II. Stable sets and colourings

1. Stable sets and colourings

Let $G = (V, E)$ be a graph. A *stable set* is a subset S of V containing no edge of G. A *clique* is a subset C of V such that any two vertices in C are adjacent. So

(1) S is a stable set of $G \iff S$ is a clique of $\overline{G}$,

where $\overline{G}$ denotes the complementary graph of G.[1]

A *vertex-colouring* or *colouring* of G is a partition Π of V into stable sets $S_1, \ldots, S_k$. The sets $S_1, \ldots, S_k$ are called the *colours* of the colouring. A *clique cover* of G is a partition Π of V into cliques.

Define:

(2)
$$\begin{aligned}
\alpha(G) &:= \max\{|S| \mid S \text{ is a stable set}\},\\
\omega(G) &:= \max\{|C| \mid C \text{ is a clique}\},\\
\chi(G) &:= \min\{|\Pi| \mid \Pi \text{ is a colouring}\},\\
\overline{\chi}(G) &:= \min\{|\Pi| \mid \Pi \text{ is a clique cover}\}.
\end{aligned}$$

These numbers are called the *stable set number*, the *clique number*, the *vertex-colouring number* or *colouring number*, and the *clique cover number* of G, respectively. We say that a graph G is *k-(vertex-)colourable* if $\chi(G) \leq k$.

Note that

(3) $\alpha(G) = \omega(\overline{G})$ and $\overline{\chi}(G) = \chi(\overline{G})$.

We have seen that in any graph $G = (V, E)$, a maximum-size matching can be found in polynomial time. This means that $\alpha(L(G))$ can be found in polynomial time, where $L(G)$ is the line graph of G.[2]

On the other hand, it is NP-complete to find a maximum-size stable set in a graph. That is, determining $\alpha(G)$ is NP-complete. Since $\alpha(G) = |V| - \tau(G)$ and $\alpha(G) = \omega(\overline{G})$, also determining the vertex cover number $\tau(G)$ and the clique number $\omega(G)$ are NP-complete problems.

Moreover, determining $\chi(G)$ is NP-complete. It is even NP-complete to decide if a graph is 3-colourable. Note that one can decide in polynomial time if a graph G is 2-colourable, as bipartiteness can be checked in polynomial time.

These NP-completeness results imply that if NP$\neq$co-NP, then one may not expect a min-max relation characterizing the stable set number $\alpha(G)$, the clique number $\omega(G)$, or the colouring number $\chi(G)$ of a graph G.

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[1] The *complement* or the *complementary graph* $\overline{G}$ of a graph $G = (V, E)$ is the graph with vertex set V, where any two distinct vertices in V are adjacent in $\overline{G}$ if and only if they are nonadjacent in G.

[2] The *line graph* $L(G)$ of a graph $G = (V, E)$ has vertex set E and edge set $\{\{e, f\} \mid e, f \in E, e \neq f, e \cap f \neq \emptyset\}$.

Well-known is the *four-colour conjecture* (*4CC*), stating that each planar graph is 4-colourable. This conjecture was proved by Appel and Haken [1] and Appel, Haken, and Koch [2] (cf. Robertson, Sanders, Seymour, and Thomas [17]), and is therefore now called the *four-colour theorem* (*4CT*).

2. Bounds

There is a trivial upper bound on the colouring number:

$$\chi(G) \leq \Delta(G) + 1, \tag{4}$$

where $\Delta(G)$ denotes the maximum degree of the vertices of G. Brooks [5] sharpened this inequality as follows:

Brooks' theorem: For any connected graph G one has $\chi(G) \leq \Delta(G)$, except if $G = K_n$ or $G = C_n$ for some odd $n \geq 3$.[3]

Another inequality relates the clique number and the colouring number:

$$\omega(G) \leq \chi(G). \tag{5}$$

This is easy, since in any clique all vertices should have different colours. It implies $\alpha(G) \leq \overline{\chi}(G)$, since $\alpha(G) = \omega(\overline{G}) \leq \chi(\overline{G}) = \overline{\chi}(G)$.

But there are several graphs which have strict inequality in (5). We mention the odd circuits C_{2k+1}, with $2k + 1 \geq 5$: then $\omega(C_{2k+1}) = 2$ and $\chi(C_{2k+1}) = 3$. Moreover, for the complement $\overline{C_{2k+1}}$ of any such graph we have: $\omega(\overline{C_{2k+1}}) = k$ and $\chi(\overline{C_{2k+1}}) = k + 1$.

It was a conjecture of Berge [4] that these graphs are crucial, which was proved by Chudnovsky, Robertson, Seymour, and Thomas [6]:[4]

Strong perfect graph theorem: Let G be a graph. If for no odd $n \geq 5$, C_n or $\overline{C_n}$ is an induced subgraph of G, then $\omega(G) = \chi(G)$.

Another conjecture is due to Hadwiger [11]. Since there exist graphs with $\omega(G) < \chi(G)$, it is not true that if $\chi(G) \geq n$ then G contains the complete graph K_n on n vertices as a subgraph. However, Hadwiger conjectured the following, where a graph H is called a *minor* of a graph G if H arises from some subgraph of G by contracting some (possible none) edges.

Hadwiger's conjecture: If $\chi(G) \geq n$ then G contains K_n as a minor.

In other words, for each n, the graph K_n is the only graph G with the property that G is not $(n-1)$-colourable and each proper minor of G is $(n-1)$-colourable.

[3] Here C_k denotes the circuit with k vertices.

[4] Let $G = (V, E)$ be a graph and let $U \subseteq V$. Then the subgraph of G *induced by* U, denoted by $G[U]$ is the graph (U, E'), where E' equals the set of all edges in E contained in U. The graph $G[U]$ is called an *induced* subgraph of G.

Hadwiger's conjecture is trivial for $n = 1, 2, 3$, and was shown by Hadwiger for $n = 4$. As planar graphs do not contain K_5 as a minor, Hadwiger's conjecture for $n = 5$ implies the four-colour theorem. In fact, Wagner [22] showed that Hadwiger's conjecture for $n = 5$ is equivalent to the four-colour conjecture. Robertson, Seymour, and Thomas [18] showed that Hadwiger's conjecture is true also for $n = 6$, by showing that also in that case it is equivalent to the four-colour theorem. For $n \geq 7$, Hadwiger's conjecture is unsettled.

Exercises

2.1. Show that if G is a bipartite graph, then $\omega(G) = \chi(G)$.

2.2. (i) Derive from Kőnig's edge cover theorem that $\alpha(G) = \overline{\chi}(G)$ if G is bipartite.

(ii) Derive Kőnig's edge cover theorem from the strong perfect graph theorem.

2.3. (i) Let H be a bipartite graph and let G be the complement of the line-graph of H. Derive from Kőnig's matching theorem that $\omega(G) = \chi(G)$.

(ii) Derive Kőnig's matching theorem from the strong perfect graph theorem.

3. Edge-colourings of bipartite graphs

For any graph $G = (V, E)$, an *edge-colouring* is a partition $\Pi = \{M_1, \ldots, M_p\}$ of the edge set E, where each M_i is a matching. Each of these matchings is called a *colour*. Define the *edge-colouring number* $\chi'(G)$ by

(6) $$\chi'(G) := \min\{|\Pi| \mid \Pi \text{ is an edge-colouring of } G\}.$$

So $\chi'(G) = \chi(L(G))$, where $L(G)$ is the line graph of G. Clearly,

(7) $$\chi'(G) \geq \Delta(G),$$

since at each vertex v, the edges incident with v should have different colours. (In other words, $\chi'(G) = \chi(L(G)) \geq \omega(L(G)) \geq \Delta(G)$.) Again the triangle K_3 has strict inequality. Kőnig [13] showed that for bipartite graphs the two numbers are equal.

Theorem 1 (Kőnig's edge-colouring theorem)**.** *For any bipartite graph $G = (V, E)$ one has*

(8) $$\chi'(G) = \Delta(G).$$

That is, the edge-colouring number of a bipartite graph is equal to its maximum degree.

Proof. First notice that the theorem is easy if $\Delta(G) \leq 2$. In that case, G consists of a number of vertex-disjoint paths and even circuits.

In the general case, colour as many edges of G as possible with $\Delta(G)$ colours, without giving the same colour to two intersecting edges. If all edges are coloured we are done, so suppose some edge $e = \{u, w\}$ is not coloured. At least one colour, say *red*, does not occur among the colours given to the edges incident with u. Similarly, there is a colour, say *blue*, not occurring at w. (Clearly, *red*$\neq$*blue*, since otherwise we could give edge e the colour *red*.)

Let H be the subgraph of G having as edges all *red* and *blue* edges of G, together with the edge e. Now $\Delta(H) = 2$, and hence $\chi'(H) = \Delta(H) = 2$. So all edges occurring in H can be (re)coloured with *red* and *blue*. In this way we colour more edges of G than before. This contradicts the maximality assumption. ∎

This proof also gives a polynomial-time algorithm to find an edge-colouring with $\Delta(G)$ colours.

We remark here that Vizing [20] proved that for general (nonbipartite) simple[5] graphs G one has

(9) $$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

(see Section 10). Here 'simple' cannot be deleted, as is shown by the graph G with three vertices, where any two vertices are connected by two parallel edges: then $\Delta(G) = 4$ while $\chi'(G) = 6$.

Exercises

3.1. (i) Let G be the line-graph of some bipartite graph H. Derive from Kőnig's edge-colouring theorem (Theorem 1) that $\omega(G) = \chi(G)$.

(ii) Derive Kőnig's edge-colouring theorem (Theorem 1) from the strong perfect graph theorem.

3.2. Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be partitions of a finite set X such that $|A_1| = \cdots = |A_n| = |B_1| = \cdots = |B_n| = k$. Show that $\mathcal{A}$ and $\mathcal{B}$ have k disjoint common transversals.[6]

4. Partially ordered sets

A *partially ordered set* is a pair $(X, \leq)$ where X is a set and where $\leq$ is a relation on X satisfying (for all $x, y, z \in X$):

(10) (i) $x \leq x$,
(ii) if $x \leq y$ and $y \leq x$ then $x = y$,
(iii) if $x \leq y$ and $y \leq z$ then $x \leq z$.

A subset C of X is called a *chain* if for all $x, y \in C$ one has $x \leq y$ or $y \leq x$. A subset A of X is called an *antichain* if for all $x, y \in A$ with $x \neq y$ one has $x \not\leq y$ and $y \not\leq x$. Note that if C is a chain and A is an antichain then

(11) $$|C \cap A| \leq 1.$$

[5] A graph is *simple* if it has no loops or parallel edges.
[6] *disjoint* always means: *pairwise* disjoint.

First we observe the following easy min-max relation:

Theorem 2. *Let $(X, \leq)$ be a partially ordered set, with X finite. Then the minimum number of antichains needed to cover X is equal to the maximum cardinality of any chain.*

Proof. The fact that the maximum cannot be larger than the minimum follows easily from (11). To see that the two numbers are equal, define for any element $x \in X$ the *height* of x as the maximum cardinality of any chain in X with maximum x. For any $i \in \mathbb{N}$, let A_i denote the set of all elements of height i.

Let k be the maximum height of the elements of X. Then $A_1, \ldots, A_k$ are antichains covering X, and moreover there exists a chain of size k. ∎

Dilworth [7] proved that the same theorem also holds when we interchange the words 'chain' and 'antichain':

Theorem 3 (Dilworth's decomposition theorem)**.** *Let $(X, \leq)$ be a partially ordered set, with X finite. Then the minimum number of chains needed to cover X is equal to the maximum cardinality of any antichain.*

Proof. We apply induction on $|X|$. The fact that the maximum cannot be larger than the minimum follows easily from (11). To see that the two numbers are equal, let α be the maximum cardinality of any antichain and let A be an antichain of cardinality α. Define

$$\begin{aligned} (12) \qquad & A^{\downarrow} := \{x \in X \mid \exists y \in A : x \leq y\}, \\ & A^{\uparrow} := \{x \in X \mid \exists y \in A : x \geq y\}. \end{aligned}$$

Then $A^{\downarrow} \cup A^{\uparrow} = X$ (since A is a maximum antichain) and $A^{\downarrow} \cap A^{\uparrow} = A$.

First assume $A^{\downarrow} \neq X$ and $A^{\uparrow} \neq X$. Then by induction $A^{\downarrow}$ can be covered with α chains. Since $A \subseteq A^{\downarrow}$, each of these chains contains exactly one element in A. For each $x \in A$, let C_x denote the chain containing x. Similarly, there exist α chains C'_x (for $x \in A$) covering $A^{\uparrow}$, where C'_x contains x. Then for each $x \in A$, $C_x \cup C'_x$ forms a chain in X, and moreover these chains cover X.

So we may assume that for each antichain A of cardinality α one has $A^{\downarrow} = X$ or $A^{\uparrow} = X$. It means that each antichain A of cardinality α is either the set of minimal elements of X or the set of maximal elements of X. Now choose a minimal element x and a maximal element y of X such that $x \leq y$. Then the maximum cardinality of an antichain in $X \setminus \{x, y\}$ is equal to $\alpha - 1$ (since each antichain in X of cardinality α contains x or y). By induction, $X \setminus \{x, y\}$ can be covered with $\alpha - 1$ chains. Adding the chain $\{x, y\}$ yields a covering of X with α chains. ∎

Exercises

4.1. Derive Kőnig's edge cover theorem from Dilworth's decomposition theorem.

4.2. Let $\mathcal{I} = (I_1, \ldots, I_n)$ be a family of intervals on $\mathbb{R}$, in such a way that each $x \in \mathbb{R}$ is contained in at most k of these intervals. Show that $\mathcal{I}$ can be partitioned into k classes $\mathcal{I}_1, \ldots, \mathcal{I}_k$ so that each $\mathcal{I}_j$ consists of disjoint intervals.

4.3. Let $D = (V, A)$ be an acyclic directed graph and let s and t be vertices of D such that each arc of D occurs in at least one $s-t$ path. Derive from Dilworth's decomposition theorem that the minimum number of $s-t$ paths needed to cover all arcs is equal to the maximum cardinality of $\delta^{\text{out}}(U)$, where U ranges over all subsets of V satisfying $s \in U, t \notin U$ and $\delta^{\text{in}}(U) = \emptyset$.

4.4. A graph $G = (V, E)$ is called a *comparability graph* if there exists a partial order $\leq$ on V such that for all u, w in V with $u \neq w$ one has:

(13) $$\{u, w\} \in E \Leftrightarrow u \leq w \text{ or } w \leq u.$$

(i) Show that if G is a comparability graph, then $\omega(G) = \chi(G)$.

(ii) Show that if G is the complement of a comparability graph, then $\omega(G) = \chi(G)$. (*Hint:* Use Dilworth's decomposition theorem (Theorem 3).)

4.5. Derive Dilworth's decomposition theorem (Theorem 3) from the strong perfect graph theorem.

5. Perfect graphs

We now consider a general class of graphs, the 'perfect' graphs, that turn out to unify several results in combinatorial optimization, in particular, min-max relations.

As we saw before, the clique number $\omega(G)$ and the colouring number $\chi(G)$ of a graph $G = (V, E)$ are related by the inequality:

(14) $$\omega(G) \leq \chi(G).$$

There are graphs that have strict inequality; for instance, the circuit C_5 on five vertices.

Having equality in (14) does not say that much about the internal structure of a graph: any graph $G = (V, E)$ can be extended to a graph $G' = (V', E')$ satisfying $\omega(G') = \chi(G')$, simply by adding to G a clique of size $\chi(G)$, disjoint from V.

However, if we require that equality in (14) holds for each induced subgraph of G, we obtain a much more powerful condition. The idea for this was formulated by Berge [4]. He defined a graph $G = (V, E)$ te be *perfect* if $\omega(G') = \chi(G')$ holds for each induced subgraph G' of G.

Several classes of graphs could be shown to be perfect, and Berge [3,4] observed the important phenomenon that for several classes of graphs that were shown to be perfect, also the class of complementary graphs is perfect.

Berge therefore conjectured that the complement of any perfect graph is perfect again. This conjecture was proved by Lovász [15], and his *perfect graph theorem* forms the kernel of perfect graph theory. It has several other theorems in graph theory as consequence. Lovász [14] gave the following stronger form of the conjecture, which we show with the elegant linear-algebraic proof found by Gasparian [8].

Theorem 4. *A graph G is perfect if and only if $\omega(G')\alpha(G') \geq |V(G')|$ for each induced subgraph G' of G.*

Proof. Necessity is easy, since if G is perfect, then $\omega(G') = \chi(G')$ for each induced subgraph G' of G, and since $\chi(G')\alpha(G') \geq |V(G')|$ for any graph G' (as $V(G')$ can be covered by $\chi(G')$ stable sets).

To see sufficiency, suppose to the contrary that there exists an imperfect graph G satisfying the condition, and choose such a graph with $|V(G)|$ minimal. So $\chi(G) > \omega(G)$, while $\chi(G') = \omega(G')$ for each induced subgraph $G' \neq G$ of G.

Let $\omega := \omega(G)$ and $\alpha := \alpha(G)$. We can assume that $V(G) = \{1, \ldots, n\}$.

We first show:

(15) there exist stable sets $C_0, \ldots, C_{\alpha\omega}$ such that each vertex is covered by exactly α of the C_i.

Let C_0 be any stable set in G of size α. By the minimality of G, we know that for each $v \in C_0$, the subgraph of G induced by $V(G) \setminus \{v\}$ is perfect, and that hence its colouring number is at most ω (as its clique number is at most ω, as it is a subgraph of G); therefore $V(G) \setminus \{v\}$ can be partitioned into ω stable sets. Doing this for each $v \in C_0$, we obtain stable sets as in (15).

Now for each $i = 0, \ldots, \alpha\omega$, there exists a clique K_i of size ω with $K_i \cap C_i = \emptyset$. Otherwise, the subgraph G' of G induced by $V(G) \setminus C_i$ would have $\omega(G') < \omega$, and hence it has colouring number at most $\omega - 1$. Adding C_i as a colour would give an ω-vertex colouring of G, contradicting the assumption that $\chi(G) > \omega(G)$.

Then, if $i \neq j$ with $0 \leq i, j \leq \alpha\omega$, we have $|K_j \cap C_i| = 1$. This follows from the fact that K_j has size ω and intersects each C_i in at most one vertex, and hence, by (15), it intersects $\alpha\omega$ of the C_i. As $K_j \cap C_j = \emptyset$, we have that $|K_j \cap C_i| = 1$ if $i \neq j$.

Now consider the $(\alpha\omega + 1) \times n$ incidence matrices $M = (m_{i,j})$ and $N = (n_{i,j})$ of $C_0, \ldots, C_{\alpha\omega}$ and $K_0, \ldots, K_{\alpha\omega}$ respectively. So M and N are $0,1$ matrices, with $m_{i,j} = 1 \Leftrightarrow j \in C_i$, and $n_{i,j} = 1 \Leftrightarrow j \in K_i$, for $i = 0, \ldots, \alpha\omega$ and $j = 1, \ldots, n$. By the above, $MN^{\mathsf{T}} = J - I$, where J is the $(\alpha\omega+1) \times (\alpha\omega+1)$ all-1 matrix, and I the $(\alpha\omega+1) \times (\alpha\omega+1)$ identity matrix. As $J - I$ has rank $\alpha\omega+1$, we have $n \geq \alpha\omega+1$. This contradicts the condition given in the theorem. ∎

This implies:

Corollary 4a ((Lovász's) perfect graph theorem)**.** *The complement of a perfect graph is perfect again.*

Proof. Directly from Theorem 4, as the condition given in it is maintained under taking the complementary graph. ∎

In fact, Berge [4] also made an even stronger conjecture, which was proved by Chudnovsky, Robertson, Seymour, and Thomas [6] (we mentioned this in Section 1 in a different but equivalent form):

Strong perfect graph theorem. A graph G is perfect if and only if G does not contain any odd circuit C_{2k+1} with $k \geq 2$ or its complement as an induced subgraph.

Exercises

5.1. Show that Corollary 4a is implied by the strong perfect graph theorem.

5.2. Give a graph G with $\omega(G) = \chi(G)$ and $\alpha(G) < \overline{\chi}(G)$.

6. Consequences of the perfect graph theorem

We now show how several theorems we have seen before follow as consequences from the perfect graph theorem. First observe that trivially, any bipartite graph G is perfect. This implies Kőnig's edge cover theorem:

Corollary 4b (Kőnig's edge cover theorem)**.** *The complement of a bipartite graph is perfect. Equivalently, the edge cover number of any bipartite graph (without isolated vertices) is equal to its stable set number.*

Proof. Directly from the perfect graph theorem. Note that if G is a bipartite graph, then its cliques have size at most 2; hence $\chi(\overline{G})$ is equal to the edge cover number of G if G has no isolated vertices.

Note moreover that the class of complements of bipartite graphs is closed under taking induced subgraphs. Hence the second statement in the Corollary indeed is equivalent to the first. ∎

We saw that, by Gallai's theorem, Kőnig's edge cover theorem directly implies Kőnig's matching theorem, saying that the matching number of a bipartite graph G is equal to its vertex cover number. That is, $\alpha(L(G)) = \overline{\chi}(L(G))$. So $\omega(\overline{L(G)}) = \chi(\overline{L(G)})$. As this is true for any induced subgraph of $L(G)$ we know that $\overline{L(G)}$ is perfect, for any bipartite graph G. Hence with the perfect graph theorem we obtain Kőnig's edge-colouring theorem (Theorem 1):

Corollary 4c (Kőnig's edge-colouring theorem)**.** *The line graph $L(G)$ of a bipartite graph G is perfect. Equivalently, the edge-colouring number of any bipartite graph is equal to its maximum degree.*

Proof. Again directly from Kőnig's matching theorem and the perfect graph theorem. ∎

We can also derive Dilworth's decomposition theorem (Theorem 3) easily from the perfect graph theorem. Let $(V, \leq)$ be a partially ordered set. Let $G = (V, E)$ be the graph with:

(16) $uv \in E$ if and only if $u < v$ or $v < u$.

Any graph G obtained in this way is called a *comparability graph*.

As Theorem 2 we saw the following easy 'dual' form of Dilworth's decomposition theorem:

Theorem 5. *In any partially ordered set $(V, \leq)$, the maximum size of any chain is equal to the minimum number of antichains needed to cover V.*

Proof. See Theorem 2. ∎

Equivalently, we have $\omega(G) = \chi(G)$ for any comparability graph. As the class of comparability graphs is closed under taking induced subgraphs we have:

Corollary 5a. *Any comparability graph is perfect.*

Proof. This is equivalent to Theorem 5. ∎

So by the perfect graph theorem:

Corollary 5b. *The complement of any comparability graph is perfect.*

Proof. Directly from Corollary 5a and the perfect graph theorem (Corollary 4a). ∎

That is:

Corollary 5c (Dilworth's decomposition theorem)**.** *In any partially ordered set* $(V, \leq)$*, the maximum size of any antichain is equal to the minimum number of chains needed to cover* V*.*

Proof. This is equivalent to Corollary 5b. ∎

A further application of the perfect graph theorem is to 'chordal graphs', which we describe in the next section.

We note here that it was shown with the help of the 'ellipsoid method' that there exists a polynomial-time algorithm for finding a maximum-size clique and a minimum vertex-colouring in any perfect graph ([9]) — see Corollary 8a. However no *combinatorial* polynomial-time algorithm is known for these problems.

7. Chordal graphs

We finally consider a further class of perfect graphs, the 'chordal graphs' (or 'rigid circuit graphs' or 'triangulated graphs'). We first consider collections of subtrees of a tree:

Theorem 6. *Let* $\mathcal{S}$ *be a collection of nonempty subtrees of a tree* T*. Then the maximum number of disjoint trees in* $\mathcal{S}$ *is equal to the minimum number of vertices of* T *intersecting each tree in* $\mathcal{S}$*.*

Proof. The maximum cannot be more than the minimum, since each subtree contains at least one of the vertices chosen.

The reverse inequality is shown by induction on $|V(T)|$, the case $|V(T)| = 1$ being trivial. If $|V(T)| \geq 2$, choose a vertex v of T of degree 1.

Case 1: $\{v\}$ *belongs (as subtree) to* $\mathcal{S}$. Let $T' := T - v$ and $\mathcal{S}' := \{S \in \mathcal{S} \mid v \notin V(S)\}$. By induction, there exist k vertices $v_1, \ldots, v_k \in V(T')$ intersecting all trees in $\mathcal{S}'$ and k disjoint subtrees $S_1, \ldots, S_k$ in $\mathcal{S}'$. Then $v, v_1, \ldots, v_k$ intersect all trees in $\mathcal{S}$ and $\{v\}, S_1, \ldots, S_k$ are pairwise disjoint.

Case 2: each tree in $\mathcal{S}$ *containing* v *also contains the neighbour* u *of* v. Let $T' := T - v$ and $\mathcal{S}' := \{S - v \mid S \in \mathcal{S}\}$. By induction, there exists k vertices $v_1, \ldots, v_k \in V(T')$ intersecting all trees in $\mathcal{S}'$ and k disjoint subtrees $S_1 - v, \ldots, S_k - v$ in $\mathcal{S}'$. Then $S_1, \ldots, S_k$

are pairwise disjoint, since if S_i and S_j intersect, they must intersect in v (since $S_i - v$ and $S_j - v$ are disjoint). But then S_i and S_j also contain u, contradicting the fact that $S_i - v$ and $S_j - v$ are disjoint. ∎

We note one important special case:

Corollary 6a. *Let $\mathcal{C}$ be a collection of pairwise intersecting subtrees of a tree T. Then there is a vertex of T contained in all subtrees in $\mathcal{C}$.*

Proof. The maximum number of pairwise disjoint subtrees in $\mathcal{C}$ is 1. Hence by Theorem 6, there is a vertex intersecting all trees in $\mathcal{C}$. ∎

The *intersection graph* of $\mathcal{S}$ is the graph with vertex set $\mathcal{S}$, where two vertices S, S' are adjacent if and only if they intersect (in at least one vertex). A graph G is called *chordal* if G is isomorphic to the intersection graph of some collection of subtrees of a tree.

Theorem 6 implies:

Corollary 6b. *Complements of chordal graphs are perfect.*

Proof. Let G be a chordal graph. So G is the intersection graph of some collection $\mathcal{S}$ of subtrees of some tree T. Let k be the maximum number of disjoint subtrees in $\mathcal{S}$. So $\alpha(G) = k$. So by Theorem 6 there are k vertices $v_1, \ldots, v_k$ of $V(T)$ intersecting all trees in $\mathcal{S}$. Now the set of subtrees in $\mathcal{S}$ containing v_i form a clique in G. So G can be covered by k cliques. Hence $\overline{\chi}(G) \leq k = \alpha(G)$. Therefore, $\alpha(G) = \overline{\chi}(G)$.

Now each induced subgraph of G is again chordal (as it is the intersection graph of a subcollection of $\mathcal{S}$). So also for induced subgraph G' we know $\alpha(G') = \overline{\chi}(G')$. So $\overline{G}$ is perfect. ∎

This implies, by the perfect graph theorem, that also chordal graphs themselves are perfect. This gives in terms of trees:

Corollary 6c. *Let $\mathcal{S}$ be a collection of subtrees of a tree T. Let k be the maximum number of times that any vertex of T is covered by trees in $\mathcal{S}$. Then $\mathcal{S}$ can be partitioned into classes $\mathcal{S}_1, \ldots, \mathcal{S}_k$ such that each $\mathcal{S}_i$ consists of disjoint trees.*

Proof. Let G be the intersection graph of $\mathcal{S}$. As G is chordal, it is perfect. So $\omega(G) = \chi(G)$. Now $\omega(G)$ is equal to the number k in the Corollary, by Corollary 6a. So $\chi(G) = k$, which implies that $\mathcal{S}$ can be partitioned as described. ∎

We note here that chordal graphs are characterized as those graphs in which each circuit of length at least 4 has a chord.[7] This is actually the original definition of chordal graphs and explains the name.

Exercises

7.1. Show that a graph $G = (V, E)$ is chordal if and only if each induced subgraph has a vertex whose neighbours form a clique.

[7] A *chord* of a circuit C is an edge e not of C connecting two vertices of C.

7.2. Derive Exercise 4.2 from Corollary 6c.

8. Lovász' ϑ-function

Lovász [16] introduced a very useful upper bound $\vartheta(G)$ on $\alpha(G)$, for any graph $G = (V, E)$. First define

(17) $\mathcal{L}_G :=$ the set of symmetric $V \times V$ matrices A with $A_{u,v} = 1$ if $u = v$ or u and v are nonadjacent,

and for any symmetric matrix A:

(18) $\Lambda(A) :=$ the largest eigenvalue of A.

Then

(19) $$\vartheta(G) := \min\{\Lambda(A) \mid A \in \mathcal{L}_G\}.$$

$\vartheta(G)$ has two important properties: it can be calculated (at least, approximated) in polynomial time, and it gives an, often close, upper bound on the stable set number $\alpha(G)$ (Lovász [16]), better than $\overline{\chi}(G)$:

Theorem 7. *For any graph $G = (V, E)$:*

(20) $$\alpha(G) \leq \vartheta(G) \leq \overline{\chi}(G).$$

Proof. To see $\alpha(G) \leq \vartheta(G)$, let S be a maximum-size stable set of G and let $A \in \mathcal{L}_G$. Let ι^S denote the incidence vector of S in $\mathbb{R}^V$.[8] Then

(21) $$|S|^2 = {\iota^S}^{\mathsf{T}} A \iota^S \leq \Lambda(A)\|\iota^S\|^2 = \Lambda(A)|S|.$$

So $\alpha(G) = |S| \leq \Lambda(A)$.

To see $\vartheta(G) \leq \overline{\chi}(G)$, consider a partition of V into cliques $C_1, \ldots, C_k$ with $k = \overline{\chi}(G)$. Define the matrix

(22) $$A := kI - \frac{1}{k}\sum_{i=1}^{k}(k\iota^{C_i} - \mathbf{1})(k\iota^{C_i} - \mathbf{1})^{\mathsf{T}}.$$

Then $\Lambda(A) \leq k$, since $kI - A$ is positive semidefinite[9]. If $v \in V$, then

[8] The *incidence vector* ι^S of $S \subseteq V$ is the vector in $\{0,1\}^V$ with $(\iota^S)_v = 1$ if and only if $v \in S$.

[9] A symmetric $n \times n$ matrix M is *positive semidefinite* if $x^{\mathsf{T}} M x \geq 0$ for each $x \in \mathbb{R}^n$. This can be proved to be equivalent to: all eigenvalues of M are nonnegative. Moreover, it is equivalent to: $M = U^{\mathsf{T}} U$ for some matrix U. For any symmetric matrix A and $t \in \mathbb{R}$: $\Lambda(A) \leq t \iff tI - A$ is positive semidefinite.

$$(23) \qquad (k\iota^{C_i} - \mathbf{1})_v = \begin{cases} k-1 & \text{if } v \in C_i, \\ -1 & \text{if } v \notin C_i. \end{cases}$$

Then a direct calculation shows that $A \in \mathcal{L}_G$. So $\vartheta(G) \le k$. ∎

Moreover, $\vartheta(G)$ can be approximated in polynomial time:

(24) there is an algorithm that for any given graph $G = (V, E)$ and any $\varepsilon > 0$, returns a rational closer than ε to $\vartheta(G)$, in time bounded by a polynomial in $|V|$ and $\log(1/\varepsilon)$.

The two theorems above imply:

Theorem 8. *For any graph G satisfying $\alpha(G) = \overline{\chi}(G)$, the stable set number can be found in polynomial time.*

Proof. Theorem 7 implies $\alpha(G) = \vartheta(G) = \overline{\chi}(G)$, and by (24) we can find a number closer than $\frac{1}{2}$ to $\vartheta(G)$ in time polynomial in $|V|$. Rounding to the closest integer yields $\alpha(G)$. ∎

To obtain an explicit maximum-size stable set, we need perfection of the graph:

Corollary 8a. *A maximum-size stable set in a perfect graph can be found in polynomial time.*

Proof. Let $G = (V, E)$ be a perfect graph. Iteratively, for each $v \in V$, replace G by $G - v$ if $\alpha(G - v) = \alpha(G)$. By the perfection of G, we can calculate these values in polynomial time, by Theorem 8.

We end up with a graph that forms a maximum-size stable set in the original graph. ∎

As perfection is closed under taking complements, also a maximum-size clique in a perfect graph can be found in polynomial time. It can also be shown that a minimum colouring of a perfect graph can be found in polynomial time.

9. The Shannon capacity $\Theta(G)$

Shannon [19] introduced the following parameter $\Theta(G)$, now called the Shannon capacity of a graph G.

The *strong product* $G \cdot H$ of graphs G and H is the graph with vertex set $VG \times VH$, with two distinct vertices (u, v) and (u', v') adjacent if and only if u and u' are equal or adjacent in G and v and v' are equal or adjacent in H.

The strong product of k copies of G is denoted by G^k. Then the *Shannon capacity* $\Theta(G)$ of G is defined by:

$$(25) \qquad \Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)}.$$

(The interpretation is that if V is an alphabet, and adjacency means 'confusable', then $\alpha(G^k)$ is the maximum number of k-letter words any two of which have unequal and inconfusable letters in at least one position. Then $\Theta(G)$ is the maximum possible 'information rate'.)

Guo and Watanabe [10] showed that there exist graphs G for which $\Theta(G)$ is not achieved by a finite product (that is, $\sqrt[k]{\alpha(G^k)} < \Theta(G)$ for each k — so the supremum is not a maximum).

Since $\alpha(G^k) \geq \alpha(G)^k$, we have

$$\alpha(G) \leq \Theta(G), \tag{26}$$

while strict inequality may hold: the 5-circuit C_5 has $\alpha(C_5) = 2$ and $\alpha(C_5^2) = 5$. (If C_5 has vertices $1, \ldots, 5$ and edges 12, 23, 34, 45, 51, then $\{(1,1),(2,3),(3,5),(4,2),(5,4)\}$ is a stable set in C_5^2.) So $\Theta(C_5) \geq \sqrt{5}$, and Shannon [19] raised the question if equality holds here. This was proved by Lovász [16] by showing:

Theorem 9. $\Theta(G) \leq \vartheta(G)$ *for each graph* G.

Proof. Since $\alpha(G) \leq \vartheta(G)$, it suffices to show that for each k: $\alpha(G^k) \leq \vartheta(G)^k$. For this it suffices to show that

$$\vartheta(G \cdot H) \leq \vartheta(G)\vartheta(H) \tag{27}$$

for any graphs G and H, as $\alpha(G^k) \leq \vartheta(G^k) \leq \vartheta(G)^k$.

By definition of ϑ, there exist matrices $A \in \mathcal{L}_G$ and $B \in \mathcal{L}_H$ such that $\vartheta(G) = \Lambda(A)$ and $\vartheta(H) = \Lambda(B)$. So $\vartheta(G)I_{V(G)} - A$ and $\vartheta(H)I_{V(H)} - B$ are positive semidefinite. (Here $I_{V(G)}$ and $I_{V(H)}$ denote the $V(G) \times V(G)$ and $V(H) \times V(H)$ identity matrices.) Since positive semidefinitess is maintained by tensor product, the matrices

$$\begin{aligned} C_1 &:= (\vartheta(G)I_{V(G)} - A) \otimes (\vartheta(H)I_{V(H)} - B), \\ C_2 &:= (\vartheta(G)I_{V(G)} - A) \otimes J_{V(H)}, \text{ and} \\ C_3 &:= J_{V(G)} \otimes (\vartheta(H)I_{V(H)} - B) \end{aligned} \tag{28}$$

are positive semidefinite. (Here $J_{V(G)}$ and $J_{V(H)}$ denote the $V(G)\times V(G)$ and $V(H)\times V(H)$ all-1 matrices. Note that these matrices are positive semidefinite.) So

$$C := \vartheta(G)\vartheta(H)I_{V(G)\times V(H)} - C_1 - C_2 - C_3 \tag{29}$$

has largest eigenvalue $\Lambda(C) \leq \vartheta(G)\vartheta(H)$. Moreover, C belongs to $\mathcal{L}_{G\cdot H}$, as is direct to check. So $\vartheta(G \cdot H) \leq \Lambda(C) \leq \vartheta(G)\vartheta(H)$. ∎

One consequence of Theorem 9 is that if $\alpha(G) = \overline{\chi}(G)$, then $\Theta(G) = \alpha(G)$ (since $\alpha(G) \leq \Theta(G) \leq \vartheta(G) \leq \overline{\chi}(G)$).

Theorem 9 moreover implies that $\Theta(C_5) = \sqrt{5}$. To see this, let $\alpha := \frac{1}{2}\sqrt{5} - \frac{3}{2} = 2\cos\frac{\pi}{5} - 2$. It can be shown that the matrix

$$A := \begin{pmatrix} 1 & \alpha & 1 & 1 & \alpha \\ \alpha & 1 & \alpha & 1 & 1 \\ 1 & \alpha & 1 & \alpha & 1 \\ 1 & 1 & \alpha & 1 & \alpha \\ \alpha & 1 & 1 & \alpha & 1 \end{pmatrix} \tag{30}$$

has largest eigenvalue $\Lambda(A) = \sqrt{5}$.[10] So $\vartheta(C_5) \leq \sqrt{5}$. As $\Theta(C_5) \geq \sqrt{\alpha(C_5^2)} = \sqrt{5}$, one has

$$\sqrt{5} \leq \Theta(C_5) \leq \vartheta(C_5) \leq \sqrt{5}, \tag{31}$$

and we have equality throughout. So $\Theta(C_5) = \sqrt{5}$.

Lovász [16] also gave the value of $\vartheta(C_n)$ for any odd circuit C_n:

$$\vartheta(C_n) = \frac{n \cos(\pi/n)}{1 + \cos(\pi/n)} \text{ for odd } n. \tag{32}$$

For odd $n \geq 7$, it is unknown if this is the value of $\Theta(C_n)$.

Lovász asked if $\Theta(G) = \vartheta(G)$ for each graph G. This was answered in the negative by Haemers [12], by giving an alternative upper bound on the Shannon capacity which for some graphs G is sharper than $\vartheta(G)$.

The proof of Theorem 9 consists of showing $\vartheta(G \cdot H) \leq \vartheta(G)\vartheta(H)$ for any two graphs G and H. In fact, Lovász [16] showed that equality holds: $\vartheta(G \cdot H) = \vartheta(G)\vartheta(H)$.

10. Vizing's theorem

Kőnig's edge-coloring theorem (Theorem 1) states that $\chi'(G) = \Delta(G)$ for any bipartite graph G. For nonbipartite graph G, strict inequality can hold, as K_3 shows. But Vizing [20,21] showed that if G is simple, $\chi'(G)$ can be at most 1 larger than $\Delta(G)$.

Theorem 10 (Vizing's theorem for simple graphs)**.** $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ *for any simple graph* G*.*

Proof. The inequality $\Delta(G) \leq \chi'(G)$ being trivial, we show $\chi'(G) \leq \Delta(G) + 1$. To prove this inductively, it suffices to show:

(33) Let $G = (V, E)$ be a simple graph and let v be a vertex such that v and all its neighbours have degree at most $k - 1$. Then if $G - v$ is k-edge-colourable, also G is k-edge-colourable.

We prove (33) by induction on k. We can assume that each vertex u in $N(v)$ has degree exactly $k-1$, since otherwise we can add a new vertex w and an edge uw without violating the conditions in (33).

[10] The following are eigenvectors: $(1, e^{2k\pi i/5}, e^{4k\pi i/5}, e^{6k\pi i/5}, e^{8k\pi i/5})^{\mathsf{T}}$, for $k = 0, \ldots, 4$.

Consider any k-edge-colouring of $G-v$. For $i=1,\ldots,k$, let X_i be the set of neighbours of v that are missed by colour i. We assume that we have chosen the colouring such that the number of i with $|X_i|=0$ is minimized.

We can assume that for each $i=1,\ldots,k$,

(34) $\qquad |X_i| \neq 1.$

To see this, assume $|X_k|=1$, say $X_k := \{u\}$. Let G' be the graph obtained from G by deleting edge vu and deleting all edges of colour k. So $G'-v$ is $(k-1)$-edge-coloured. Moreover, in G', vertex v and all its neighbours have degree at most $k-2$. So by the induction hypothesis, G' is $(k-1)$-edge-colourable. Restoring colour k, and giving edge vu colour k, gives a k-edge-colouring of G. So we can assume (34).

Now each $u \in N(v)$ is in precisely two of the X_i. Hence $\sum_{i=1}^{k}|X_i| = 2|N(v)| = 2\deg(v) < 2k$, and so $|X_i|=0$ for some i, say $|X_1|=0$. This implies

(35) $\qquad |X_i| \in \{0,2\}$ for each i.

For if, say, $|X_2|\geq 3$, consider the subgraph H made by all edges of colours 1 and 2. Consider a component P of H containing a vertex in X_2. As $X_1=\emptyset$, P is a path starting with an edge of colour 1. Exchanging colours 1 and 2 on P reduces the number of i with $|X_i|=0$, contradicting our minimality assumption. This proves (35).

So the nonempty X_i form a 2-regular graph on $N(v)$. Hence there is a one-to-one function $f: N(v)\to\{1,\ldots,k\}$ such that $u \in X_{f(u)}$ for each $u \in N(v)$. Hence giving edge vu colour $f(u)$ for each $u \in N(v)$, yields a proper k-edge-colouring of G. ∎

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