet \\ (n, Pv_1 \cdots v_m; \phi)^\bullet & := & Pv_1 \cdots v_m; (n, \phi)^\bullet \\ (n, v_1 \doteq v_2; \phi)^\bullet & := & v_1 \doteq v_2; (n, \phi)^\bullet \\ (n, \neg\phi_1; \phi_2)^\bullet & := & \neg(n, \phi_1)^\bullet; (n, \phi_2)^\bullet. \end{array}$$

E.g., L-formula $(2, \exists; Ru_1u_3; \exists; Su_3u_4)$ gets translated by $^\bullet$ into the DPL formula $\exists u_3; Ru_1u_3; \exists u_4; Su_3u_4$.

Note that the DPL translations of L formulas are rather special, for they will contain *no destructive assignments*, since all quantifications are over 'fresh' variables.

To show that the translation function is correct in the sense that it preserves satisfaction of formulas, assume a DPL language over the set of variables $V = \{u_i \mid i \in \mathbb{N}^+\}$. Then a DPL state over a model $\mathcal{M} = (M, I)$ is a member of M^V. We use $\mathcal{M}, s, s' \models_{dpl} \phi$ for: the state pair s, s' satisfies the DPL formula ϕ in model $\mathcal{M}$.

If s is a DPL state over $\mathcal{M}$, and $\sigma \in M^*$ we define the state s_σ as follows. If $i \leq l(\sigma)$ then $s_\sigma(u_i) := \sigma[i]$, otherwise $s_\sigma(u_i) := s(u_i)$.

Proposition 4 *For all $(n, \phi) \in L$, all models $\mathcal{M}$, all $\sigma, \tau \in M^*$, all $s \in M^V$: ${}_\sigma[\![(n, \phi)]\!]^{\mathcal{M}}_\tau$ iff $\mathcal{M}, s_\sigma, s_\tau \models_{dpl} (n, \phi)^\bullet$.*

The proof is by induction on the structure of ϕ.

To illustrate the considerable expressive power of dynamic logic without variables, here is a translation function from FOL to L. We assume a set of first order variables $V = \{u_i \mid i \in \mathbb{N}^+\}$. If f is an assignment to V in some domain M and $\sigma \in M^k$, f_σ is defined in the obvious way, by putting $f_\sigma(u_i) := \sigma[i]$ for $i \leq k$, $f_\sigma(u_i) := f(u_i)$ for $i > k$.

To translate a FOL formula ϕ, let $k := \sup\{i \mid u_i \in FV(\phi)\}$. Then the L translation of ϕ is (k, ϕ^k), with the translation functions ^{k}, for $k \in \mathbb{N}$, defined as follows:

$$
\begin{aligned}
(\top)^k &:= \top \\
(Pv_1 \cdots v_m)^k &:= Pv_1 \cdots v_m \\
(v_1 \doteq v_2)^k &:= v_1 \doteq v_2 \\
(\neg\phi)^k &:= \neg(\phi)^k \\
(\phi_1 \wedge \phi_2)^k &:= \neg\neg(\phi_1)^k ; (\phi_2)^k \\
(\exists u_i \phi)^k &:= \exists ; ([u_{k+1}/u_i]\phi)^{k+1}.
\end{aligned}
$$

The substitution $[u_{k+1}/u_i]\phi$ is subject to the usual condition that u_{k+1} should be free for u_i in ϕ. If necessary, replace ϕ by an alphabetic variant that meets the condition.

Proposition 5 *Let ϕ be a FOL formula, and let*

$$k := \sup\{i \mid u_i \in FV(\phi)\}.$$

For all models $\mathcal{M} = (M, I)$, all stacks $\sigma \in M^k$, all variable assignments f:

$$\mathcal{M}, f_\sigma \models \phi \textit{ iff there is a } \tau \textit{ with } {}_\sigma[\![k, \phi^k]\!]^{\mathcal{M}}_\tau.$$

This is again proved by induction on the structure of ϕ.

Combining the well-known translation from DPL to FOL with the above translation, we get in two steps a translation from DPL to L. Still, it is instructive to define a direct translation. The fixed occurrences of variables in a DPL formula are the variable occurrences that are neither classically bound nor in the dynamic scope of an existential quantifier. Let ϕ be a DPL formula, and let

$$k := \sup\{i \mid u_i \text{ has a fixed occurrence in } \phi\}.$$

Then ϕ translates into $(k, \phi^{(k)})$, with $^{(k)}$ given by:

$$\begin{aligned}
(\top)^{(k)} &:= \top \\
(\exists u_i)^{(k)} &:= \exists \\
(Pv_1 \cdots v_m)^{(k)} &:= Pv_1 \cdots v_m \\
(v_1 \doteq v_2)^{(k)} &:= v_1 \doteq v_2 \\
(\neg\phi)^{(k)} &:= \neg(\phi)^{(k)} \\
((\phi_1; \phi_2); \phi_3)^{(k)} &:= (\phi_1; (\phi_2; \phi_3))^{(k)} \\
(A; \phi)^{(k)} &:= A^{(k)}; \phi^{(k)} \\
(\exists u_i; \phi)^{(k)} &:= \exists; ([u_{k+1}/u_i]\phi)^{(k+1)}.
\end{aligned}$$

Here A denotes a DPL formula of the form $Pv_1 \cdots v_m$, $v_1 \doteq v_2$ or $\neg\phi$. Moreover, $[u_{k+1}/u_i]\phi$ denotes dynamic substitution, i.e., substitution of u_{k+1} for all occurrences of u_i that are neither classically or dynamically bound, while taking care, through appropriate switches to alphabetic variants, that the replacing occurrences of u_{k+1} are dynamically free in the result (i.e., are not in the dynamic scope of an existential quantifier).

Finally, we can prove by induction on the structure of ϕ:

Proposition 6 *For all models $\mathcal{M} = (M, I)$, all $\sigma \in M^k$, all $s \in V \to M$, where $V = \{u_i \mid i \in \mathbb{N}^+\}$, the following holds:*

$$\exists t \in V \to M : \ \mathcal{M}, s_\sigma, t \models_{dpl} \phi \iff \exists \tau \in M^* : \ {}_\sigma[\![k, \phi^{(k)}]\!]_\tau^{\mathcal{M}}.$$

3. Incremental Dynamic Logic and Discourse Representation

Next, we want to show that L formulas correspond exactly to canonical forms of Discourse Representation Structures in the sense of (Kamp, 1981) (so-called *pure* DRSs). DRSs are defined by the following mutual recursion. Again, we assume for simplicity that there are no individual constants (and again, nothing hinges on this).

$$\begin{aligned}
v &::= u_1 \mid u_2 \mid \cdots \\
C &::= Pv_1 \cdots v_n \mid v_1 \doteq v_2 \mid \neg D \\
D &::= (\{v_1, \ldots, v_n\}, \{C_1, \ldots, C_m\})
\end{aligned}$$

If $D = (\{v_1, \dots, v_n\}, \{C_1, \dots, C_m\})$ and D' are DRSs, then $D \Rightarrow D'$ abbreviates the condition

$$\neg(\{v_1, \dots, v_n\}, \{C_1, \dots, C_m, \neg D'\}).$$

We give a translation function $^\oplus$ from L formulas to DRSs, as follows (using $(n, \phi)^\oplus_0$ and $(n, \phi)^\oplus_1$ for the first and second components of $(n, \phi)^\oplus$):

$$\begin{array}{rcl}
(n, \top)^\oplus & := & (\emptyset, \emptyset) \\
(n, \exists; \phi)^\oplus & := & (\{u_{n+1}\} \cup (n+1, \phi)^\oplus_0, (n+1, \phi)^\oplus_1) \\
(n, Pv_1 \cdots v_m; \phi)^\oplus & := & ((n, \phi)^\oplus_0, \{Pv_1 \cdots v_m\} \cup (n, \phi)^\oplus_1) \\
(n, v_1 \doteq v_2; \phi)^\oplus & := & ((n, \phi)^\oplus_0, \{v_1 \doteq v_2\} \cup (n, \phi)^\oplus_1) \\
(n, \neg(\phi); \psi)^\oplus & := & ((n, \psi)^\oplus_0, \{\neg(n, \phi)^\oplus\} \cup (n, \psi)^\oplus_1).
\end{array}$$

An example:

$$(2, Ru_1u_2; \neg(\exists; Ru_1u_3))^\oplus \quad = \quad \begin{array}{|l|}\hline \\ \hline Ru_1u_2 \\ \\ \neg\ \begin{array}{|c|}\hline u_3 \\ \hline Ru_1u_3 \\ \hline\end{array} \\ \hline\end{array}$$

We will now show that the translation is adequate. Assume a model $\mathcal{M} = (M, I)$. An embedding function in the sense of DRT is a function from a finite subset of the set of variables $\{u_i \mid i \in \mathbb{N}^+\}$ to M. We use $\sigma^\oplus$ for the function in $\{u_1, \dots u_{l(\sigma)}\} \to M$ that corresponds to stack $\sigma \in M^*$, in the obvious sense (namely, by setting $\sigma^\oplus(u_i) := \sigma[i]$).

Proposition 7 *If $(n, \phi) \in L$ then:*

1. *$(n, \phi)^\oplus$ is a DRS.*
2. *${}_\sigma[\![n, \phi]\!]^{\mathcal{M}}_\tau$ iff $\tau^\oplus$ verifies $(n, \phi)^\oplus$ in $\mathcal{M}$ with respect to $\sigma^\oplus$ (in the sense of DRT).*

Both claims are proved by induction on the structure of ϕ.

The DRS translations have the additional property that they yield *pure* DRSs: If K' is a sub-DRS of K then their sets of introduced markers will be disjoint. A special case is the case of L formulas of the form $(0, \phi)$. These correspond precisely to so-called *proper* pure DRSs, i.e., pure DRSs without 'fixed' variable occurrences.

To capture the precise meaning of 'fixed' variables in a DRS, we need to distinguish three kinds of variable occurrences in a DRS: (1) fixed by the larger context (or: free in the current context), (2) introduced in the current context (or: dynamically bound by a quantifier in the current context), and (3) fixed in a subordinate context (or: classically bound).

Here are the definitions of these sets.

Definition 8 (fix, intro, cbnd)

	fix	*intro*	*cbnd*
$(U, \{C_1, \dots, C_m\})$	$\bigcup_i \mathit{fix}(C_i) - U$	U	$\bigcup_i \mathit{cbnd}(C_i)$
$Pv_1 \cdots v_n$	$\{v_1, \dots, v_n\}$	$\emptyset$	$\emptyset$
$v_1 \doteq v_2$	$\{v_1, v_2\}$	$\emptyset$	$\emptyset$
$\neg D$	$\mathit{fix}(D)$	$\emptyset$	$\mathit{intro}(D) \cup \mathit{cbnd}(D)$

To define a translation from DRSs to L formulas, we can use a technique similar to the mapping of FOL to L. Let D be a DRS, and let k be $\sup\{i | u_i \in \mathit{fix}(D)\}$. Then the L-translation of D is the formula $(k, D^{[k]})$, where the functions $^{[k]}$ are given by:

$(\emptyset, \{C_1, \dots, C_m\})^{[k]} := C_1^{[k]}; \dots ; C_m^{[k]}$
$(\{v_1, \dots, v_n\}, \{C_1, \dots, C_m\})^{[k]} :=$
$\quad \exists; (\{v_2, \dots, v_n\}, \{[u_{k+1}/v_1]C_1, \dots, [u_{k+1}/v_1]C_m\})^{[k+1]}$
$(\bot)^{[k]} := \bot$
$(Pv_1 \cdots v_m)^{[k]} := Pv_1 \cdots v_m$
$(v_1 \doteq v_2)^{[k]} := v_1 \doteq v_2$
$(\neg\phi)^{[k]} := \neg(\phi)^{[k]}$.

Note that there is an element of indeterminism in this translation instruction, for $\{v_1, \dots, v_n\}$ is a set, and the translation recipe instructs us to take its elements one by one. If the reader does not like this, she can use the order on $\{v_1, \dots, v_n\}$ imposed by the indices of the variables to always pick the smallest element from this set.

Again, we have to ensure that u_{k+1} is free for v_i in $[u_{k+1}/v_i]C$, i.e., that u_{k+1} does not have contextually bound occurrences in C. If this condition is not met, we have to replace C by an alphabetic variant first. For all purposes, contextually bound variables in DRT behave exactly like bound variables in FOL. Note that u_{k+1} cannot have fixed occurrences in C, by an inductive argument based on the fact that the initial choice of index is the highest index of the initial set of fixed occurrences, and that fixed occurrences of variables that are not in the initial set would have caused a variable clash at the level where they got introduced.

For the next proposition, we have to relate embedding functions to stacks of elements of a domain. If f is a function in

$$\{u_i \mid 1 \leq i \leq k\} \to M$$

for some $k \in \mathbb{N}$, then $f^* \in M^k$ is given by $f^*[i] := f(u_i)$.

Proposition 9 *Let D be a DRS, and let k be $\sup\{i \mid u_i \in \mathit{fix}(D)\}$. Then the following hold:*

1. $(k, D^{[k]}) \in L$,
2. *For all models $\mathcal{M} = (M, I)$, all functions $f : \mathit{fix}(D) \to M$: there is a $g : \mathit{fix}(D) \cup \mathit{intro}(D) \to M$ such that g verifies D with respect to f in $\mathcal{M}$ (in the sense of DRT) iff there is a $\tau \in M^*$ with ${}_{f^*}[\![k, D^{[k]}]\!]^{\mathcal{M}}_{\tau}$.*

Both claims are proved by induction on the structure of D.

In this section we have shown that L and DRT have exactly the same expressive power. Moreover, L formulas are isomorphic to DRSs in canonical form, in the following sense. L formulas correspond to pure DRSs, and L formulas of the form $(0, \phi)$ correspond to proper pure DRSs. The advantage of L over DRT will reveal itself when we are going to define logical consequence for L, in Section 5.

4. Merging Formulas and Merging Representation Structures

Suppose we want to 'merge' two formulas (n, ϕ) and (m, ψ) in left-to-right order, in such a way that the output of (n, ϕ) serves as input to (m, ψ). One could introduce a merge operation $\bullet$ as a partial operation on L formulas, as follows:

$$(n, \phi) \bullet (m, \psi) := \begin{cases} (n, \phi; \psi) & \text{if } m = n + e(\phi), \\ \uparrow & \text{otherwise.} \end{cases}$$

In case the result of merging (n, ϕ) and (m, ψ) is undefined all is not lost, however. The undefinedness may be due to the fact that the context is too large or to the fact that the context is too small. The context is too large in case $n + e(\phi) > m$. In this case, the problem can be remedied by performing a 'write memory shift operation' on (m, ψ), as follows:

$$\frac{(m, \psi)}{(m + k, [^{m}_{+k}]\psi)}$$

Here, $[^{m}_{+k}]\psi$ is the index substitution which replaces every $i > m$ by $i + k$.

Proposition 10 *${}_{\sigma}[\![m,\psi]\!]^{\mathcal{M}}_{\sigma\hat{}\tau}$ iff for all $\theta \in M^k$: ${}_{\sigma\hat{}\theta}[\![m+k, [^{\;m}_{+k}]\psi]\!]^{\mathcal{M}}_{\sigma\hat{}\theta\hat{}\tau}$.*

The other case where the result of merging (n,ϕ) and (m,ψ), in that order, is undefined, is the case where the context is too small. This is the case when $n + e(\phi) < m$. In this case we can use 'existential padding'. A useful abbreviation for this is $\exists^k$, defined recursively by $\exists^0 := \top$, $\exists^{k+1} := \exists; \exists^k$. Existential padding is applied to conclude from $m+k, \psi$ that $m, \exists^k; \psi$.

Proposition 11 *${}_{\sigma}[\![m, \exists^k; \psi]\!]^{\mathcal{M}}_{\tau}$ iff ${}_{\sigma\hat{}\tau[m+1..m+k]}[\![m+k, \psi]\!]^{\mathcal{M}}_{\tau}$.*

The rules for memory shift and existential padding are built into the calculus of Section 6.

As Propositions 7 and 9 have shown us, the variable free dynamic logic L can be viewed as a rational reconstruction of DRT (in a way that DPL *cannot* be viewed as such). In fact, the reconstruction has made us sensitive to a distinction which often remains implicit in DRT: the distinction between representation structures which contain reference markers not introduced in the structure itself but imported from a pre-existing representation on one hand and representation structures which do not contain such imported markers on the other. These are the representations in which no reference markers are imported from outside; every marker gets introduced in the structure itself.

The variable constraint imposed in DRT ("always take fresh variables when extending a DRT structure") avoids the destructive assignment problem from DPL, but this variable constraint comes with a heavy penalty. It necessitates a top-down DRS construction algorithm. The variable free notation, on the other hand, points the way to a very natural bottom-up perspective.

Several possible solutions to the merge problem for DRT are discussed in (van Eijck and Kamp, 1996). If one wants merge to be a total operation on DRSs, the merge of DRSs D and D', in that order, may involve substitution of the introduced variables of D'. The present variable free perspective on dynamic logic suggests a particular choice for the merge operation. The DRS translations of L-formulas have the following general form (this is the general form of a pure DRS):

$$\begin{array}{|c|}\hline u_{m+1}, \ldots, u_n \\ \hline C_1 \\ \vdots \\ C_k \\ \hline \end{array}$$

Here it is assumed that all the markers occurring in $C_1, \dots, C_k$ are among $u_1, \dots, u_n$. The markers $u_1, \dots, u_m$ are the fixed markers of the DRS, the markers $u_{m+1} \cdots u_n$ the introduced reference markers.

Assuming that DRSs are all in this canonical form, we can merge them as follows, using substitution to avoid variable clashes:

$$\begin{array}{|c|}\hline u_{p+1} \cdots u_{p+m} \\ \hline C_1 \\ \vdots \\ C_k \\ \hline \end{array} \bullet_{[u_{p+m+1}/u_{q+1}, \dots, u_{p+m+n}/u_{q+n}]} \begin{array}{|c|}\hline u_{q+1} \cdots u_{q+n} \\ \hline C'_1 \\ \vdots \\ C'_r \\ \hline \end{array}$$

$$\Longrightarrow \quad \begin{array}{|c|}\hline u_{p+1} \cdots u_{p+m+n} \\ \hline C_1 \\ \vdots \\ C_k \\ C'_1[u_{p+m+1}/u_{q+1}, \dots, u_{p+m+n}/u_{q+n}] \\ \vdots \\ C'_r[u_{p+m+1}/u_{q+1}, \dots, u_{p+m+n}/u_{q+n}] \\ \hline \end{array}$$

Here is an example:

$$\begin{array}{|c|}\hline u_2, u_3 \\ \hline Ru_1u_2 \\ Su_2u_3 \\ \hline \end{array} \bullet_{[u_4/u_3, u_5/u_4]} \begin{array}{|c|}\hline u_3, u_4 \\ \hline Tu_1u_4 \\ Vu_3u_4 \\ \hline \end{array} \Longrightarrow \begin{array}{|c|}\hline u_2, u_3, u_4, u_5 \\ \hline Ru_1u_2 \\ Su_2u_3 \\ Tu_1u_5 \\ Vu_4u_5 \\ \hline \end{array}$$

This example corresponds to the merge of $(1, \exists; \exists; Ru_1u_2; Su_2u_3)$ and $(2, \exists; \exists; Tu_1u_4; Vu_3u_4)$, in that order, after memory shift right of the second formula over one position, to get $(3, \exists; \exists; Tu_1u_5; Vu_4u_5)$, with end result:

$$(1, \exists; \exists; Ru_1u_2; Su_2u_3; \exists; \exists; Tu_1u_5; Vu_4u_5).$$

The switch rules of the calculus of Section 6 permit to transform this in turn into:

$$(1, \exists; \exists; \exists; \exists; Ru_1u_2; Su_2u_3; Tu_1u_5; Vu_4u_5),$$

which again corresponds to (a canonical, i.e., pure, representation of) the result DRS.

5. A Transitive Notion of Dynamic Consequence

A piece of text containing anaphoric references can either be self-contained, in case all anaphora find their antecedent in the text itself, or it can be linked to a context, in case some pronouns refer to an antecedent outside the text itself, e.g. an antecedent mentioned in previous discourse, or introduced by another speaker, or introduced by an act of pointing, and so on. We can say that texts of the latter kind have an *anaphoric presupposition.* In order to establish the truth conditions of such a text one needs access to the context that provides antecedents for the outward pointing anaphora, and in that sense the anaphoric context is presupposed.

Still, it is clear that we can make (minimal) sense of a piece of text containing unresolved anaphora, even without access to the context. We can *abstract* from the context, in the usual way, by viewing the meaning of a piece of text with anaphoric presupposition as a *function* from contexts to denotations. The full information content of the text reveals itself once the anaphoric context is plugged in. As long as the context is unknown, the anaphora with an outside link have the weakest possible information content: they carry the same information as a wide scope existential quantifier. (Note how this is different from the recent proposal in (Beaver, 1999): the relation $\rightsquigarrow$ of 'possible entailment' defined there existentially quantifies over possible ways of resolving pronoun references, whereas we define the consequence relation for essentially unresolved pronouns, i.e., 'anaphora with an outside link'.)

The modeling of anaphoric presupposition as context in the scope of an existential quantification, with this context in turn treated as a piece of 'read-only memory', suggests a very natural consequence notion for 'reasoning under anaphoric presupposition'.

The anaphoric presupposition of a formula (n, ϕ) is given by its 'offset' n, for the number of anaphoric elements that need (possibly different) outside referents. It should be noted, though, that not every index i in $\{1, \dots, n\}$ need occur in ϕ. We can now say that (n, ϕ) entails (m, ψ) iff for all models, the interpretation of (n, ϕ) is 'more informative' than that of (m, ψ). The formula (n, ϕ) will export $n+e(\phi)$ anaphoric elements, for $e(\phi)$ measures the number of new referents that are introduced by ϕ. In order to ensure that all of these can be absorbed by (m, ψ), we have to assume that $n+e(\phi) \leq m$. Making the assumption $n + e(\phi) \leq m$ into a presupposition boils down to the statement that if

$n + e(\phi) > m$ then it is vacuously *false* that (m, ψ) follows from (n, ϕ). These considerations lead to the following formal definition of logical consequence for L:

Definition 12 (L Consequence)
$(n, \phi) \models (m, \psi)$ $:\Longleftrightarrow$ $n + e(\phi) \leq m$ and for all $\mathcal{M}, \sigma, \tau$: if ${}_\sigma[\![n, \phi]\!]^{\mathcal{M}}_\tau$ then there are θ, ρ with $\tau \sqsubseteq \theta$ and ${}_\theta[\![m, \psi]\!]^{\mathcal{M}}_\rho$.

This consequence relation is truly dynamic in that it allows carrying anaphoric links from premise to conclusion. For example: from 'a man walks and he talks' it follows that 'he talks':

$$(0, \exists; Mu_1; Wu_1; Tu_1) \models (1, Tu_1).$$

The following lemma shows that L consequence has a very desirable property.

Lemma 13 (Transitivity) *For all $(n, \phi), (m, \psi), (k, \chi) \in L$:*

If $(n, \phi) \models (m, \psi)$ and $(m, \psi) \models (k, \chi)$ then $(n, \phi) \models (k, \chi)$.

Proof. Suppose $(n, \phi) \models (m, \psi)$ and $(m, \psi) \models (k, \chi)$, and assume ${}_\sigma[\![n, \phi]\!]^{\mathcal{M}}_\tau$. We have to show that there are θ and ρ with $\tau \sqsubseteq \theta$ and ${}_\theta[\![k, \chi]\!]^{\mathcal{M}}_\rho$.

By $(n, \phi) \models (m, \psi)$ and the assumption there are $\tau' \sqsupseteq \tau$ and θ with ${}_{\tau'}[\![m, \psi]\!]^{\mathcal{M}}_\theta$. From this and $(m, \psi) \models (k, \chi)$ we get $\theta' \sqsupseteq \theta$ and ρ with ${}_{\theta'}[\![k, \chi]\!]^{\mathcal{M}}_\rho$. By incrementality and by transitivity of $\sqsubseteq$, we get $\tau \sqsubseteq \theta'$ and we are done.

One of the problems with the dynamic consequence relation of DPL (Groenendijk and Stokhof, 1991) is the fact that it is not transitive, as witnessed by Van Benthem's example (van Benthem, 1987):

Suppose a man owns a house. Then he owns a garden.
Suppose he owns a garden. Then he sprinkles it.
BUT NOT: Suppose a man owns a house. Then he sprinkles it.

Right now, let us gloss over the role of pronominal reference resolution in interpreting this natural language example. We will return to that aspect below, at the end of the section.

The DPL version of this example can be taken as a playful reminder of the lack of transitivity of DPL consequence:

$$\frac{\exists x; Mx; \exists y; Hy; Oxy \models_{dpl} \exists z; Gz; Oxz \quad \exists z; Gz; Oxz \models_{dpl} Sxz}{\exists x; Mx; \exists y; Hy; Oxy \not\models_{dpl} Sxz}$$

What this illustrates is that the DPL consequence relation $\models_{dpl}$ is not transitive. Here $\models_{dpl}$ is defined by: $\phi \models_{dpl} \psi$ iff for all $\mathcal{M}, \sigma, \sigma'$: if $\mathcal{M}, \sigma, \sigma' \models_{dpl} \phi$ then there is some σ'' with $\mathcal{M}, \sigma', \sigma'' \models_{dpl} \psi$.

Interestingly, the same example can be brought to bear on DRT. Let us adopt as the DRT consequence relation the obvious meta-language counterpart of the DRS operation $\rightarrow$, namely the relation $\models$ for which we have: $D_1 \rightarrow D_2$ is true in all models iff $D_1 \models D_2$. Now assume we represent the example like this:

$$\frac{\begin{array}{|c|}\hline x, y \\ \hline Mx \\ Hy \\ Oxy \\ \hline \end{array} \models_{drt} \begin{array}{|c|}\hline z \\ \hline Gz \\ Oxz \\ \\ \hline \end{array} \qquad \begin{array}{|c|}\hline z \\ \hline Gz \\ Oxz \\ \\ \hline \end{array} \models_{drt} \begin{array}{|c|}\hline \quad \\ \hline Sxz \\ \\ \\ \hline \end{array}}{\begin{array}{|c|}\hline x, y \\ \hline Mx \\ Hy \\ Oxy \\ \hline \end{array} \not\models_{drt} \begin{array}{|c|}\hline \quad \\ \hline Sxz \\ \\ \\ \hline \end{array}}$$

The problem exhibited by the example hinges on the fact that reference markers used in a constraint of a discourse representation structure D without being introduced in the set of reference markers of that structure D are assumed to get their reference from context, but in the example case this leads to the wrong reading, for they should be existentially quantified over.

This suggests that the DRT representation of the example may have been wrong in the first place. If we take care to introduce markers from an antecedent box again in the consequence box the problem vanishes, so admittedly, the DRS rules may be construed in such a way that they engender transitivity of the DRT consequence relation **for this particular example**. But only at a cost. The cost is that the distinction between markers that get their reference from context (the marker occurrences that we have called *fixed*) and markers occurrences that we have called *introduced* has to go. For it is precisely the fixed marker z in

$$\begin{array}{|c|}\hline \quad \\ \hline Sxz \\ \hline \end{array}$$

that causes trouble. It seems to me that the distinction between fixed and introduced markers is an essential feature of the framework. Indeed, how else could one give an account of the difference between picking up a reference from context on one hand and introducing a new topic of conversation on the other? I take it, therefore, that the DRT consequence relation is **not** transitive.

The incremental dynamics framework fares better than both DPL and DRT here, for it allows us to both maintain the distinction between references picked up from context and newly introduced references, and to have a transitive consequence relation. We get the following new version of Van Benthem's example:

$$\frac{(0, \exists; Mu_1; \exists; Hu_2; Ou_1u_2) \models (2, \exists; Gu_3; Ou_1u_3) \quad (2, \exists; Gu_3; Ou_1u_3) \models (3, Su_1u_3)}{(0, \exists; Mu_1; \exists; Hu_2; Ou_1u_2) \models (3, Su_1u_3)}$$

This is a valid argument, for as we have seen the consequence notion of L is transitive. Note that in the definition of valid consequence for L *existential padding* is used to provide an antecedent for the index 3. The conclusion should be read as:

> Suppose a man owns a house. Then there is a thing which he sprinkles.

Thus we see that 'existential padding' preserves the logical meaning of $(3, Su_1u_3)$.

The following proposition, proved by induction on k, shows that we can always choose to make existential padding explicit:

Proposition 14 *For all* $(n, \phi), (m, \psi) \in L$ *with* $n + e(\phi) \leq m$*:*

$(n, \phi) \models (m, \psi)$ *iff* $(n, \phi) \models (n + e(\phi), \exists^k; \psi)$*, where* $k = m - (n + e(\phi))$*.*

Of course, the incremental dynamics version of the garden sprinkling example is slightly at odds with linguistic intuition about pronoun reference resolution, for the contextual element u_3 is not linguistically salient in a context set up by $(0, \exists; Mu_1; \exists; Hu_2; Ou_1u_2)$. But then salience for anaphoric resolution is not our concern here.

A concluding remark on Van Benthem's example is in order. I have focussed on the logical *pointe* about failure of transitivity, but one might wish to read more in the example (as one of the reviewers did). "Van Benthem's Natural Language example is evidence for the claim that NL reasoning with anaphora is not transitive. DPL and DRT correctly reflect this fact. In other words, from a 'linguistic' point of view, the lack of transitivity in DPL is good, and the author's logic is not good." Needless to say, I cannot agree. There is more to Van Benthem's example than meets the eye at the first glance. The intermediate conclusion *he owns a garden* introduces a new context element, and this context element is picked up by the pronoun *it* in *he sprinkles it.* For this context link to be established, the intermediate phrase *he owns a garden* is indispensable. Pruning this intermediate conclusion also prunes the context, with a dangling reference for *it* as a result.

A natural way to read the concluding argument is with *it* picking up a reference to the only appropriate item in the antecedent of this argument, *a house*. Here linguistic intuition ('the only appropriate referent for *it* is *a house*') and logical intuition ('the meaning-in-context of *then he sprinkles it* should be preserved in the application of the transitivity rule') are at odds. Our logical intuition says that in the application of the move from $A \models B, B \models C$ to $A \models C$ the meaning of C should not change. Incremental dynamics agrees with this logical intuition, DPL and DRT do not.

Still, there is something odd about the incremental dynamics solution: it resolves the pronoun to a context item that is not linguistically salient. It is clear, then, that a full account of all that goes on in the example has to take the process of pronominal reference resolution on board. See the remarks in Section 10 for how this could be done.

6. A Calculus for Incremental Dynamic Reasoning

In this section, we will give a set of sequent deduction rules for incremental dynamic reasoning. We postpone the treatment of equality to Section 9. As an aside, we mention here that as a fringe benefit the calculus of this section has served as a basis for sequent axiomatizations of some frameworks criticized in the present paper, witness the sequent style calculi for DPL and DRT in (van Eijck, 1999).

We will write sequents as $(n, \phi) \Longrightarrow (m, \psi)$, where $\Longrightarrow$ is the sequent separator. Note that $(n, \phi) \Longrightarrow (m, \bot)$, for any $m \geq n + e(\phi)$, expresses that (n, ϕ) is inconsistent.

In the calculus we are about to present, we need some further notation for substitutions, in addition to $[^{m}_{+k}]$. Recall that $[^{m}_{+k}]$ is the substitution that replaces every index $n > m$ by $n + k$. This is useful to create room for k new indices starting from position $m + 1$. What we also need is an operation that removes a gap after the substitution of a referent for an $\exists$ that binds position m. The operation for this is $[^{m}_{-1}]$; this replaces every index $n > m$ by $n - 1$. Finally, $[k/m]$ has the usual meaning: replace index m by k everywhere. We abbreviate $[^{m}_{-1}][k/m]$ as $[k/m]^{-}$ ('substitute k for m and close the gap'). And that's all we need.

In the calculus, we use C, with and without subscripts, as a variable over contexts (formula lists composed with ;, including the empty list). We extend the function e to contexts by stipulating that $e(C) = 0$ if C is the empty list. Substitution is extended to contexts in a similar way. In the rules below we will use T as an abbreviation of formulas ϕ with $e(\phi) = 0$ (T for *Test* formula).

Structural Rules

6.0.0.1. *Test Axiom*

$$\frac{}{(n,T) \Longrightarrow (n,T)}\ (n,T) \in L$$

6.0.0.2. *Soundness of Test Axiom* If ${}_\sigma[\![n,T]\!]^{\mathcal{M}}_\tau$ then $\sigma = \tau$ (because T is a test) and therefore ${}_\tau[\![n,T]\!]^{\mathcal{M}}_\tau$. Thus, $(n,T) \models (n,T)$.

6.0.0.3. *Transitivity Rule*

$$\frac{(n,\phi) \Longrightarrow (m,\psi) \qquad\qquad (m,\psi) \Longrightarrow (k,\chi)}{(n,\phi) \Longrightarrow (k,\chi)}$$

6.0.0.4. *Soundness of Transitivity Rule* This was established in Lemma 13.

6.0.0.5. *Test Swap Rule*

$$\frac{(n, C_1T_1; T_2C_2) \Longrightarrow (m,\phi)}{(n, C_1T_2; T_1C_2) \Longrightarrow (m,\phi)}$$

Note: a test swap rule for the right hand side is derivable, because a formula in the right hand side of a sequent always can be moved over to the left hand side by means of the negation and double negation rules (see below).

6.0.0.6. *Soundness of Test Swap Rule* Immediate from the fact that $[\![n, T_1; T_2]\!]^{\mathcal{M}} = [\![n, T_2; T_1]\!]^{\mathcal{M}}$.

6.0.0.7. $\exists$ *Swap Rule*

$$\frac{n, C_1T; \exists\ C_2 \Longrightarrow (m,\phi)}{n, C_1\exists; ([^k_{+1}]T)C_2 \Longrightarrow (m,\phi)}\ k = n + e(C_1)$$

This rule allows us to pull $\exists$ leftward through a test T, provided we increment the appropriate indices in T.

Pulling $\exists$ through a test T in the opposite direction is allowed in those cases where $\exists$ does not bind anything in T. Now we must adjust T by decrementing the appropriate indices:

$$\frac{(n, C_1\exists; TC_2) \Longrightarrow (m, \phi)}{(n, C_1([^k_{-1}]T); \exists\ C_2) \Longrightarrow (m, \phi)}\ k = n + e(C_1),\ k+1 \text{ not in } T$$

6.0.0.8. *Soundness of $\exists$ Swap Rules* Soundness of the rule for moving $\exists$ to the left follows from the fact that $[\![k, T; \exists]\!]^{\mathcal{M}} = [\![k, \exists; [^k_{+1}]T]\!]^{\mathcal{M}}$.

Soundness of the rule for moving $\exists$ to the right follows from the fact that if index $k{+}1$ does not occur in T, then $[\![k, \exists; T]\!]^{\mathcal{M}} = [\![k, ([^k_{-1}]T); \exists]\!]^{\mathcal{M}}$.

Context Rules

6.0.0.9. *Memory Shift Rules*

$$\frac{(n, \phi) \Longrightarrow (m, \psi)}{(n+1, [^n_{+1}]\phi) \Longrightarrow (m+1, [^n_{+1}]\psi)} \qquad \frac{(n, \phi) \Longrightarrow (m, \psi)}{(n, \phi) \Longrightarrow (m+1, [^m_{+1}]\psi)}$$

6.0.0.10. *Soundness of Memory Shift Rules* Memory shift on left hand side: If ${}_{\sigma}[\![n, \phi]\!]^{\mathcal{M}}_{\sigma}\hat{\ }\tau$ then for all $a \in M$, ${}_{\sigma\hat{\ }a}[\![n{+}1, [^n_{+1}]\phi]\!]^{\mathcal{M}}_{\sigma}\hat{\ }a\hat{\ }\tau$. Soundness of memory shift on right hand side is established similarly.

6.0.0.11. *Context Extension*

$$\frac{(n, \exists; \phi) \Longrightarrow (m, \psi)}{(n+1, \phi) \Longrightarrow (m, \psi)}$$

The counterpart to the rule of context extension (i.e., context absorption) is the rule for introducing an existential quantifier in the antecedent (see the logical rules below).

What context extension and absorption express is that linking information to an outside context (of which nothing further is known) is equivalent, for all purposes of reasoning, to assuming that your information is existentially quantified over.

This is how one can make sense of a ongoing conversation about an unknown 'he': instead of asking questions of identification that might interrupt the flow of the gossip one simply inserts an existential quantifier and listens to what is being said.

6.0.0.12. *Soundness of Context Extension* Follows from the fact that

$${}_{\sigma}[\![n, \exists; \phi]\!]^{\mathcal{M}}_{\tau} \text{ iff for some } a \in M,\ {}_{\sigma\hat{\ }a}[\![n+1, \phi]\!]^{\mathcal{M}}_{\tau}.$$

Logical Rules

The rule for $\exists$ Left (the converse of context extension) is a special case of the rule ; Left. See below.

6.0.0.13. $\exists$ *Right*

$$\frac{(n,\phi) \Longrightarrow (m, [^k/_{m+1}]^- \psi)}{(n,\phi) \Longrightarrow (m, \exists; \psi)}$$

This format is familiar from the Gentzen format of $\exists$-right in standard predicate logic. Here is an example application:

$$\frac{(1, Ru_1u_1; \neg(\exists; Su_1u_2)) \Longrightarrow (1, Ru_1u_1; \neg(\exists; Su_1u_2))}{(1, Ru_1u_1; \neg(\exists; Su_1u_2)) \Longrightarrow (1, \exists; Ru_1u_2; \neg(\exists; Su_2u_3))}$$

$Ru_1u_1; \neg(\exists; Su_1u_2)$ equals $[^1/_2]^-(Ru_1u_2; \neg(\exists; Su_2u_3))$, so this is indeed a correct application of the rule.

6.0.0.14. *Soundness of* $\exists$ *Right* Assume a model $\mathcal{M}$ with input and output assignments σ, τ such that ${}_\sigma[\![n,\phi]\!]_\tau^{\mathcal{M}}$. Then by the soundness of the premise there is a $\theta \sqsupseteq \tau$ and a ρ with

$${}_\theta[\![m, [^k/_{m+1}]^- \psi]\!]_\rho^{\mathcal{M}}.$$

Let $[\![k]\!]_\theta^{\mathcal{M}} = a$. Then, by the definition of the substitution $[^k/_{m+1}]^-$, ${}_{\theta\hat{\ }a}[\![m{+}1, \psi]\!]_\rho^{\mathcal{M}}$. It follows that ${}_\theta[\![m, \exists; \psi]\!]_\rho^{\mathcal{M}}$. This proves $n, \phi \models (m, \exists; \psi)$.

6.0.0.15. *; Left and Right*

$$\frac{(n + e(\phi), \psi) \Longrightarrow (m, \chi)}{(n, \phi; \psi) \Longrightarrow (m, \chi)}$$

$$\frac{(n,\phi) \Longrightarrow (m,\psi) \qquad\qquad (n,\phi) \Longrightarrow (m,\chi)}{(n,\phi) \Longrightarrow (m, \psi; [^m_{+e(\psi)}]\chi)}$$

The first of these does double duty as a left weakening rule. Antecedent weakening is always extension on the left hand side. This is because extension on the right hand side might affect the stack. Weakening with a test is valid anywhere in the antecedent; the swap rules account for that.

An example application of the rule for ; right is:

$$\frac{(1, Ru_1u_1) \Longrightarrow (1, \exists; Ru_1u_2) \qquad\qquad (1, Ru_1u_1) \Longrightarrow (1, \exists; Ru_2u_1)}{(1, Ru_1u_1) \Longrightarrow (1, \exists; Ru_1u_2; \exists; Ru_3u_1)}$$

6.0.0.16. *Soundness of ; Left* Suppose ${}_{\sigma}[\![n, \phi; \psi]\!]^{\mathcal{M}}_{\tau}$. Let $\sigma' := \tau[1..e(\phi)]$. Then ${}_{\sigma'}[\![n{+}e(\phi), \psi]\!]^{\mathcal{M}}_{\tau}$. By the soundness of the premise, there are $\theta \sqsupseteq \tau$ and ρ with ${}_{\theta}[\![m, \chi]\!]^{\mathcal{M}}_{\rho}$. This establishes

$$(n, \phi; \psi) \models (m, \chi).$$

6.0.0.17. *Soundness of ; Right* Assume ${}_{\sigma}[\![n, \phi]\!]^{\mathcal{M}}_{\tau}$. Then by the soundness of the second premise, there are $\theta \sqsupseteq \tau$ and ρ with ${}_{\theta}[\![m, \chi]\!]^{\mathcal{M}}_{\rho}$. By Proposition 10, for any $\theta' \in M^{e(\psi)}$,

$${}_{\theta\hat{}\theta'}[\![m + e(\psi), [^{m}_{+e(\psi)}]\chi]\!]^{\mathcal{M}}_{\theta\hat{}\theta'\hat{}\rho}$$

By the soundness of the first premise, combined with Lemma 3, there is a $\theta' \in M^{e(\psi)}$ with

$${}_{\theta}[\![m, \psi]\!]^{\mathcal{M}}_{\theta\hat{}\theta'}.$$

It follows that ${}_{\theta}[\![m, \psi; [^{m}_{+e(\psi)}]\chi]\!]^{\mathcal{M}}_{\theta\hat{}\theta'\hat{}\rho}$. This establishes

$$(n, \phi) \models (m, \psi; [^{m}_{+e(\psi)}]\chi).$$

6.0.0.18. $\neg$ *Left and Right*

$$\frac{(n, \phi) \Longrightarrow (n + e(\phi), \psi)}{(n, \phi; \neg\psi) \Longrightarrow (m, \bot)}\; m \geq n + e(\phi)$$

$$\frac{(n, \phi; \psi) \Longrightarrow (m, \bot)}{(n, \phi) \Longrightarrow (n + e(\phi), \neg\psi)}$$

6.0.0.19. *Soundness of* $\neg$ *Left* Assume ${}_{\sigma}[\![n, \phi; \neg\psi]\!]^{\mathcal{M}}_{\tau}$. Then ${}_{\sigma}[\![n, \phi]\!]^{\mathcal{M}}_{\tau}$ and there is no θ with ${}_{\tau}[\![n + e(\phi), \psi]\!]^{\mathcal{M}}_{\theta}$. Contradiction with the soundness of the premise. This establishes $(n, \phi; \neg\psi) \models (m, \bot)$.

6.0.0.20. *Soundness of* $\neg$ *Right* Assume ${}_{\sigma}[\![n, \phi]\!]^{\mathcal{M}}_{\tau}$. Then by the soundness of the premise, there is no θ with ${}_{\tau}[\![n{+}e(\phi), \psi]\!]^{\mathcal{M}}_{\theta}$. This establishes $(n, \phi) \models (n + e(\phi), \neg\psi)$.

6.0.0.21. *Double Negation Rules*

$$\frac{(n, \phi) \Longrightarrow (m, \neg\neg\psi)}{(n, \phi) \Longrightarrow (m, \psi)} \qquad \frac{(n, \phi; \neg\neg\psi) \Longrightarrow (m, \bot)}{(n, \phi; \psi) \Longrightarrow (m, \bot)}$$

6.0.0.22. *Soundness of Double Negation Rules* For Double Negation Left, assume ${}_{\sigma}[\![n, \phi]\!]_{\tau}^{\mathcal{M}}$. Then by the soundness of the premise, for no $\theta \sqsupseteq \tau$ is there a ρ with

$${}_{\theta}[\![m, \neg\psi]\!]_{\rho}^{\mathcal{M}}.$$

In particular, we do not have ${}_{\theta}[\![m, \neg\psi]\!]_{\theta}^{\mathcal{M}}$. Therefore, there is a ρ with

$${}_{\theta}[\![m, \psi]\!]_{\rho}^{\mathcal{M}}.$$

This establishes $(n, \phi) \models (m, \psi)$. The soundness of Double Negation Right is established similarly.

This completes the presentation of the calculus. Since we have checked the soundness of of all the axioms and rules, we have established the following:

Theorem 15 *The Calculus of Incremental Dynamic Reasoning is sound.*

7. Derivable Rules for Incremental Dynamic Reasoning

We derive some extra rules that we need for the completeness reasoning in Section 8.

Proposition 16 (Contradiction Rule) *The following rule is derivable:*

$$\frac{(n, \phi; \neg\psi) \Longrightarrow (n + e(\phi), \neg\chi) \qquad (n, \phi; \neg\psi) \Longrightarrow (n + e(\phi), \chi)}{(n, \phi) \Longrightarrow (n + e(\phi), \psi)}$$

Proof. Consider the following derivations:

$$\frac{(n, \phi; \neg\psi) \Longrightarrow (n + e(\phi), \neg\chi) \qquad (n, \phi; \neg\psi) \Longrightarrow (n + e(\phi), \chi)}{(n, \phi; \neg\psi) \Longrightarrow (n + e(\phi), \neg\chi; \chi)}\ ;\text{r}$$

$$\cfrac{\cfrac{\cfrac{}{(n + e(\phi), \neg\chi) \Longrightarrow (n + e(\phi), \neg\chi)}\ \text{test axiom}}{(n + e(\phi), \neg\chi; \neg\neg\chi) \Longrightarrow (n + e(\phi), \bot)}\ \neg\text{l}}{(n + e(\phi), \neg\chi; \chi) \Longrightarrow (n + e(\phi), \bot)}\ \text{dn}$$

From these two, by transitivity, we get $(n, \phi, \neg\psi) \Longrightarrow (n + e(\phi), \bot)$. From this, we derive the desired conclusion as follows:

$$\cfrac{\cfrac{\cfrac{(n, \phi; \neg\psi) \Longrightarrow (n + e(\phi), \neg\chi) \qquad (n, \phi; \neg\psi) \Longrightarrow (n + e(\phi), \chi)}{(n, \phi; \neg\psi) \Longrightarrow (n + e(\phi), \bot)}\ \text{see above}}{(n, \phi) \Longrightarrow (n + e(\phi), \neg\neg\psi)}\ \neg\text{r}}{(n, \phi) \Longrightarrow (n + e(\phi), \psi)}\ \text{dn}$$

Proposition 17 (Cases Rule) *The following rule is derivable:*

$$\frac{(n,\phi;\neg\psi) \Longrightarrow (n+e(\phi),\chi) \qquad (n,\phi;\neg\neg\psi) \Longrightarrow (n+e(\phi),\chi)}{(n,\phi) \Longrightarrow (n+e(\phi),\chi)}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{(n,\phi;\neg\psi) \Longrightarrow (n+e(\phi),\chi)}{(n,\phi;\neg\psi;\neg\chi) \Longrightarrow (n+e(\phi),\bot)}\,\neg\text{l}}{(n,\phi;\neg\chi;\neg\psi) \Longrightarrow (n+e(\phi),\bot)}\,\text{swap}}{(n,\phi;\neg\chi) \Longrightarrow (n+e(\phi),\neg\neg\psi)}\,\neg\text{r} \quad \dfrac{\dfrac{\dfrac{(n,\phi;\neg\neg\psi) \Longrightarrow (n+e(\phi),\chi)}{(n,\phi;\neg\neg\psi;\neg\chi) \Longrightarrow (n+e(\phi),\bot)}\,\neg\text{l}}{(n,\phi;\neg\chi;\neg\neg\psi) \Longrightarrow (n+e(\phi),\bot)}\,\text{swap}}{(n,\phi;\neg\chi) \Longrightarrow (n+e(\phi),\neg\neg\neg\psi)}\,\neg\text{r}}{(n,\phi) \Longrightarrow (n+e(\phi),\chi)}\,\text{contrad}$$

Proposition 18 (Ex Falso Rule) *The following rule is derivable:*

$$\frac{}{(n,\bot) \Longrightarrow (n,\phi)}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(n,\bot) \Longrightarrow (n,\bot)}\,\text{start}}{(n,\neg\phi;\bot) \Longrightarrow (n,\bot)}\,;\ \text{l}}{(n,\bot;\neg\phi) \Longrightarrow (n,\bot)}\,\text{swap}}{(n,\bot) \Longrightarrow (n,\neg\neg\phi)}\,\neg\text{r}}{(n,\bot) \Longrightarrow (n,\phi)}\,\text{dn}$$

Proposition 19 (Inconsistency Rule) *The following rule is derivable:*

$$\frac{(n,\phi) \Longrightarrow (m,\bot)}{(n,\phi) \Longrightarrow (m,\psi)}$$

Proof.

$$\dfrac{(n,\phi) \Longrightarrow (m,\bot) \qquad \dfrac{}{(m,\bot) \Longrightarrow (m,\psi)}\,\text{ex falso}}{(n,\phi) \Longrightarrow (m,\psi)}\,\text{tr}$$

Proposition 20 (Modus Ponens) *The following rule is derivable:*

$$\frac{(n,\phi) \Longrightarrow (n+e(\phi),(\neg\neg\psi) \rightarrow \chi) \quad (n,\phi) \Longrightarrow (n+e(\phi),\psi)}{(n,\phi) \Longrightarrow (n+e(\phi),\chi)}$$

Proof.

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{(n,\phi) \Longrightarrow (n+e(\phi), \neg(\neg\neg\psi;\neg\chi))}{(n,\phi;\neg\neg(\neg\neg\psi;\neg\chi)) \Longrightarrow (n+e(\phi),\bot)}\,\neg\mathrm{l}}{(n,\phi;\neg\neg\psi;\neg\chi) \Longrightarrow (n+e(\phi),\bot)}\,\mathrm{dn}}{(n,\phi;\neg\neg\psi) \Longrightarrow (n+e(\phi),\neg\neg\chi)}\,\neg\mathrm{r}}{(n,\phi;\neg\neg\psi) \Longrightarrow (n+e(\phi),\chi)}\,\mathrm{dn}
\quad
\dfrac{\dfrac{(n,\phi) \Longrightarrow (n+e(\phi),\psi)}{(n,\phi;\neg\psi) \Longrightarrow (n+e(\phi),\bot)}\,\neg\mathrm{r}}{(n,\phi;\neg\psi) \Longrightarrow (n+e(\phi),\chi)}\,\mathrm{incons}
}{(n,\phi) \Longrightarrow (n+e(\phi),\chi)}\,\mathrm{cases}
$$

8. Completeness of the Calculus

To establish the completeness of the calculus, we assume that

$$(n,\phi) \not\Longrightarrow (m,\psi).$$

The definition of L consequence immediately yields that $(n,\phi) \not\models (m,\psi)$ for all m with $m < n + e(\phi)$, so we may assume that $m \geq n + e(\phi)$. Since we can shift memory on the right hand side, we may furthermore assume without loss of generality that $m = n + e(\phi)$. Because of the context extension rule, we may even assume that the context is initially empty. Indeed, from $(n,\phi) \not\Longrightarrow (m,\psi)$, with $n > 0$, it follows by context extension that $(0,\exists^n;\phi) \not\Longrightarrow (m,\psi)$.

We will construct a counter model by a slight modification of the s-tandard Henkin construction for the completeness of classical predicate logic. It is convenient to use k for $e(\phi)$ throughout the reasoning that follows. Also, in the following, we extend the language with individual constants.

Definition 21 *A set of L formulas is k-bounded if every member of the set is in L_k, i.e., every formula in the set has the form (k,ϕ). We use (k,Γ) to refer to k-bounded sets of formulas.*

$\phi \vdash_\Gamma \psi$:⇔ *there are $(k,\phi_1),\dots,(k,\phi_n) \in (k,\Gamma)$ with*

$$(0,\phi;\neg\neg\phi_1;\cdots;\neg\neg\phi_n) \Longrightarrow (k,\psi).$$

(k,Γ) is consistent with $(0,\phi)$ if there is a (k,ψ) with $\phi \not\vdash_\Gamma \psi$.

(k,Γ) is negation complete with respect to $(0,\phi)$ if for every $(k,\psi) \in L$ either $\phi \vdash_\Gamma \psi$ or $\phi \vdash_\Gamma \neg\psi$.

(k,Γ) has witnesses for $(0,\phi)$ if for every $(k,\exists;\psi)$ such that $\phi \vdash_\Gamma \exists;\psi$ there is a c for which $(k, \neg\neg\exists\psi \to [{}^c/_{k+1}]^-\psi) \in (k,\Gamma)$.

Note that in the definition of $\phi \vdash_\Gamma \psi$ the extra premises from Γ do not extend the 'anaphoric context': the context change potential of the premises from Γ is blocked off by means of double negation signs.

Proposition 22 *If* $\phi \nvdash_\Gamma \psi$ *then at least one of*

$$(k, \Gamma) \cup \{(k, \psi)\}.(k, \Gamma) \cup \{(k, \neg\psi)\}$$

is consistent with $(0, \phi)$*.*

Proof. Use the Cases Rule.

Let $(k, \exists; \chi_1), \ldots$ be a list of all k-bounded formulas of L that start with $\exists$. Let $C_0 := c_1^0, \ldots$ be a list of fresh individual constants. Let L_0 be $L(C_0)$ (the result of adding the constants C_0 to L).

$$(k, \Delta_0) := \{(k, \neg\neg\exists\chi_i \to [^{c_i^0}/_{k+1}]^- \chi_i) \mid i \in \mathbb{N}^+\}.$$

Let $(k, \exists\chi_1^m), \ldots$ be a list of all k-bounded existential formulas which occur in L_m. Let $C_{m+1} := c_1^{m+1}, \ldots$ be a list of fresh individual constants. Let $L_{m+1} := L_m(C_{m+1})$.

$$(k, \Delta_{m+1}) := \{(k, \neg\neg\exists\chi_i^{m+1} \to [^{c_i^{m+1}}/_{k+1}]^- \chi_i^{m+1}) \mid i \in \mathbb{N}^+\}.$$

Let $C := \bigcup_m C_m$, and let (k, Δ) be the set of $L(C)$ formulas given by:

$$(k, \Delta) := \bigcup_m (k, \Delta_m).$$

The presence of constants forces a slight extension of our notation for substitutions; we will use $[^c/_{k+1}]^-$ for the substitution that puts constant c in place of all occurrences of u_{k+1} and closes the gap.

Proposition 23 *If* (k, Γ) *consists of* $L(C)$ *formulas, and*

$$(k, \Gamma) \supseteq (k, \Delta),$$

then (k, Γ) *has witnesses for* $(0, \phi)$*.*

Proof. Take some $(k, \exists\psi)$ with $\phi \vdash_\Gamma \exists\psi$. Then $\exists\psi \in L_m$ for some m. So there is some $c \in C$ with $\neg\neg\exists\psi \to [^c/_{k+1}]^-\psi \in \Delta_{m+1}$. So $(k, \neg\neg\exists\psi \to [^c/_{k+1}]^-\psi) \in (k, \Delta) \subseteq (k, \Gamma)$.

Proposition 24 *If* (k, Γ) *has witnesses for* $(0, \phi)$ *and* $\phi \vdash_\Gamma \exists\psi$*, then there is some* $c \in C$ *with* $\phi \vdash_\Gamma [^c/_{k+1}]^-\psi$*.*

Proof. By the fact that Modus Ponens is a derivable rule (Proposition 20).

Proposition 25 *If* (k, Γ) *is consistent with* $(0, \phi)$ *then there is a*

$$(k, \Gamma') \supseteq (k, \Gamma)$$

which is consistent with $(0, \phi)$*, negation complete with respect to* $(0, \phi)$*, and has witnesses for* $(0, \phi)$*.*

Proof. Assume (k, Γ) consistent with $(0, \phi)$. Let $(k, \chi_1), \ldots, (k, \chi_i), \ldots$ be an enumeration of all k-bounded formulas of the language $L(C)$. Extend (k, Γ) as follows to a (k, Γ') with the required properties.

$$(k, \Gamma_0) := (k, \Gamma) \cup (k, \Delta)$$

$$(k, \Gamma_{m+1}) := \begin{cases} (k, \Gamma_m \cup \{\chi_m\}) & \text{if } (k, \Gamma_m \cup \{\chi_m\}) \text{ consistent with } (0, \phi), \\ (k, \Gamma_m) & \text{otherwise.} \end{cases}$$

$$(k, \Gamma') := (k, \bigcup_m \Gamma_m)$$

$(k, \Gamma') \supseteq (k, \Delta)$, so by Proposition 23 (k, Γ') has witnesses for $(0, \phi)$.

Assume (k, Γ') is inconsistent with $(0, \phi)$. Then some (k, Γ_m) has to be inconsistent with $(0, \phi)$ and contradiction with Proposition 22. So (k, Γ') is consistent with $(0, \phi)$.

Finally, (k, Γ') is negation complete by construction.

Definition 26 (Canonical Model) *Let* (k, Γ) *be a set of* k*-bounded formulas that is consistent with* $(0, \phi)$*, negation complete with respect to* $(0, \phi)$*, and has witnesses for* $(0, \phi)$*. Then* $\mathcal{M}_\Gamma = (D, I)$ *is defined as follows.* $D :=$ *the set of natural numbers* $\{1, \ldots, k\}$ *together with the set of constants* C *occurring in* $\Gamma \cup \{\phi\}$*. For all terms of the language, let* $I(t) := t$*. Let* $I(P) := \{\langle t_1, \ldots, t_k\rangle \mid \phi \vdash_\Gamma Pt_1 \cdots t_k)\}$ *(where it is given that all the* t_i *are either constants or indices in the range* $1, \ldots, k$*).*

Lemma 27 (Satisfaction Lemma) *Let* (k, Γ) *be a set of* k*-bounded formulas that is consistent with* $(0, \phi)$*, negation complete with respect to* $(0, \phi)$*, and has witnesses for* $(0, \phi)$*. For all* k*-bounded* ξ*:*
$\phi \vdash_\Gamma \xi$ *iff* $\exists \tau$ *with* ${}_{\langle 1..k\rangle}[\![k, \xi]\!]^{M_\Gamma}_{\langle 1..k\rangle\hat{}\tau}$.

Proof. Induction on the structure of ξ.

$\phi \vdash_\Gamma \top$ by $(k, \top) \Longrightarrow (k, \top)$ plus ; left.

$\phi \vdash_\Gamma \exists; \xi$ iff (Γ has witnesses) $\phi \vdash_\Gamma [^c/_{k+1}]^- \xi$ iff (i.h.)

$$\text{there is a } \tau \text{ with } {}_{\langle 1..k\rangle}[\![k, [^c/_{k+1}]^- \xi]\!]^{M_\Gamma}_{\langle 1..k\rangle\hat{}\tau}$$

iff ${}_{\langle 1..k\rangle}[\![k,\exists;\xi]\!]^{M_\Gamma}_{\langle 1..k\rangle\hat{}c\hat{}\tau}$.

$\phi \vdash_\Gamma Pt_1 \cdots t_n;\xi$ iff $\langle t_1..t_n\rangle \in I(P)$ and $\phi \vdash_\Gamma \xi$ iff

$$ {}_{\langle 1..k\rangle}[\![k, Pt_1 \cdots t_n]\!]^{M_\Gamma}_{\langle 1..k\rangle} $$

and (i.h.)

$$ \text{there is a } \tau \text{ with } {}_{\langle 1..k\rangle}[\![k,\xi]\!]^{M_\Gamma}_{\langle 1..k\rangle\hat{}\tau} $$

iff

$$ \text{there is a } \tau \text{ with } {}_{\langle 1..k\rangle}[\![k, Pt_1 \cdots t_n;\xi]\!]^{M_\Gamma}_{\langle 1..k\rangle\hat{}\tau}. $$

$\phi \vdash_\Gamma \neg\xi_1;\xi_2$ iff (Γ negation complete) $\phi \not\vdash_\Gamma \xi_1$ and ϕ $vdash_\Gamma \xi_2$ iff (i.h. twice) ${}_{\langle 1..k\rangle}[\![k,\neg\xi_1]\!]^{M_\Gamma}_{\langle 1..k\rangle}$ and

$$ \text{there is a } \tau \text{ with } {}_{\langle 1..k\rangle}[\![k,\xi_2]\!]^{M_\Gamma}_{\langle 1..k\rangle\hat{}\tau} $$

iff

$$ \text{there is a } \tau \text{ with } {}_{\langle 1..k\rangle}[\![k,\neg\xi_1;\xi_2]\!]^{M_\Gamma}_{\langle 1..k\rangle\hat{}\tau}. $$

Proposition 28 *Let (k,Γ) be consistent with $(0,\phi)$, be negation complete with respect to $(0,\phi)$, and have witnesses for $(0,\phi)$. Then*

$$ {}_\epsilon[\![0,\phi]\!]^{M_\Gamma}_{\langle 1..k\rangle}. $$

Proof. Let $\exists^k\phi'$ be the result of applying the rule for moving $\exists$ leftward as many times as necessary to ϕ to ensure that $e(\phi') = 0$. Then $(0,\phi)$ and the formula $(0,\exists^k\phi')$ are proof equivalent. Furthermore, we have:

$$ \dfrac{\dfrac{\dfrac{}{(k,\phi') \Longrightarrow (k,\phi')}\text{ test axiom}}{(0,\exists^k\phi') \Longrightarrow (k,\phi')}\ \exists\text{l}, k \text{ times}}{(0,\phi) \Longrightarrow (k,\phi')}\ \text{swap rules} $$

Therefore, $\phi \vdash_\Gamma \phi'$, and by the satisfaction lemma, ${}_{\langle 1..k\rangle}[\![k,\phi']\!]^{M_\Gamma}_{\langle 1..k\rangle}$. Also, by definition of the semantics for $\exists$, we have that ${}_\epsilon[\![0,\exists^k]\!]^{M_\Gamma}_{\langle 1..k\rangle}$. By the semantic equivalence of ϕ and $\exists^k\phi'$ we get ${}_\epsilon[\![0,\phi]\!]^{M_\Gamma}_{\langle 1..k\rangle}$.

Theorem 29 (Completeness)
If $(n,\phi) \models (m,\psi)$ then $(n,\phi) \Longrightarrow (m,\psi)$.

Proof. Assume $(n, \phi) \not\Longrightarrow (m, \psi)$. Without loss of generality we may assume that $(n, \phi) \not\Longrightarrow (k, \psi)$, where $k := n + e(\phi)$, because from $(n, \phi) \Longrightarrow (k, \psi)$ it follows by a suitable number of applications of memory shift on the right hand side that $(n, \phi) \Longrightarrow (m, \psi')$, where ψ' is the result of shifting the 'write registers' of ψ to the right.

By context extension, it follows from $(n, \phi) \not\Longrightarrow (k, \psi)$ that

$$(0, \exists^n \phi) \not\Longrightarrow (k, \psi).$$

Set $\phi' := \exists^n \phi$. Then $k = e(\phi')$, and $\{(k, \neg\psi)\}$ is consistent with $(0, \phi')$. By proposition 25, there is a $(k, \Gamma) \supseteq \{(k, \neg\psi)\}$ which is consistent with $(0, \phi')$, is negation complete with respect to $(0, \phi')$, and has witnesses for $(0, \phi')$. Construct the canonical model and apply the satisfaction lemma to get:

$$_{\langle 1..k \rangle}[\![k, \neg\psi]\!]^{M_\Gamma}_{\langle 1..k \rangle}.$$

By the semantic clause for negation we have that for all τ:

$$\langle 1..k \rangle, \tau) \notin [\![k, \psi]\!]^{M_\Gamma}.$$

By proposition 28:

$$_{\epsilon}[\![0, \phi']\!]^{M_\Gamma}_{\langle 1..k \rangle}.$$

This proves $(0, \phi') \not\models (k, \psi)$, i.e., $(0, \exists^n; \phi) \not\models (k, \psi)$, and therefore, $(n, \phi) \not\models (k, \psi)$.

9. Anaphoric Reasoning with Equality

Anaphoric linking makes extensive use of equality. See (van Eijck, 1993) for an in-depth analysis of the use of equality in anaphoric descriptions. An anaphoric definite description like *the garden* can be treated as a definiteness quantifier followed by a link to a contextually available index. The translation of *He sprinkles the garden* would then be something like $2, \iota : (u_3 \doteq u_2; Gu_3); Su_1u_3$. Also, the determiner *another* often has an implicit anaphoric element. In such cases, the treatment involves non-identity links to contextually available referents. *He met another woman* gets a translation like $2, \exists; u_3 \neq u_2; Wu_3; Mu_1u_3$. Below we indicate how to handle equality, while leaving the axiomatization of definiteness in the present framework for another occasion.

The following rules must be added to the calculus to deal with equality statements (we now assume the presence of a set *Cons* of individual constants):

9.0.0.23. *Reflexivity Axiom*

$$\frac{}{(n,\phi) \Longrightarrow (m, t \doteq t)} \; n + e(\phi) \leq m, \; t \in \text{Cons} \cup \{u_1, \dots, u_m\}$$

9.0.0.24. *Soundness of Reflexivity Axiom* The axiom expresses that equality is reflexive.

9.0.0.25. *Substitution Rule*

$$\frac{(n,\phi) \Longrightarrow (m, [^{t_1}/_{t_2}]\psi)}{(n,\phi; t_1 \doteq t_2) \Longrightarrow (m,\psi)} \; t_1, t_2 \in \text{Cons} \cup \{u_1, \dots, u_m\}$$

Example application:

$$\cfrac{\cfrac{}{(0,\top) \Longrightarrow (0, a \doteq a)}\text{ refl}}{(0, a \doteq b) \Longrightarrow (0, b \doteq a)}\text{ subst}$$

For the correctness of this application, note that $a \doteq a$ is of the form $[^a/_b]b \doteq a$.

$$\cfrac{\cfrac{}{(0, a \doteq b) \Longrightarrow (0, a \doteq b)}\text{ test axiom}}{(0, a \doteq b; b \doteq c) \Longrightarrow (0, a \doteq c)}\text{ subst}$$

For the correctness of this application, note that $a \doteq b$ is of the form $[^b/_c]a \doteq c$.

The quantified version of the transitivity rule can be derived as follows:

$$\cfrac{\cfrac{\cfrac{}{(3, u_1 \doteq u_2) \Longrightarrow (3, u_1 \doteq u_2)}\text{ test axiom}}{(3, u_1 \doteq u_2; u_2 \doteq u_3) \Longrightarrow (3, u_1 \doteq u_3)}\text{ subst}}{(0, \exists; \exists; \exists; u_1 \doteq u_2; u_2 \doteq u_3) \Longrightarrow (3, u_1 \doteq u_3)}\; \exists\text{l}, 3\text{ times}$$

9.0.0.26. *Soundness of the Substitution Rule* Assume

$$_\sigma[\![n, \phi; t_1 \doteq t_2]\!]^{\mathcal{M}}_\tau.$$

Then $_\sigma[\![n,\phi]\!]^{\mathcal{M}}_\tau$, and $[\![t_1]\!]^{\mathcal{M}}_\tau = [\![t_2]\!]^{\mathcal{M}}_\tau$. By the soundness of the premise, there are $\theta \sqsupseteq \tau$ and ρ with $_\theta[\![m, [^{t_1}/_{t_2}]\psi]\!]^{\mathcal{M}}_\rho$. Therefore, $_\theta[\![m,\psi]\!]^{\mathcal{M}}_\rho$. This shows $(n, \phi; t_1 \doteq t_2) \models (m,\psi)$.

The completeness of the anaphoric calculus with equality is proved by modifying the Henkin construction in the usual way: instead of terms we take equivalence classes of terms under provable equality as elements of the canonical model.

10. Conclusion

Semanticists sympathetic to DRT do not tend to worry about the top down construction algorithm for DRSs, with a novelty condition to ensure incrementality of interpretation. Those interested in carrying out a Montagovian or Fregean enterprise of building representations for complex constituents out of representations for its components insist on the formulation of a bottom up procedure, however. This interest in a Montagovian perspective on dynamic interpretation has led to the emergence of various dynamic logics intended as rational reconstructions of the DRT programme. Unfortunately, the most well known of these, DPL, has a problem of destructive assignment, and is therefore not the best candidate for a reformulation of DRT in Fregean or Montagovian terms. This flaw is remedied in the present proposal.

Moreover, the incremental dynamics framework advocated here has as advantage over the DRT framework that it comes with a clearer distinction between the following two kinds of actions:

- picking up a reference from context,
- introducing a new topic of conversation for future reference.

We have seen that attempts to formulate a transitive relation of logical consequence relation for DRT force one to blur this distinction. The framework of incremental dynamics presented in this paper improves on this, for it yields both a clear distinction between 'picking up an old reference' and 'introducing a new reference' and a transitive relation of logical consequence.

When looking at the general picture of frameworks for dynamic interpretation, there may not emerge a single 'best' framework. Instead, it might well be the case that various proposals shed light on different aspects of the dynamics of text processing that all merit study in their own right. The present 'calculus of incremental dynamics' focuses on the abstraction over anaphoric context in reasoning. It gives an explicit account of anaphoric links between premises and conclusion in reasoning, and is more well behaved than previous DRT-like frameworks (if one accepts transitivity of the consequence relation as a mark of good behaviour, that is). To be sure, the present sequent approach to axiomatizing dynamic logic can also be used to get still closer to standard DRT, or to axiomatize DPL and its variants: see the proof systems that are given in (van Eijck, 1999).

Here is a mere sketch of what an incremental Montague grammar in the spirit of the present system might look like. Such a system would use type scheme patterns that allow context type schemes of the form $[i]$, for a context consisting of i elements, with i an index variable. With these we can represent the meaning of *a man* as:

$$\lambda Q_{\triangleright(i+1,K)}.\lambda c_{[i]}\lambda c'_{[i+K+1]}.\exists x_e(\text{man } x \wedge (((Q(i+1))c\hat{\ }x)c')).$$

Intuitively, *a man* introduces a new context element, states that this element is a man, points at that man with an index $i+1$, and extends the context by an unspecified number of further items to be provided by the verb phrase.

In the formula, $Q_{\triangleright(i+1,K)}$ represents an index into a context transition with the index indicated by $\triangleright$, the input context of size $i+1$, i.e., non-empty, and a context increment of size K, i.e., the transition extends the context with K elements. Intuitively, this is the type of the verb phrase that combines with *a man*.

The c, c' represent contexts, and we see from the translation that the indefinite article extends the input context by one element, for there is an element x such that extending the input context c with it (denoted by $c\hat{\ }x$) yields an intermediate context to which the verb phrase meaning is to be applied. Finally, note that the index $i+1$ in the formula is used to keep track of the subject; it refers to the item introduced by the indefinite, for this occupies position $i+1$ in the context.

Note that the presence of type schemes for contexts provides a neat interface for anaphoric reference resolution: pronouns can be translated as invitations to pick a reference from the current context. The author is aware that this sketch is far too concise, but a more detailed account of all this will have to wait for another occasion.

To wind up our story we mention some connections to related work. Via the translation to DRT in Section 3 (proposition 7) we have a proof system for a streamlined version of DRT. The calculus makes the discipline of using and modifying the anaphoric context and of handling dynamically bound indices fully explicit. It can be viewed as a proof system for 'pure' DRSs, a proof system that avoids the award reference to alphabetic variance in the rules of proof proposed in (Kamp and Reyle, 1996). Like the DRT calculus of (van Eijck, 1999), the present calculus differs from the earlier proof system for DRT in (Saurer, 1993) in the fact that it does not rely on an implicit translation to FOL.

Finally we mention the connections with (Dekker, 1994), where a similar plea is made for incremental dynamics but the problem of a calculus for reasoning is not addressed, with (Visser and Vermeulen, 1995) and (Visser, 1994; Visser, 1997), and with (Blackburn and Venema, 1995) and (Hollenberg, 1997). Indeed, the Hollenberg equational axioms of dynamic negation and relational composition are all derivable in the calculus of incremental dynamics.

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