Comments on 'Modal Fixed Point Logic and Changing Models'

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This is indeed a *very nice* draft that I have read with great pleasure, and that has helped me to better understand the completeness proof for LCC. Modal fixed point logic allows for an illuminating new version (and a further extension) of that proof.

But still. My main comment is that I think the perspective on substitutions in the draft paper is flawed.

The general drift of the paper is that relativization, (predicate) substitution and product update are general operations on models, and that it is important to check whether given logical languages are closed under these operations. FO logic is closed under relativization, predicate substitution and product constructions (such as those involved in relative interpretation).

The minimal modal logic is closed under relativization, which explains the reduction of epistemic logic (withhout common knowledge) + public announcement to epistemic logic simpliciter (as observed in Van Benthem, [2]). The reduction breaks down as soon as one adds common knowledge.

The minimal modal logic is also closed under substitution, which explains the reduction of epistemic logic plus (publicly observable) factual change to epistemic logic simpliciter, via the following reduction axioms (I use $!p := \phi$ for the operation of publicly changing the truth value of p to ϕ):

$$\begin{array}{rcl}
[!p := \phi]p & \leftrightarrow & \phi \\
[!p := \phi]q & \leftrightarrow & q \quad (p \text{ and } q \text{ syntactically different }) \\
[!p := \phi]\neg\psi & \leftrightarrow & \neg[!p := \phi]\psi \\
[!p := \phi](\psi_1 \wedge \psi_2) & \leftrightarrow & [!p := \phi]\psi_1 \wedge [!p := \phi]\psi_2 \\
[!p := \phi][i]\psi & \leftrightarrow & [i][!p := \phi]\psi
\end{array}$$

Unlike the case of relativisation, this can be extended to the case of epistemic logic with common knowledge, by means of:

$$[!p := \phi]C_G\psi \quad \leftrightarrow \quad C_G[!p := \phi]\psi$$

We get the following

Theorem 1 *Public change logic with common knowledge is axiomatised by the minimal modal* *logic plus the mix and induction axioms for common knowledge,*

$$\begin{array}{rcl} C_G\phi & \leftrightarrow & \phi \wedge \bigwedge_{i\in G}[i]C_G\phi \\ (\phi \wedge C_G(\phi \rightarrow \bigwedge_{i\in G}[i]\phi)) & \rightarrow & C_G\phi \end{array}$$

plus the reduction axioms above.

A proof is given in Kooi [3], but it should be noted that the result also follows almost directly from the completeness theorem for LCC in Van Benthem cs [1]: just observe what happens when the common knowledge operator $[(i \cup j \cup \cdots)^*]$ gets pulled through the action model for public substitution $p := \phi$.

It is observed in the paper that the reduction axioms for public announcement just spell out the recursive clauses for performing syntactic relativization. Just so, the reduction axioms for public change just spell out the recursive clauses for performing syntactic substitution. Indeed, it is one of the themes of LCC [1] that factual change in epistemic update logic can be handled by means of substitution.

Now let us focus on the logic of public announcement and public change, but without common knowledge. Since the minimal modal logic is closed under both relativization and substitution, one can prove by construction that there is an appropriate syntactic notion of substitution for the language:

Theorem 2 *The logic of public announcement (without common knowledge) is closed under substitution, in the sense that there is a syntactic operation $(\psi)^{p:=\phi}$ with the property that*

$$M, s \models (\psi)^{p:=\phi} \text{ iff } M[p := \phi^M], s \models \psi.$$

Proof. Just add the operator $[!p := \phi]$ to the language, with the obvious meaning, and observe that the following reduction axioms can be used to eliminate it again:

$$\begin{array}{rcl} [!p := \phi]p & \leftrightarrow & \phi \\ [!p := \phi]q & \leftrightarrow & q \quad (p \text{ and } q \text{ syntactically different }) \\ [!p := \phi]\neg\psi & \leftrightarrow & \neg[!p := \phi]\psi \\ [!p := \phi](\psi_1 \wedge \psi_2) & \leftrightarrow & [!p := \phi]\psi_1 \wedge [!p := \phi]\psi_2 \\ [!p := \phi][i]\psi & \leftrightarrow & [i][!p := \phi]\psi \\ & & \\ [!p := \phi][!\chi]q & \leftrightarrow & [!p := \phi](\chi \rightarrow q) \quad (p \text{ and } q \text{ same or different }) \\ [!p := \phi][!\chi]\neg\psi & \leftrightarrow & [!p := \phi]\neg[!\chi]\psi \\ [!p := \phi][!\chi](\psi_1 \wedge \psi_2) & \leftrightarrow & [!p := \phi]([!\chi]\psi_1 \wedge [!\chi]\psi_2) \\ [!p := \phi][!\chi][i]\psi & \leftrightarrow & [!p := \phi](\phi \rightarrow [i](\phi \rightarrow [!\chi]\psi)) \end{array}$$

This effectively defines an appropriate notion of syntactic substitution for the language. □

Remark If one takes 'syntactic substitution' to mean 'an operation that is defined on the language atoms and that commutes with all constructs of the language', then the above does not qualify as a definition of substitution. But let's not quibble over words. As in the case of relativization, the point is that there is a syntactic definition that works. Those who dislike calling this operation a substitution may rephrase the theorem as saying that the logic of public announcement without common knowledge is closed under factual change.

The syntactic definition of substitution (or: factual change) in the previous theorem may seem rather involved, but there is a simple method behind it. It is easily seen that the following holds if we have a base logic with a reducible 'update superstructure': If the base logic is closed under substitution then the full logic is closed under factual change. The method: first 'iron out' the update operators, and next use closedness under substitution for the base logic. This gives:

Theorem 3 *LCC is closed under substitution (factual change): There is a syntactic operation* $(\psi)^{p:=\phi}$ *with the property that*

$$M, s \models (\psi)^{p:=\phi} \text{ iff } M[p := \phi^M], s \models \psi.$$

And finally, on the same grounds:

Theorem 4 *The modal mu calculus with announcement modalities is closed under substitution (factual change).*

Theorem 5 *The modal mu calculus with product updates for all finite action models is closed under substitution (factual change).*

The supposed counterexample in Section 3 of your draft is based on a flawed procedure for carrying out the factual change operation. Indeed, if by 'performing the procedure of factual change syntactically' we take care to eliminate all update operations before carrying out the substitutions, no problems will ever arise.

E.g., I believe that the correct way of carrying out the operation

$$(\langle !\Diamond\top\rangle\Diamond p)^{p:=\langle !\Diamond\top\rangle\top}$$

is as follows:

$$\begin{aligned}
(\langle !\Diamond\top\rangle\Diamond p)^{p:=\langle !\Diamond\top\rangle\top} &\leftrightarrow (\Diamond(\Diamond\top \wedge p))^{p:=\langle !\Diamond\top\rangle\top} \\
&\leftrightarrow (\Diamond(\Diamond\top \wedge \langle !\Diamond\top\rangle\top) \\
&\leftrightarrow (\Diamond(\Diamond\top \wedge \Diamond\top) \\
&\leftrightarrow \Diamond\Diamond\top.
\end{aligned}$$

Using the correct procedure for carrying out the factual change, we see that for the systems you are considering the smallest fixpoints can still be defined as the limit of ordinal approximations starting from $\bot$. The approximation sequences for your examples $\langle !\Diamond\top\rangle\mu q \cdot \Box q$ and $\Diamond\top \wedge \mu q \cdot \langle !\Diamond\top\rangle\Box q$ are the same.

I believe the final remark of the section should be changed to: "Model changing operators are nice devices, but they demand some extra caution in generalizing familiar notions like substitution."

Finally, a typo: On page 7, middle (".. compare the following identities .."): $F^M_{\langle P\rangle\phi}(X)$ is equivalent to $F^{M|P}_{\phi}(X)$ (not $F^{M|P}_{\langle P\rangle\phi}(X)$).

Hopefully my remarks are still useful to you.

References

[1] J. van Benthem, J. van Eijck, and B. Kooi. Logics of communication and change. *Information and Computation*, 204(11):1620–1662, 2006.

[2] Johan van Benthem. Update as relativization. Technical report, ILLC, Amsterdam, 2000.

[3] B.P. Kooi. Expressivity and completeness for public update logics via reduction axioms. *Journal of Applied Non-Classical Logics*, 16(2), 2007.