Belief, Uncertainty, and Probability

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Qualitative and Quantitative Methods in Formal Epistemology

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Abstract

For reasoning about uncertain situations, we have probability theory, and we have logics of knowledge and belief. How does elementary probability theory relate to epistemic logic and the logic of belief? The talk will argue that uncertainty caused by ignorance and uncertainty caused by physical indeterminism can be handled both by probability theory and by the logic of knowledge and belief, and that these two perspectives are systematically related, even intimately connected.

How are Logic and Probability Theory Related?

- Logic = Reasoning about Certainty
- Probability Theory = Reasoning about Uncertainty
- Epistemic or Bayesian probability can be viewed as an extension of propositional logic with hypotheses, i.e., basic propositions whose truth or falsity is uncertain.
- But logic has something to say, too, about reasoning under uncertainty: epistemic logic, doxastic logic, default logic, logic of conditionals ...
- Frank Ramsey : “In this Essay the Theory of Probability is taken as a branch of logic, the logic of partial belief and inconclusive argument [...]” [Ram31].

Probability, Betting, Ignorance

Probability, Betting, Ignorance

Christiaan Huygens, 1629–1695

Van
Rekeningh
in
Spelen van Geluck

1660

[Huy60]

beroemden Heer CHRISTIANUS HUGENIUS aengaende 't reeckenen in Spelen van Geluck uytgevonden ende my in schrift van hem mede gedeelt is, alhier met desselfs brief, in plaets van 'tgeene my overig was, by-voegde. Welck sijn Tractaet ick dan UE. des te aengenamer acht te sullen wesen, als 'tgeen daer in verhandelt wordt te subtijlder en ongemeender sal bevonden worden; insonderheyt door dien hy tot desselfs vinding deselve *Analysis*, welckers fondamenten hy eertijts van my geleert heeft, als ick, gebruyckt; en alsoo de bevlytigers van die de weg baent om diergelijcke Voorstellen te ontbinden. Waer in, soo ick nevens mijnen andren arbeyt aen U, Beminde Leser, in dese soort van Studie genoegsame stof van oeffening gegeven hebbe; soo sult ghy daer uyt (gelijck ick hoop) mijne bereytwilligheyt t'uwaerts konnen afnemen, en dienvolgende oock mijnen arbeyt, t'uwen en der Studien besten aengenomen, ten goeden duyden. Vaert wel.

Aen

Aen mijn Heer

FRANCISCUS van SCHOOTEN.

Mijn Heer,

NAer dien ick weet dat U E., de loffelijcke vruchten van sijn vernuft ende arbeyt in 't licht gevende, onder anderen dit ooghmerck heeft: namentlijck, om door de verscheydenheyt der verhandelde stoffen te betoonen hoe wijt onse uytnemende Konst van Algebra *sich uytstreckt; soo en twijffele ick oock niet, of het geene ick van de Rekeningh in Spelen van geluck beschreven heb, sal tot U E. opset niet ondienstig zijn. Want soo veel te swaerder als het schijnt, door reden te konnen bepalen het geene onseker is ende het geval onderworpen, soo veel te meer verwonderinghs waerdigh sal die wetenschap schijnen, waer door sulcx kan werden te wege gebracht. Dewijl ick dan op U E. versoeck ende aen-maeninghe, dese Rekening eerst heb beginnen by geschrift te stellen, ende U E. deselve waerdigh acht om te gelijck met syne diepsinnige vonden in 't licht te komen; soo en sal ick niet alleen het selve geerne toestaen, maer oock tot mijn voordeel duyden, dat die op dese maniere te voorschijn werde gebracht. Want of sommige mochten dencken dat ick ontrent geringhe dingen en van weynigh gewichte mijn moeyte besteedt hadde, soo en sullen sy noch tans niet t'eenemael voor onnut ende onprijsselijck houden, het geene U E. in dier voegen als voor het syne is aen-nemende, en niet sonder arbeyt uyt onse spraeck in de Latijnsche heeft overgeset. Alhoewel ick wil gelooven, so iemandt dese dinghen wat naerder begint in te sien, dat hy haest sal bevinden, geen enckel spel te zijn het geene hier wert verhandelt, maer datter de beginselen en gronden geleyt werden van een seer aerdige en diepe speculatie. Soo sullen oock, meyne ick, de Voorstellen die in dese Materie voorvallen, geensins lichter als die van* Diophantus *geacht werden, doch wel vermaeckelijcker misschien, door dienst tets meer inhouden als bloote eygenschappen der getallen. Voorts is te weten,*

dat

Huygens’ Proposal for the Foundations of Probability

“Ick neeme tot beyder fondament, dat in het speelen de kansse, die yemant ergens toe heeft, even soo veel weerdt is als het geen, het welck hebbende hy weder tot deselfde kansse kan geraecken met rechtmatigh spel, dat is, daer in niemandt verlies geboden werdt. By exempel. So yemandt sonder mijn weeten in d’eene handt 3 schellingen verbergt, en in d’ander 7 schellingen, ende my te kiesen geeft welck van beyde ick begeere te hebben, ick segge dit my even soo veel weerdt te zijn, als of ick 5 schellingen seecker hadde. Om dat, als ik 5 schellingen hebbe, ick wederom daer toe kan geraecken, dat ick gelijcke kans sal hebben, om 3 of 7 schellingen te krijgen, en dat met rechtmatigh spel: gelijck hier naer sal betoont werden.”

Translation

“I take as the foundation of both [calculating what non-finished hazard games are worth, and calculating winning chances in such games] that in playing the chance that someone has in some matter, is worth just as much as the amount that, if he possesses it, will give him the same chances in a fair game, that is a game where no loss is offered to anyone. For instance. Suppose someone without my knowing hides in one hand 3 shillings, and in the other 7 shillings, and he offers me the choice between the two hands. Then I would say that this offer is worth the same as having 5 shillings for sure. Because, if I have 5 shillings, I can wager them in such manner that I have equal chances of getting 3 or 7 shillings, and that in a fair game, as will be explained hereafter.”

“[. . .] Indien ick gelijcke kans heb om 3 te hebben of 7, soo is door dit Voorstel mijn kansse 5 weerdt; ende het is seecker dat ick 5 hebbende weder tot de selfde kansse kan geraecken. Want speelende om de selve tegen een ander die daer 5 tegen set, met beding dat de geene die wint den anderen 3 sal geven; soo is dit rechtmaetig spel, ende het blijckt dat ick gelijcke kans hebbe om 3 te hebben, te weeten, als ick verlies, of 7 indien ick win; want alsdan treck ick 10, daer van ick hem 3 geef.”

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“[…] If I have equal chances to have 3 or 7, then by my Proposal this chance is worth 5; and it is sure that if I have 5, I will get to the same chance. Because putting 5 at stake against someone who stakes 5 against it, with condition that the one who wins will give the other 3, one has a fair game, and it becomes clear that I have equal chance of getting 3, namely, if I lose, or 7 if I win; because if I win I draw 10, of which I give 3 to him.”

Huygens' Reconstruction

- Starting point is expectation of single individual in lottery-like situation.
- Reconstruction uses n-person game, where n is the number of proposed chances, with equal stakes, and symmetric roles.
- Value of the stakes equals expectation.
- "Equal Chance" is validly defined as free choice for the player in a symmetric situation. See Freudenthal [Fre80].
- This game foundation of probability predates the Dutch book argument of Ramsey (1926) [Ram31] by more than two and a half centuries.

Decision Making under Uncertainty

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An agent faces a choice between a finite number of possible courses of action $a_1, \ldots, a_n$. The agent is uncertain about the state of the world: she considers states $s_1, \ldots, s_m$ possible. There is a table of consequences c, with $c(s_i, a_j)$ giving the consequences of performing action a_j in state s_i.

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Suppose there is a preference ordering R on the consequences, with cRc' expressing that either the agent is indifferent between c and c', or the agent strictly prefers c to c'. Assume R is transitive and reflexive. Then define cPc' as $cRc' \wedge \neg c'Rc$, so that cPc' expresses that the agent strictly prefers c to c'. The relation P is transitive and irreflexive.

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How can the agent pick the best available action?

THEORY OF GAMES AND ECONOMIC BEHAVIOR

By JOHN VON NEUMANN, and

OSKAR MORGENSTERN

PRINCETON
PRINCETON UNIVERSITY PRESS
1944

The Von Neuman and Morgenstern Decision Tool

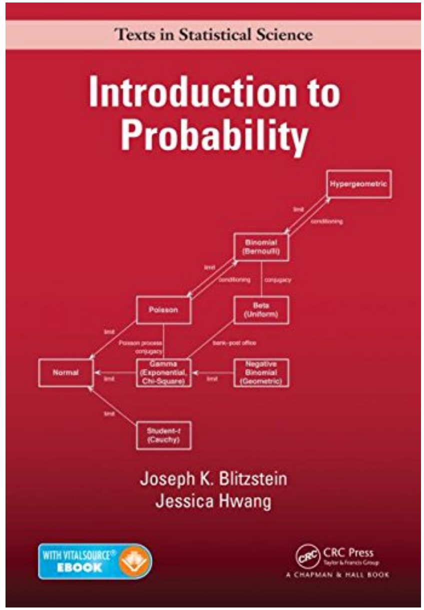
Von Neumann and Morgenstern [NM44] showed how to turn this into a tool for decision making if one adds a probability measure P on the state set. So assume $P(s_i) \geq 0$ and $\sum_{i=1}^{n} P(s_i) = 1$. Then a utility function u on the consequences induces a utility function U on the actions, by means of

$$U(a_j) = \sum_{i=1}^{n} P(s_i) u(s_i, a_j).$$

A rational agent who disposes of a utility function u representing her preferences and a probability measure on what she thinks is possible will perform the action a_j that maximizes $U(a_j) \ldots$

This is the reason why expositions of probability theory often make strong claims about the applicability of their subject.

Strong Claims . . .



"Life: Life is uncertain, and probability is the logic of uncertainty. While it isn't practical to carry out a formal probability calculation for every decision made in life, thinking hard about probability can help us avert some common fallacies, shed light on coincidences, and make better predictions." [BH14]

Success Story: The German Tank Problem

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Statistical estimate for tanks produced in August 1942: 327. Intelligence estimate: 1550. German records: 342.

Qualitative and Quantitative Beliefs at Odds

You are from a population with a statistical chance of 1 in 100 of having disease D. The initial screening test for this has a false positive rate of 0.2 and a false negative rate of 0.1. You tested positive (T).

Should you believe you have disease D?

- You reason: "if I test positive then, given that the test is quite reliable, the probability that I have D is quite high."
- So you believe that you have D.
- Now you recall: "True positives dwarfed by false positives"
- You pick up pen and paper and calculate:

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\neg D)P(\neg D)}$$

- Filling in $P(T|D) = 0.9$, $P(D) = 0.01$, $P(\neg D) = 0.99$, $P(T|\neg D) = 0.2$ gives $P(D|T) = \frac{1}{23}$.
- You don't believe you have D but you agree to further testing.

- Richard Jeffrey: Give up qualitative belief. It is misleading. [Jef04]

RICHARD JEFFREY

Subjective Probability

The Real Thing

Some Varieties of Belief

- Betting belief (or: Bayesian belief) in φ: $P(\varphi) > P(\neg\varphi)$. Van Eijck & Renne [ER14].
- Threshold belief in φ: $P(\varphi) > t$, for some specific t with $\frac{1}{2} \leq t < 1$. Also known as Lockean belief.
- Stable belief in φ: For all consistent ψ: $P(\varphi|\psi) > P(\neg\varphi|\psi)$ (Leitgeb [Lei10]).
- Strong belief in φ. Defined for plausibility models, e.g., locally connected well-preorders. An agent strongly believes in φ if φ is true in all most plausible accessible worlds. This yields a KD45 notion of belief (reflexive, euclidean, and serial). Baltag & Smets [BS06, BS08]
- Subjective certainty belief in φ: $P(\varphi) = 1$. Used in epistemic

game theory (Aumann [Aum99]).

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But actually, she believes, of course, that one of the tickets is winning:

$$B_a \bigvee_{t=000001}^{111111} t.$$

This is a contradiction.

Three Possible Reactions to the Lottery Puzzle

1. Deny that Alice believes that her ticket is not winning.
2. Block the inference from $\bigwedge_{t=000001}^{111111} B_a \neg t$ to $B_a \bigwedge_{t=000001}^{111111} \neg t$.
3. ...?? ...Deny that Alice believes that there is a winning ticket.

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3. . . . ?? . . . Deny that Alice believes that there is a winning ticket.

Discussion

- Advantage of (1): no need to sacrifice closure of belief under conjunction.
- Disadvantage of (1): severe restriction of what counts as belief.
- Advantage of (2): sacrifice closure of belief under conjunction is maybe not so bad after all. Lots of nice logical properties remain (see below).
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  Easy to recognize: they call the lottery puzzle the lottery paradox.
- Proponents of (2): Subjective Probabilists like Jeffrey [Jef04]. Decision theorists like Kyburg [Kyb61].

How can we drop the closure of belief under conjunction?

We need an operator B_i that does not satisfy (Dist).

$$B_i(\varphi \to \psi) \to B_i\varphi \to B_i\psi \tag{Dist-B}$$

This means: B_i is not a normal modal operator.

See also [Zve10].

Epistemic Neighbourhood Models

An **Epistemic Neighbourhood Model** $\mathcal{M}$ is a tuple

$$(W, R, V, N)$$

where

- W is a non-empty set of worlds.
- R is a function that assigns to every agent $i \in Ag$ an equivalence relation $\sim_i$ on W. We use $[w]_i$ for the $\sim_i$ class of w, i.e., for the set $\{v \in W \mid w \sim_i v\}$.
- V is a valuation function that assigns to every $w \in W$ a subset of *Prop*.
- N is a function that assigns to every agent $i \in Ag$ and world $w \in W$ a collection $N_i(w)$ of sets of worlds—each such set called a neighbourhood of w—subject to a set of conditions.

Conditions

(c) $\forall X \in N_i(w) : X \subseteq [w]_i$. This ensures that agent i does not believe any propositions $X \subseteq W$ that she knows to be false.

(f) $\emptyset \notin N_i(w)$. This ensures that no logical falsehood is believed.

(n) $[w]_i \in N_i(w)$. This ensures that what is known is also believed.

(a) $\forall v \in [w]_i : N_i(v) = N_i(w)$. This ensures that if X is believed, then it is known that X is believed.

(m) $\forall X \subseteq Y \subseteq [w]_i$: if $X \in N_i(w)$, then $Y \in N_i(w)$. This says that belief is monotonic: if an agent believes X, then she believes all propositions $Y \supseteq X$ that follow from X.

Extra conditions

(d) If $X \in N_i(w)$ then $[w]_i - X \notin N_i(w)$. This says that if i believes a proposition X then i does not believe the negation of that proposition.

(sc) $\forall X, Y \subseteq [w]_a$: if $[w]_a - X \notin N_a(w)$ and $X \subsetneq Y$, then $Y \in N_a(w)$. If the agent does not believe the complement $[w]_a - X$, then she must believe any strictly weaker Y implied by X.

Language

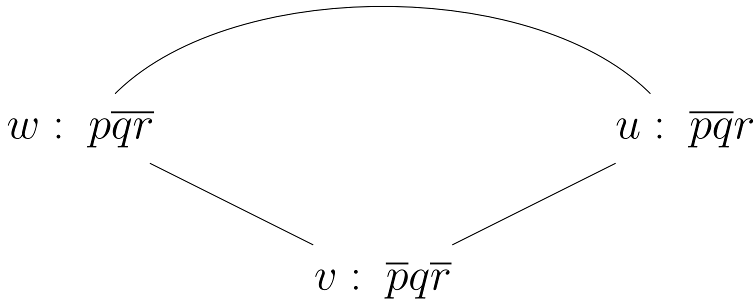
$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid B_i\varphi.$$

Semantics:

$$\mathcal{M}, w \models K_i\varphi \text{ iff } \quad \text{for all } v \in [w]_i : \mathcal{M}, v \models \varphi.$$

$$\mathcal{M}, w \models B_i\varphi \text{ iff } \quad \text{for some } X \in N_i(w), \text{for all } v \in X : \mathcal{M}, v \models \varphi.$$

Example



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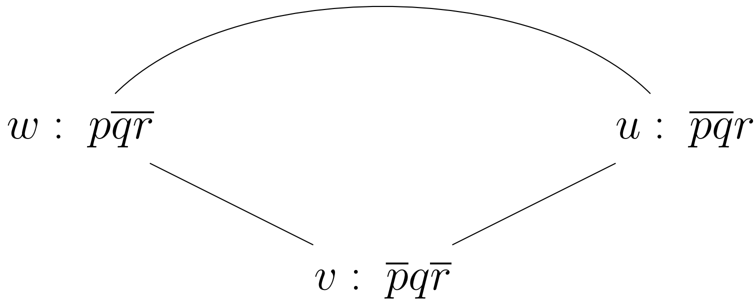

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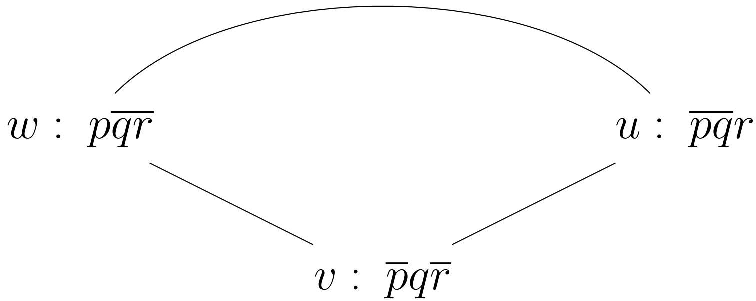

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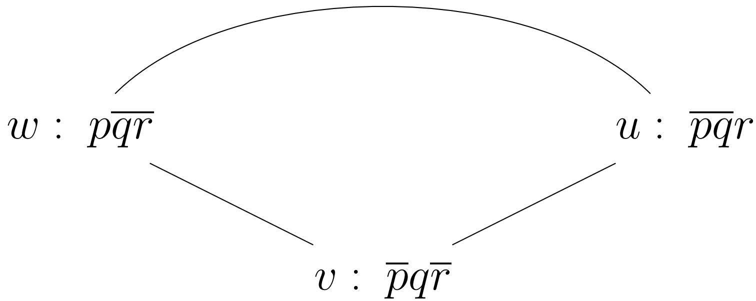

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Example

$$\begin{tikzcd}
w:\ p\overline{qr} \arrow[rr, dash, bend left] \arrow[dr, dash] & & u:\ \overline{pq}r \arrow[dl, dash] \\
& v:\ \overline{p}q\overline{r} &
\end{tikzcd}$$

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The lottery puzzle is solved in neighbourhood models for belief by non-closure of belief under conjunction.

(Taut)	All instances of propositional tautologies
(Dist-K)	$K_i(\varphi \to \psi) \to K_i\varphi \to K_i\psi$
(T)	$K_i\varphi \to \varphi$
(PI-K)	$K_i\varphi \to K_iK_i\varphi$
(NI-K)	$\neg K_i\varphi \to K_i\neg K_i\varphi$
(F)	$\neg B_i\bot.$
(PI-KB)	$B_i\varphi \to K_iB_i\varphi$
(NI-KB)	$\neg B_i\varphi \to K_i\neg B_i\varphi$
(KB)	$K_i\varphi \to B_i\varphi$
(M)	$K_i(\varphi \to \psi) \to B_i\varphi \to B_i\psi$
(D)	$B_i\varphi \to \neg B_i\neg\varphi.$
(SC)	$\check{B}_a\varphi \wedge \check{K}_a(\neg\varphi \wedge \psi) \to B_a(\varphi \vee \psi)$

RULES

$$\frac{\varphi \to \psi \quad \varphi}{\psi}\ \text{(MP)} \qquad \frac{\varphi}{K_i\varphi}\ \text{(Nec-K)}$$

Completeness for Epistemic Neighbourhood Models

See [ER14] and [BvBvES14].

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Epistemic Probability Models

Epistemic probability models are the result of replacing the neighbourhood function of an epistemic neighbourhood model by a weight function L.

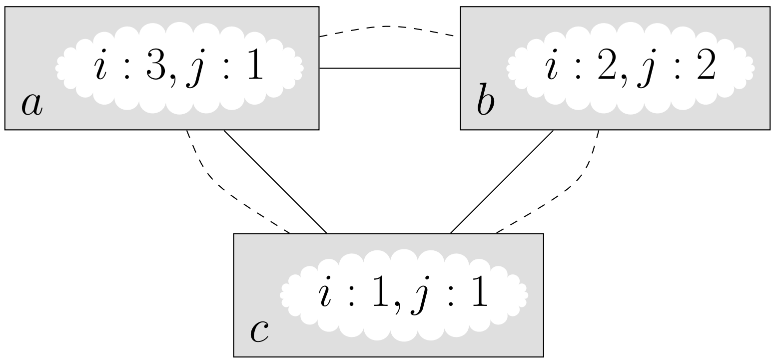
A weight function L assigns to every agent i a function $L_i : W \to \mathbb{Q}^+$ (the positive rationals), subject to the constraint that the sum of the L_i values over each epistemic partition cell of i is bounded.

If $X \subseteq W$ then let $L_i(X)$ be shorthand for $\sum_{x \in X} L_i(x)$.

Boundedness: for each i and w: $L_i([w]_i) < \infty$.

Example: Horse Racing

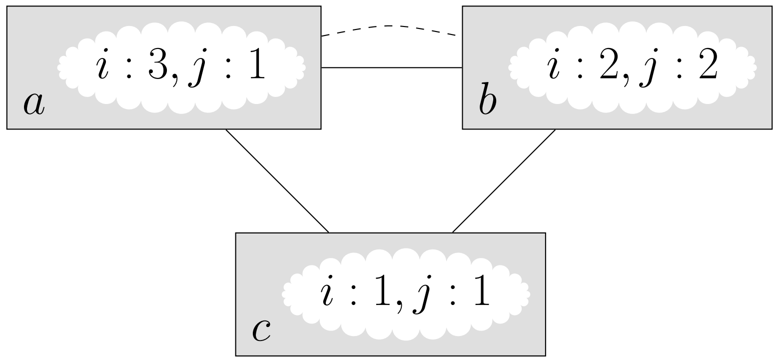
Two agents i, j consider betting on a horse race. Three horses a, b, c take part in the race, and there are three possible outcomes: a for "a wins", b for " b wins", and c for "c wins." i takes the winning chances to be $3 : 2 : 1$, j takes them to be $1 : 2 : 1$.



In all worlds, i assigns probability $\frac{1}{2}$ to a, $\frac{1}{3}$ to b and $\frac{1}{6}$ to c, while j assigns probability $\frac{1}{4}$ to a and to c, and probability $\frac{1}{2}$ to b.

Agent j has learnt something

Agent j (dashed lines) now considers c impossible.



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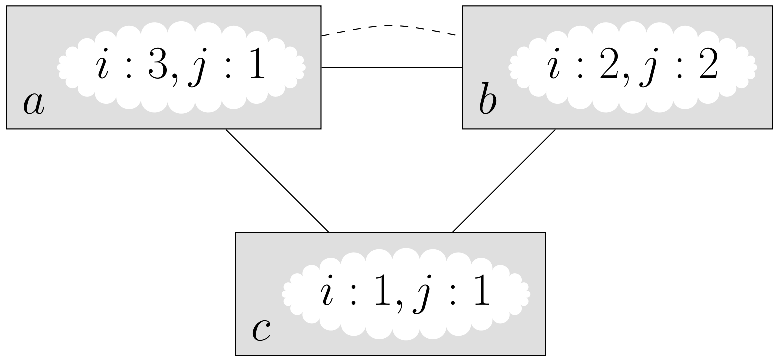

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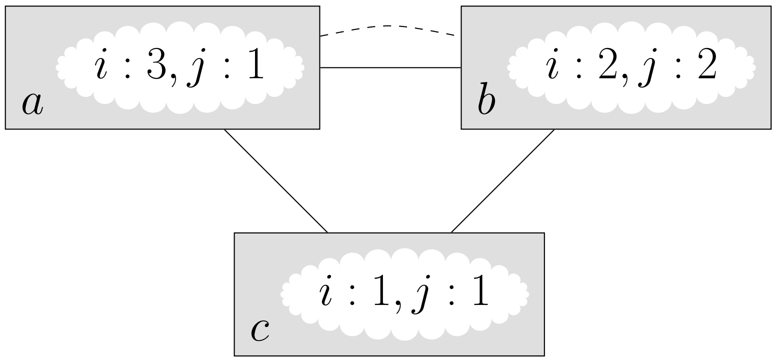

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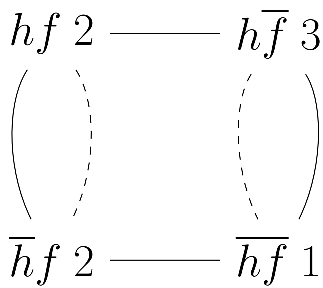
The probabilities assigned by j have changed, as follows. In worlds a and b, j assigns probability $\frac{1}{3}$ to a and $\frac{2}{3}$ to b. In world c, j is sure of c.

Fair or Biased?

Two agents i (solid lines) and j (dashed lines) are uncertain about the toss of a coin. i holds it for possible that the coin is fair f and that it is biased $\overline{f}$, with a bias $\frac{2}{3}$ for heads h. j can distinguish f from $\overline{f}$. The two agents share the same weight (so this is a single weight model), and the weight values are indicated as numbers in the picture.



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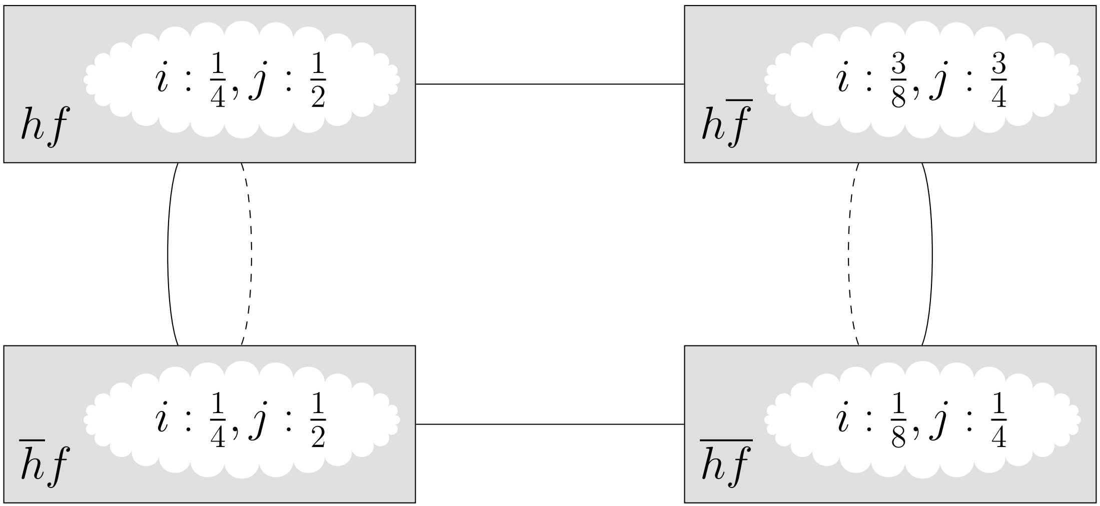
$$
\begin{tikzcd}
hf\ 2 \arrow[r, no head] \arrow[d, no head, bend right] \arrow[d, no head, dashed, bend left] & h\overline{f}\ 3 \arrow[d, no head, dashed, bend right] \arrow[d, no head, bend left] \\
\overline{h}f\ 2 \arrow[r, no head] & \overline{hf}\ 1
\end{tikzcd}
$$

In world hf, i assigns probability $\frac{5}{8}$ to h and probability $\frac{1}{2}$ to f, and j assigns probability $\frac{1}{2}$ to h and probability 1 to f.

Normalized Version

Give each agent its own weight, and normalize the weight functions using the epistemic accessibilities.



Interpretation of KB language in Epistemic Probability Models

$$\mathcal{M}, w \models K_i\varphi \text{ iff } \quad \text{for all } v \in [w]_i : \mathcal{M}, v \models \varphi.$$

$$\mathcal{M}, w \models B_i\varphi \text{ iff}$$
$$\sum\{L_i(v) \mid v \in [w]_i, \mathcal{M}, v \models \varphi\} > \sum\{L_i(v) \mid v \in [w]_i, \mathcal{M}, v \models \neg\varphi\}.$$

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Agreement

Let $\mathcal{M} = (W, R, V, N)$ be a neighbourhood model and let L be a weigth function for $\mathcal{M}$. Then L agrees with $\mathcal{M}$ if it holds for all agents i and all $w \in W$ that

$$X \in N_i(w) \text{ iff } L_i(X) > L_i([w]_i - X).$$

Incompleteness of KB Calculus for Probability Models

There exists an epistemic neighbourhood model $\mathcal{M}$ that has no agreeing weight function.

Adaptation of example 2 from [WF79, pp. 344-345]

Let *Prop* $:= \{a, b, c, d, e, f, g\}$. Assume a single agent 0. Define:

$$\mathcal{X} := \{efg, abg, adf, bde, ace, cdg, bcf\}.$$

$$\mathcal{X}' := \{abcd, cdef, bceg, acfg, bdfg, abef, adeg\}.$$

Notation: xyz for $\{x, y, z\}$.

$$\mathcal{Y} := \{Y \mid \exists X \in \mathcal{X} : X \leq Y \leq W\}.$$

Let $\mathcal{M} := (W, R, V, N)$ be defined by $W :=$ ***Prop***, $R_0 = W \times W$, $V(w) = \{w\}$, and for all $w \in W$, $N_0(w) = \mathcal{Y}$.

Check that $\mathcal{X}' \cap \mathcal{Y} = \emptyset$. So $\mathcal{M}$ is a neighbourhood model.

Toward a contradiction, suppose there exists a weight function L that agrees with $\mathcal{M}$.

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Since each letter $p \in W$ occurs in exactly three of the seven members of $\mathcal{X}$, we have:

$$\sum_{X \in \mathcal{X}} L_0(X) = \sum_{p \in W} 3 \cdot L_0(\{p\}).$$

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Since each letter $p \in W$ occurs in exactly four of the seven members of $\mathcal{X}'$, we have:

$$\sum_{X \in \mathcal{X}'} L_0(X) = \sum_{p \in W} 4 \cdot L_0(\{p\}).$$

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On the other hand, from the fact that $L_0(X) > L_o(W - X)$ for all members X of $\mathcal{X}$ we get:

$$\sum_{X \in \mathcal{X}} L_0(X) > \sum_{X \in \mathcal{X}} L_0(W - X) = \sum_{X \in \mathcal{X}'} L_0(X).$$

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Contradiction. So no such L_0 exists.

Strengthening the Axiom System

Scott Axioms, intuitively:

If agent a knows the number of true φ_i is less than or equal to the number of true ψ_i, agent a believes φ_1, and the remaining φ_i are each consistent with her beliefs, then agent a believes one of the ψ_i.

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It turns out this is expressible in the KB language.

Segerberg notation [Seg71]:

$(\varphi_1, \ldots, \varphi_m \mathbb{I}_a \psi_1, \ldots, \psi_m)$

abbreviates a KB formula expressing that agent a knows that the number of true φ_i's is less than or equal to the number of true ψ_i's.

Put another way, $(\varphi_i \mathbb{I}_a \psi_i)_{i=1}^m$ is true if and only if every one of a's epistemically accessible worlds satisfies at least as many ψ_i as φ_i.

Scott Axioms

(Scott) $\left[(\varphi_i \mathbb{I}_a \psi_i)_{i=1}^{m} \wedge B_a\varphi_1 \wedge \bigwedge_{i=2}^{m} \check{B}_a\varphi_i\right] \to \bigvee_{i=1}^{m} B_a\psi_i$

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Fact 1 *Adding the Scott axioms to the KB calculus yields a system that is sound and complete for epistemic probability models [ER14].*

Epistemic Probability Language

Let i range over *Ag*, p over *Prop*, and q over $\mathbb{Q}$. Then the language of epistemic probability logic is given by:

$$\begin{array}{lcl} \varphi & ::= & \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid t_i \geq 0 \mid t_i = 0 \\ t_i & ::= & q \mid q \cdot P_i\varphi \mid t_i + t_i \text{ where all indices } i \text{ are the same.} \end{array}$$

Truth for Epistemic Probability Logic

Let $\mathcal{M} = (W, R, V, L)$ be an epistemic weight model and let $w \in W$.

$$
\begin{array}{rcl}
\mathcal{M}, w \models \top & & \text{always} \\
\mathcal{M}, w \models p & \text{iff} & p \in V(w) \\
\mathcal{M}, w \models \neg\varphi & \text{iff} & \text{it is not the case that } \mathcal{M}, w \models \varphi \\
\mathcal{M}, w \models \varphi_1 \wedge \varphi_2 & \text{iff} & \mathcal{M}, w \models \varphi_1 \text{ and } \mathcal{M}, w \models \varphi_2 \\
\mathcal{M}, w \models t_i \geq 0 & \text{iff} & [\![t_i]\!]^{\mathcal{M}}_w \geq 0 \\
\mathcal{M}, w \models t_i = 0 & \text{iff} & [\![t_i]\!]^{\mathcal{M}}_w = 0.
\end{array}
$$

$$
\begin{array}{rcl}
[\![q]\!]^{\mathcal{M}}_w & := & q \\
[\![q \cdot P_i\varphi]\!]^{\mathcal{M}}_w & := & q \times P^{\mathcal{M}}_{i,w}(\varphi) \\
[\![t_i + t'_i]\!]^{\mathcal{M}}_w & := & [\![t_i]\!]^{\mathcal{M}}_w + [\![t'_i]\!]^{\mathcal{M}}_w
\end{array}
$$

$$
P^{\mathcal{M}}_{i,w}(\varphi) = \frac{L_i(\{u \in [w]_i \mid \mathcal{M}, u \models \varphi\})}{L_i([w]_i)}.
$$

Fact 2 *A sound and complete calculus for the language of epistemic probability logic, interpreted in epistemic probability models, is given in [ES14]. See also [FH94] and [Koo03].*

From Epistemic Probability Models to Epistemic Neighbourhood Models

If $\mathcal{M} = (W, R, V, L)$ is an epistemic weight model, then $\mathcal{M}^\bullet$ is the tuple (W, R, V, N) given by replacing the weight function by a function N, where N is defined as follows, for $i \in Ag$, $w \in W$.

$$N_i(w) = \{X \subseteq [w]_i \mid L_i(X) > L_i([w]_i - X)\}.$$

Fact 3 *For any epistemic weight model $\mathcal{M}$ it holds that $\mathcal{M}^\bullet$ is a neighbourhood model.*

Translating Knowledge and Belief

If φ is a KB formula, then $\varphi^\bullet$ is the formula of the language of epistemic probability logic given by the following instructions:

$$
\begin{aligned}
\top^\bullet &= \top \\
p^\bullet &= p \\
(\neg\varphi)^\bullet &= \neg\varphi^\bullet \\
(\varphi_1 \wedge \varphi_2)^\bullet &= \varphi_1^\bullet \wedge \varphi_2^\bullet \\
(K_i\varphi)^\bullet &= P_i(\varphi^\bullet) = 1 \\
(B_i\varphi)^\bullet &= P_i(\varphi^\bullet) > P_i(\neg\varphi^\bullet).
\end{aligned}
$$

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(K_i\varphi)^\bullet &= P_i(\varphi^\bullet) = 1 \\
(B_i\varphi)^\bullet &= P_i(\varphi^\bullet) > P_i(\neg\varphi^\bullet).
\end{aligned}$$

Theorem 4 *For all KB formulas φ, for all epistemic probability models $\mathcal{M}$, for all worlds w of $\mathcal{M}$:*

$$\mathcal{M}^\bullet, w \models \varphi \textit{ iff } \mathcal{M}, w \models \varphi^\bullet.$$

Theorem 5 *Let* $\vdash$ *denote derivability in the neighbourhood calculus for KB. Let* $\vdash'$ *denote derivability in the calculus of EPL. Then* $\vdash \varphi$ *implies* $\vdash' \varphi^\bullet$.

Conclusions and Connections

- Representation of probability information by means of weight functions was designed with implementation of model checking in mind. Just extend epistemic model checkers for S5 logics with a weight table for each agent.
- Implementations of model checkers for these logics can be found in [Eij13] and in [San14] ...
- The implementations can deal with Monty Hall style puzzles, urn puzzles, Bayesian updating by drawing from urns or tossing (possibly biased) coins, and 'paradoxes' such as the puzzle of the three prisoners.
- Efficiency was not a goal, but these implementation can be made very efficient with a little effort.

How to Move on From Here

- Further analysis of the connection between neighbourhood logics and probabilistic logics [ER14]. This is also connected to work of Wes Holliday and Thomas Icard [HI13]. Are there applications where neighbourhoods without agreeing weight functions are natural? Is there a natural interpretation for the incompleteness example for $\{a, b, c, d, e, f, g\}$?

- Combine EPL with network information for the agents, where the network is given by a relation, and where links starting from an agent can be added (“start following”) and deleted (“stop following, unfollow”). Interpret announcements as group messages to all followers. See [RT11] and current work by Jerry Seligman and Thomas Agotnes. But: this can all be done with epistemic PDL with a binary follow relation F added.

- Add bias variables X for the representation of unknown biases. Collaboration in progress with Joshua Sack.
- Work with the epistemic PDL version of the probabilistic logic, as an extension of LCC from [BvEK06]. This gives us common knowledge, and a nice axiomatisation by means of epistemic program transformation [Ach14].
- Achieve better efficiency, by using methods proposed by Kaile Su.
- Towards analysis of real-life protocols. Compare the use of epistemic model checking by Malvin Gattinger [Gat13, Gat14b, Gat14a].

- Consider weak weight models, where the weight functions assign pairs of values (x, y), with x giving the lower probability L and $x + y$ the upper probability U. Belief of i in φ is now modelled as $L_i(\varphi) > H_i(\neg\varphi)$. This connects up to weak Bayesianism and imprecise probability theory [Wal91].
- Consolidate what we know about the topic in a state-of-the-art textbook [BvBvES14].

Aside: The Puzzle of the Three Prisoners

Alice, Bob and Carol are in prison. It is known that two of them will be shot, the other freed. The warden knows what is going to happen, so Alice asks him to reveal the name of one other than herself who will be shot, explaining to him that since there must be at least one, this will not reveal any new information. The warden agrees and says that Bob will be shot. Alice is cheered up a little by this, for she concludes that her chance of surviving has now improved from $\frac{1}{3}$ to $\frac{1}{2}$. Is this correct? How does this agree with the intuition that the warden has not revealed new information?

Many sources, e.g. [Jef04].

References

[Ach14] Andreea Christina Achimescu. Games and logics for informational cascades. Master's thesis, ILLC, Amsterdam, February 2014.

[Aum99] R.J. Aumann. Interactive epistemology I: Knowledge. International Journal of Game Theory, 28(3):263–300, 1999.

[BH14] Joseph K. Blitzstein and Jessica Hwang. Introduction to Probability. CRC Press, 2014.

[BS06] A. Baltag and S. Smets. Conditional doxastic models: A qualitative approach to dynamic belief revision. Electronic Notes in Theoretical Computer Science (ENTCS), 165:5–21, 2006.

[BS08] A. Baltag and S. Smets. A qualitative theory of dynamic interactive belief revision. In G. Bonanno, W. van der Hoek, and M. Wooldridge, editors, Texts in Logic and Games. Amsterdam University Press, 2008. To appear.

[BvBvES14] Alexandru Baltag, Johan van Benthem, Jan van Eijck, and Sonja Smets. Reasoning about communication and action. Book Manuscript, ILLC, 2014.

[BvEK06] J. van Benthem, J. van Eijck, and B. Kooi. Logics of communication and change. Information and Computation, 204(11):1620–1662, 2006.

[Eij13] Jan van Eijck. Learning about probability. available from `homepages.cwi.nl:~/jve/software/prodemo`, 2013.

[ER14] Jan van Eijck and Bryan Renne. Belief as willingness to bet. Manuscript, ILLC, Amsterdam, 2014.

[ES14] Jan van Eijck and François Schwarzentruber. Epistemic probability logic simplified. In Advances in Modal Logic, 2014.

[FH94] R. Fagin and J.Y. Halpern. Reasoning about knowledge and probability. Journal of the ACM, pages 340–367, 1994.

[Fre80] Hans Freudenthal. Huygens' foundations of probability. Historia Mathematica, 7:113–117, 1980.

[Gat13] Malvin Gattinger. Epistemic crypto logic — functional programming and model checking of cryptographic protocols. Technical report, ILLC, Amsterdam,

2013. Exam paper for the course 'Functional Specification of Algorithms'.

[Gat14a] Malvin Gattinger. Dynamic epistemic logic for guessing games and cryptographic protocols. Master's thesis, ILLC, University of Amsterdam, June 2014.

[Gat14b] Malvin Gattinger. More epistemic crypto logic. Technical report, ILLC, Amsterdam, February 2014.

[HI13] Wesley H. Holliday and Thomas F. Icard. Measure semantics and qualitative semantics for epistemic modals. In Proceedings of SALT, volume 23, pages 514–534, 2013.

[Huy60] Christiaan Huygens. Van Rekeningh in Spelen van Geluck. 1660. About Calculation in Hazard Games; Latin: De ratociniis in Ludo aleae.

[Jef04] Richard Jeffrey. Subjective Probability — The Real Thing. Cambridge University Press, 2004.

[Koo03] Barteld P. Kooi. Probabilistic dynamic epistemic logic. Journal of Logic, Language and Information, 12(4):381–408, 2003.

[Kyb61] H.E. Kyburg. Probability and the Logic of Rational Belief. Wesleyan University Press, Middletown, CT, 1961.

[Lei10] Hannes Leitgeb. Reducing belief simpliciter to degrees of belief. Technical report, Ludwig Maximilian University of Munich, 2010.

[NM44] John von Neumann and Oscar Morgenstern. Theory of Games and Economic Behavior. Princeton University Press, 1944.

[Ram31] F.P. Ramsey. Truth and probability. In R. Braithwaite, editor, The Foundations of Mathematics and Other Essays. Humanities Press, 1931.

[RT11] Ji Ruan and Michael Thielscher. A logic for knowledge flow in social networks. In D. Wang and M. Reynolds, editors, AI 2011, number 7106 in LNAI, pages 511–520. Springer, 2011.

[San14] Thomas Santoli. Haskell project epistemic logic. Technical report, ILLC, Summer 2014.

[Seg71] K. Segerberg. Qualitative probability in a modal setting. In J. Fenstad, editor, Proceedings of the 2nd Scandinavian Logic Symposium, pages 341–352, Amsterdam, 1971. North Holland.

[Wal91] Peter Walley. Statistical Reasoning with Imprecise Probabilities. Chapman and Hall, London, 1991.

[WF79] P. Walley and T.L. Fine. Varieties of modal (classificatory) and comparative probability. Synthese, 41:321–374, 1979.

[Zve10] Jonathan A. Zvesper. Playing with Information. PhD thesis, ILLC, University of Amsterdam, 2010.