

A Pythagoras proof of Szemerédi’s regularity lemma

Notes for our seminar — Alexander Schrijver¹

Abstract. We give a short proof of Szemerédi’s regularity lemma, based on elementary geometry.

The ‘regularity lemma’ of Endre Szemerédi [1] roughly asserts that, for each $\varepsilon > 0$, there exists a number k such that the vertex set V of any graph $G = (V, E)$ can be partitioned into at most k *almost* equal-sized classes so that between *almost* any two classes, the edges are distributed *almost* homogeneously. Here *almost* depends on ε . The important issue is that k (though generally extremely huge) only depends on ε , and not on the size of the graph. The lemma has several applications in graph and number theory, discrete geometry, and theoretical computer science.

We give a short proof based on elementary Euclidean geometry. The general line of the proof is like that of the standard proof (in fact, Szemerédi’s original proof), but most of the technicalities are swallowed by Pythagoras’ theorem. We prove two lemmas, one on ‘ ε -balanced’ partitions, the other on ‘ ε -regular’ partitions.

Let V be a finite set. A *partition* of V is a collection of disjoint nonempty sets (called *classes*) with union V . Partition Q of V is a *refinement* of partition P if each class of Q is contained in some class of P . For $\varepsilon > 0$, partition P of V is called ε -*balanced* if P contains a subcollection C such that all sets in C have the same size and such that $|V \setminus \bigcup C| \leq \varepsilon|V|$.

Lemma 1. *Each partition P of V has an ε -balanced refinement Q with $|Q| \leq (1 + \varepsilon^{-1})|P|$.*

Proof. Define $t := \varepsilon|V|/|P|$. Split each class of P into classes, each of size $\lceil t \rceil$, except for at most one of size less than t . This gives Q . Then $|Q| \leq |P| + |V|/t = (1 + \varepsilon^{-1})|P|$. Also, the union of the classes of Q of size less than t has size at most $|P|t = \varepsilon|V|$. So Q is ε -balanced. ■

Let $G = (V, E)$ be a graph. For nonempty $I, J \subseteq V$, the *density* $d(I, J)$ of (I, J) is the number of adjacent pairs of vertices in $I \times J$, divided by $|I \times J|$. Call the pair (I, J) ε -*regular* if for all $X \subseteq I, Y \subseteq J$:

$$(1) \quad \text{if } |X| > \varepsilon|I| \text{ and } |Y| > \varepsilon|J| \text{ then } |d(X, Y) - d(I, J)| \leq \varepsilon.$$

A partition P of V is called ε -*regular* if

$$(2) \quad \sum_{\substack{I, J \in P \\ (I, J) \text{ } \varepsilon\text{-irregular}}} |I||J| \leq \varepsilon|V|^2.$$

For Lemma 2 we need the following. Consider the matrix space $\mathbb{R}^{V \times V}$, with the Frobenius norm $\|M\| = \text{Tr}(M^T M)^{1/2}$ for $M \in \mathbb{R}^{V \times V}$. For nonempty $I, J \subseteq V$, let $L_{I, J}$ be the 1-dimensional subspace of $\mathbb{R}^{V \times V}$ consisting of all matrices that are constant on $I \times J$ and 0 outside $I \times J$. For any $M \in \mathbb{R}^{V \times V}$, let $M_{I, J}$ be the orthogonal projection of M onto $L_{I, J}$. So the entries of $M_{I, J}$ on $I \times J$ are all equal to the average value of M on $I \times J$.

¹ CWI and University of Amsterdam. Mailing address: CWI, Science Park 123, 1098 XG Amsterdam, The Netherlands. Email: lex@cwi.nl.

If P is a partition of V , let L_P be the sum of the spaces $L_{I,J}$ with $I, J \in P$, and let M_P be the orthogonal projection of M onto L_P . So $M_P = \sum_{I,J \in P} M_{I,J}$. Note that if Q is a refinement of P , then $L_P \subseteq L_Q$, hence $\|M_P\| \leq \|M_Q\|$.

Lemma 2. *Let $\varepsilon > 0$ and $G = (V, E)$ be a graph, with adjacency matrix A . Then each ε -irregular partition P has a refinement Q with $|Q| \leq |P|4^{|P|}$ and $\|A_Q\|^2 > \|A_P\|^2 + \varepsilon^5|V|^2$.*

Proof. Let $(I_1, J_1), \dots, (I_n, J_n)$ be the ε -irregular pairs in P^2 . For each $i = 1, \dots, n$, we can choose (by definition (1)) subsets $X_i \subseteq I_i$ and $Y_i \subseteq J_i$ with $|X_i| > \varepsilon|I_i|$, $|Y_i| > \varepsilon|J_i|$ and $|d(X_i, Y_i) - d(I_i, J_i)| > \varepsilon$. For any fixed $K \in P$, there exists a partition Q_K of K such that each X_i with $I_i = K$ and each Y_i with $J_i = K$ is a union of classes of Q_K and such that $|Q_K| \leq 2^{2|P|} = 4^{|P|}$.² Let $Q := \bigcup_{K \in P} Q_K$. Then Q is a refinement of P such that each X_i and each Y_i is a union of classes of Q . Moreover, $|Q| \leq |P|4^{|P|}$.

Now note that for each i , since $(A_Q)_{X_i, Y_i} = A_{X_i, Y_i}$ (as $L_{X_i, Y_i} \subseteq L_Q$) and since A_{X_i, Y_i} and A_P are constant on $X_i \times Y_i$, with values $d(X_i, Y_i)$ and $d(I_i, J_i)$, respectively:

$$(3) \quad \|(A_Q - A_P)_{X_i, Y_i}\|^2 = \|A_{X_i, Y_i} - (A_P)_{X_i, Y_i}\|^2 = |X_i||Y_i|(d(X_i, Y_i) - d(I_i, J_i))^2 > \varepsilon^4|I_i||J_i|.$$

Then negating (2) gives with Pythagoras, as A_P is orthogonal to $A_Q - A_P$ (as $L_P \subseteq L_Q$), and as the spaces L_{X_i, Y_i} are pairwise orthogonal,

$$(4) \quad \|A_Q\|^2 - \|A_P\|^2 = \|A_Q - A_P\|^2 \geq \sum_{i=1}^n \|(A_Q - A_P)_{X_i, Y_i}\|^2 \geq \sum_{i=1}^n \varepsilon^4|I_i||J_i| > \varepsilon^5|V|^2. \quad \blacksquare$$

Define $f_\varepsilon(x) := (1 + \varepsilon^{-1})x4^x$ for $\varepsilon, x > 0$. For $n \in \mathbb{N}$, f_ε^n denotes the n -th iterate of f_ε .

Szemerédi's regularity lemma. For each $\varepsilon > 0$ and graph $G = (V, E)$, each partition P of V has an ε -balanced ε -regular refinement of size $\leq f_\varepsilon^{\lfloor \varepsilon^{-5} \rfloor}((1 + \varepsilon^{-1})|P|)$.

Proof. Let A be the adjacency matrix of G . Starting with P , iteratively apply Lemmas 1 and 2 alternately. At each application of Lemma 1, $\|A_P\|^2$ does not decrease, and at each application of Lemma 2, $\|A_P\|^2$ increases by more than $\varepsilon^5|V|^2$. Now, for any partition Q of V , $\|A_Q\|^2 \leq \|A\|^2 \leq |V|^2$. Hence, after at most $\lfloor \varepsilon^{-5} \rfloor$ iterations we must have an ε -balanced ε -regular partition as required. \blacksquare

We note that if P is an ε -balanced ε -regular partition of V , and $C \subseteq P$ is such that all sets in C have the same size and such that $|V \setminus \bigcup C| \leq \varepsilon|V|$, then the number s of ε -irregular pairs in C^2 is at most $\varepsilon(1 - \varepsilon)^{-2}|C|^2$. For let t be the common size of the sets in C . Then, by (2), $st^2 \leq \varepsilon|V|^2 \leq \varepsilon(1 - \varepsilon)^{-2}|\bigcup C|^2 = \varepsilon(1 - \varepsilon)^{-2}(t|C|)^2 = \varepsilon(1 - \varepsilon)^{-2}|C|^2t^2$.

Reference

- [1] E. Szemerédi, Regular partitions of graphs, in: *Problèmes combinatoires et théorie des graphes* (Proceedings Colloque International C.N.R.S., Paris-Orsay, 1976) [Colloques Internationaux du C.N.R.S. N° 260], Éditions du C.N.R.S., Paris, 1978, pp. 399–401.

²For any collection C of subsets of a finite set S , there is a partition R of S such that any set in C is a union of classes of R and such that $|R| \leq 2^{|C|}$: take $R := \{\bigcap_{X \in D} X \cap \bigcap_{Y \in C \setminus D} S \setminus Y \mid D \subseteq C\} \setminus \{\emptyset\}$.