Positive Semidefinite Matrix Completion, Universal Rigidity and the Strong Arnold Property

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Abstract

This paper addresses the following three topics: positive semidefinite (psd) matrix completions, universal rigidity of frameworks, and the Strong Arnold Property (SAP). We show some strong connections among these topics, using semidefinite programming as unifying theme. Our main contribution is a sufficient condition for constructing partial psd matrices which admit a unique completion to a full psd matrix. Such partial matrices are an essential tool in the study of the Gram dimension $\mathrm{gd}(G)$ of a graph G, a recently studied graph parameter related to the low psd matrix completion problem. Additionally, we derive an elementary proof of Connelly's sufficient condition for universal rigidity of tensegrity frameworks and we investigate the links between these two sufficient conditions. We also give a geometric characterization of psd matrices satisfying the Strong Arnold Property in terms of nondegeneracy of an associated semidefinite program, which we use to establish some links between the Gram dimension $\mathrm{gd}(\cdot)$ and the Colin de Verdière type graph parameter $\nu^{=}(\cdot)$.

Keywords: Matrix completion, tensegrity framework, universal rigidity, semidefinite programming, Strong Arnold Property, nondegeneracy

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1. Introduction

The main motivation for this paper is the positive semidefinite (psd) matrix completion problem, defined as follows: Given a graph $G = (V = [n], E)$ and a vector $a \in \mathbb{R}^{E \cup V}$ indexed by the nodes and the edges of G, decide whether there exists a real symmetric $n \times n$ matrix X satisfying

$$X_{ij} = a_{ij} \text{ for all } \{i, j\} \in V \cup E, \text{ and } X \text{ is positive semidefinite.} \quad (1)$$

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Throughout the paper we identify V with the set of diagonal pairs $\{i, i\}$ for $i \in V$. Any vector $a \in \mathbb{R}^{V \cup E}$ can be viewed as a partial symmetric matrix whose entries are determined only at the diagonal positions (corresponding to the nodes) and at the off-diagonal positions corresponding to the edges of G. A vector $a \in \mathbb{R}^{V \cup E}$ is called a *G-partial psd matrix* when (1) is feasible, i.e., when the partial matrix a admits at least one completion to a full psd matrix.

The psd matrix completion problem is an instance of the semidefinite programming feasibility problem, and as such its complexity is still unknown [33]. A successful line of attack embraced in the literature has been to identify graph classes for which some of the handful of known necessary conditions that guarantee that a G-partial matrix is completable are also sufficient (see e.g. [5, 17, 22]).

In this paper we develop a systematic method for constructing partial psd matrices with the property that they admit a *unique* completion to a fully specified psd matrix. Such partial matrices are a crucial ingredient for the study of two new graph parameters considered in [14, 24, 25], defined in terms of ranks of psd matrix completions of G-partial matrices. The first one is the Gram dimension $\mathrm{gd}(\cdot)$ which we will introduce in Section 5.2 and whose study is motivated by the low rank psd matrix completion problem [24, 25]. The second one is the extreme Gram dimension $\mathrm{egd}(\cdot)$ whose study is motivated by its relevance to the bounded rank Grothendieck constant of a graph [14]. Several instances of partial matrices with a unique psd completion were constructed in [14, 24, 25], but the proofs were mainly by direct case checking. In this paper we give a sufficient condition for constructing partial psd matrices with a unique psd completion (Theorem 3.2) and using this condition we can recover most examples of [14, 24, 25] (see Section 3.3).

The condition for uniqueness of a psd completion suggests a connection to the theory of universally rigid frameworks. A *framework* $G(\mathbf{p})$ consists of a graph $G = (V = [n], E)$ together with an assignment of vectors $\mathbf{p} = \{p_1, \dots, p_n\}$ to the nodes of the graph. The framework $G(\mathbf{p})$ is said to be *universally rigid* if it is the only framework having the same edge lengths in any space, up to congruence. A related concept is that of global rigidity of frameworks. A framework $G(\mathbf{p})$ in $\mathbb{R}^d$ is called *globally rigid in* $\mathbb{R}^d$ if up to congruence it is the only framework in $\mathbb{R}^d$ having the same edge lengths. Both concepts have been extensively studied and there exists an abundant literature about them (see e.g. [9, 10, 11, 12, 16] and references therein).

The analogue of the notion of global rigidity, in the case when Euclidean distances are replaced by inner products, was recently investigated in [35]. There it is shown that many of the results that are valid in the setting of Euclidean distances can be adapted to the so-called 'spherical setting'. The latter terminology refers to the fact that when the vectors $p_1, \dots, p_n \in \mathbb{R}^d$ are restricted to lie on the unit sphere then their pairwise inner products lead to the study of the spherical metric space, where the distance between two points p_i, p_j is given by $\arccos(p_i^{\mathsf{T}} p_j)$, i.e., the angle formed between the two vectors [34]. Taking this analogy further, our sufficient condition for constructing partial psd matrices with a unique psd completion can be interpreted as the analogue in the spherical setting of Connelly's celebrated sufficient condition for universal rigidity of frameworks (see the respective results from Theorem 3.2 and Theorem 4.4).

The unifying theme of this paper is semidefinite programming (SDP). In particular, the notions of SDP nondegeneracy and strict complementarity play a crucial role in this paper. This should come as no surprise as there are already well established links between semidefinite programming and universal rigidity [3] and psd matrix completion with SDP nondegeneracy [31]. To arrive at our results we develop a number of tools that build upon fundamental results guaranteeing the uniqueness of optimal solutions to SDP's.

Using this machinery we can also give new proofs of some known results, most notably a short and elementary proof of Connelly's sufficient condition for universal rigidity (Theorem 4.4). With the intention to make Section 4 a self contained treatment of universal rigidity we also address the case of generic universally rigid frameworks (Section 4.2). Lastly, we investigate the relation between our sufficient condition and Connelly's sufficient condition and show that in some special cases they turn out to be equivalent (Section 4.3).

In this paper we also revisit a somewhat elusive matrix property called the *Strong Arnold Property* (SAP) whose study is motivated by the celebrated Colin de Verdière graph parameter $\mu(\cdot)$ introduced in [8]. We present a geometric characterization of matrices fulfilling the SAP by associating them with the extreme points of a certain spectrahedron (Theorem 5.2). Furthermore, we show that psd matrices having the SAP can be understood as nondegenerate solutions of certain SDP's (Theorem 5.3).

Lastly, using our tools we can shed some more light and gain insight on the relation between two graph parameters that have been recently studied in the literature. The first one is the parameter $\nu^=(\cdot)$ of [18, 19], whose study is motivated by its relation to the Colin de Verdiére graph parameter $\mu(\cdot)$. The second one is the *Gram dimension* $\mathrm{gd}(\cdot)$ of a graph, introduced in [24, 25], whose study is motivated by its relation to the low rank psd matrix completion problem. In particular we reformulate $\nu^=(\cdot)$ in terms of the maximum Gram dimension of certain G-partial psd matrices satisfying a nondegeneracy property (Theorem 5.9), which enables us to recover that $\mathrm{gd}(G) \geq \nu^=(G)$ for any graph G (Corollary 5.10).

Contents.

The paper is organized as follows. In Section 2 we group some basic facts about semidefinite programming that we need in the paper. In Section 3 we present our sufficient condition for the existence of unique psd completions (in the general setting of tensegrities, i.e., allowing equalities and inequalities), and we illustrate its use by several examples. In Section 4 we present a simple proof for Connelly's sufficient condition for universally rigid tensegrities (generic and non-generic) and we investigate the links between these two sufficient conditions for the spherical and Euclidean distance settings. Finally in Section 5 we revisit the Strong Arnold Property, we present a geometric characterization of psd matrices having the SAP in terms of nondegeneracy of semidefinite programming, which we use to establish a link between the graph parameters $\mathrm{gd}(\cdot)$ and $\nu^{=}(\cdot)$.

Notation.

Let C be a closed convex set. A convex subset $F \subseteq C$ is called a *face* of C if, for any $x, y \in C$, $\lambda x + (1-\lambda)y \in F$ for some scalar $\lambda \in (0,1)$ implies $x, y \in F$. A point $x \in C$ is called an *extreme point* of C if the set $\{x\}$ is a face of C. A vector z is said to be a *perturbation* of $x \in C$ if $x \pm \epsilon z \in C$ for some $\epsilon > 0$. The set of perturbations of $x \in C$ form a linear space which we denote as $\mathrm{Pert}_C(x)$. Clearly, x is an extreme point of C if and only if $\mathrm{Pert}_C(x) = \{0\}$.

We denote by $e_1, \cdots, e_n \in \mathbb{R}^n$ the standard unit vectors in $\mathbb{R}^n$ and for $1 \leq i \leq j \leq n$, we define the symmetric matrices $E_{ij} = (e_i e_j^{\mathsf{T}} + e_j e_i^{\mathsf{T}})/2$ and $F_{ij} = (e_i - e_j)(e_i - e_j)^{\mathsf{T}}$. Given vectors $p_1, \ldots, p_n \in \mathbb{R}^d$, $\mathrm{lin}\{p_1, \ldots, p_n\}$ denotes their linear span which is a vector subspace of $\mathbb{R}^d$. We also use the shorthand notation $[n] = \{1, \ldots, n\}$.

Throughout $\mathcal{S}^n$ denotes the set of real symmetric $n \times n$ matrices and $\mathcal{S}^n_+$ the subcone of positive semidefinite matrices. For a matrix $X \in \mathcal{S}^n$ its kernel is denoted as $\mathrm{Ker}\, X$ and its range as $\mathrm{Ran}\, X$. The *corank* of a matrix $X \in \mathcal{S}^n$ is the dimension of its kernel. For a matrix $X \in \mathcal{S}^n$, the notation $X \succeq 0$ means that X is positive semidefinite (abbreviated as psd). The space $\mathcal{S}^n$ is equipped with the trace inner product given by $\langle X, Y \rangle = \mathrm{Tr}(XY) = \sum_{i,j=1}^n X_{ij} Y_{ij}$. We will use the following property: For two positive semidefinite matrices $X, Y \in \mathcal{S}^n_+$, $\langle X, Y \rangle \geq 0$, and $\langle X, Y \rangle = 0$ if and only if $XY = 0$.

Given vectors $p_1, \cdots, p_n \in \mathbb{R}^d$, their *Gram matrix* is the $n \times n$ symmetric matrix $\mathrm{Gram}(p_1, \ldots, p_n) = (p_i^{\mathsf{T}} p_j)_{i,j=1}^n$. Clearly, the rank of the Gram matrix $\mathrm{Gram}(p_1, \ldots, p_n)$ is equal to the dimension of the linear span of $\{p_1, \ldots, p_n\}$. Moreover, two systems of vectors $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_n\}$ in $\mathbb{R}^d$ have the same Gram matrix, i.e., $p_i^{\mathsf{T}} p_j = q_i^{\mathsf{T}} q_j$ for all $i, j \in [n]$, if and only if there exists a $d \times d$ orthogonal matrix O such that $q_i = O p_i$ for all $i \in [n]$.

2. Semidefinite programming

In this section we recall some basic facts about semidefinite programming. Our notation and exposition follow [4] (another excellent source is [30]).

A semidefinite program is a convex program defined as the minimization of a linear function over an affine section of the cone of positive semidefinite matrices. In this paper we will consider semidefinite programs of the form:

$$p^* = \sup_X \{\langle C, X \rangle : X \succeq 0,\ \langle A_i, X \rangle = b_i \ (i \in I),\ \langle A_i, X \rangle \leq b_i \ (i \in J)\}. \tag{P}$$

Standard semidefinite programs are usually defined involving only linear equalities; we also allow here linear inequalities, since they will be used to model tensegrity frameworks in Sections 3 and 4. The dual program of (P) reads:

$$d^* = \inf_{y,Z} \left\{ \sum_{i \in I \cup J} b_i y_i \ : \ \sum_{i \in I \cup J} y_i A_i - C = Z \succeq 0,\ y_i \geq 0 \ (i \in J) \right\}. \tag{D}$$

Here, $C \in \mathcal{S}^n, A_i \in \mathcal{S}^n$ $(i \in I \cup J)$ and $b \in \mathbb{R}^{|I|+|J|}$ are given and $I \cap J = \emptyset$.

We denote the primal and dual feasible regions by $\mathcal{P}$ and $\mathcal{D}$, respectively. The primal feasible region

$$\mathcal{P} = \{X \in \mathcal{S}^n : X \succeq 0,\ \langle A_i, X\rangle = b_i\ (i \in I),\ \langle A_i, X\rangle \leq b_i\ (i \in J)\} \tag{2}$$

is a convex set defined as the intersection of the cone of positive semidefinite matrices with an affine subspace and some affine half-spaces. For $J = \emptyset$, such sets are known as *spectrahedra* and, for $J \neq \emptyset$, they are called *semidefinite representable* (i.e., they can be obtained as projections of spectrahedra, by using slack variables). Recently, there has been a surge of interest in the study of semidefinite representable sets since they constitute a rich class of convex sets for which there exist efficient algorithms for optimizing linear functions over them [6].

As is well known (and easy to see), weak duality holds: $p^* \leq d^*$. Moreover, if the dual (resp. primal) is strictly feasible and $d^* > -\infty$ (resp. $p^* < \infty$), then *strong duality* holds: $p^* = d^*$ and the primal (resp. dual) optimum value is attained.

A pair $X, (y, Z)$ of primal and dual optimal solutions are called *complementary* if $XZ = 0$ and *strict complementary* if moreover $\operatorname{rank} X + \operatorname{rank} Z = n$. For a matrix $X \in \mathcal{P}$, let $J_X = \{i \in J : \langle A_i, X\rangle = b_i\}$ denote the set of inequality constraints that are active at X. Similarly, for a matrix $Z \in \mathcal{D}$ set $J_Z = \{i \in J : y_i > 0\}$. Assuming strong duality, a pair of primal and dual feasible solutions $X, (y, Z)$ are both optimal if and only if $\langle X, Z\rangle = 0$ and $J_Z \subseteq J_X$, i.e., if $y_i > 0$ for some $i \in J$ then $\langle A_i, X\rangle = b_i$. We refer to these two conditions as the *complementary slackness* conditions.

The following theorem provides an explicit characterization of the space of perturbations of an element of the primal feasible region $\mathcal{P}$.

Theorem 2.1. *[26, 13] Consider a matrix $X \in \mathcal{P}$, written as $X = PP^{\mathsf{T}}$, where $P \in \mathbb{R}^{n\times r}$ and $r = \operatorname{rank} X$. Then,*

$$\operatorname{Pert}_{\mathcal{P}}(X) = \left\{PRP^{\mathsf{T}} : R \in \mathcal{S}^r,\ \langle PRP^{\mathsf{T}}, A_i\rangle = 0\ (i \in I \cup J_X)\right\}. \tag{3}$$

As a direct application, we obtain a characterization for extreme points of the primal feasible region $\mathcal{P}$.

Corollary 2.2. *Consider a matrix $X \in \mathcal{P}$, written as $X = PP^{\mathsf{T}}$, where $P \in \mathbb{R}^{n\times r}$ and $r = \operatorname{rank} X$. The following assertions are equivalent:*

(i) X is an extreme point of $\mathcal{P}$.

(ii) If $R \in \mathcal{S}^r$ satisfies $\langle P^{\mathsf{T}} A_i P, R\rangle = 0$ for all $i \in I \cup J_X$, then $R = 0$.

(iii) $\operatorname{lin}\{P^{\mathsf{T}} A_i P : i \in I \cup J_X\} = \mathcal{S}^r$.

We denote by $\mathcal{R}_r$ the manifold of symmetric $n \times n$ matrices with rank equal to r. Given a matrix $X \in \mathcal{R}_r$, let $X = Q\Lambda Q^{\mathsf{T}}$ be its spectral decomposition, where Q is an orthogonal matrix whose columns are the eigenvectors of X and Λ is the diagonal matrix with the corresponding eigenvalues as diagonal entries. Without loss of generality we may assume that $\Lambda_{ii} \neq 0$ for $i \in [r]$.

The *tangent space* of $\mathcal{R}_r$ at X is given by

$$\mathcal{T}_X = \left\{ Q \begin{pmatrix} U & V \\ V^{\mathsf{T}} & 0 \end{pmatrix} Q^{\mathsf{T}} : U \in \mathcal{S}^r, V \in \mathbb{R}^{r \times (n-r)} \right\}. \tag{4}$$

Hence, its orthogonal complement is defined by

$$\mathcal{T}_X^{\perp} = \left\{ Q \begin{pmatrix} 0 & 0 \\ 0 & W \end{pmatrix} Q^{\mathsf{T}} : W \in \mathcal{S}^{n-r} \right\}. \tag{5}$$

We will also use the equivalent description:

$$\mathcal{T}_X^{\perp} = \{ M \in \mathcal{S}^n : XM = 0 \}. \tag{6}$$

We now introduce the notions of nondegeneracy and strict complementarity for the semidefinite programs (P) and (D) in standard form.

Definition 2.3. *[4] Consider the pair of primal and dual semidefinite programs (P) and (D). A matrix $X \in \mathcal{P}$ is called* primal nondegenerate *if*

$$\mathcal{T}_X + \mathrm{lin}\{A_i : i \in I \cup J_X\}^{\perp} = \mathcal{S}^n. \tag{7}$$

The pair $(y, Z) \in \mathcal{D}$ is called dual nondegenerate *if*

$$\mathcal{T}_Z + \mathrm{lin}\{A_i : i \in I \cup J_Z\} = \mathcal{S}^n. \tag{8}$$

Next we present some well known results that provide necessary and sufficient conditions for the unicity of optimal solutions in terms of the notions of primal or dual nondegeneracy and strict complementarity. With the intention to make the section self-contained we have also included short proofs.

Theorem 2.4. *[4] Assume that the optimal values of* (P) *and* (D) *are equal and that both are attained. If* (P) *has a nondegenerate optimal solution, then* (D) *has a unique optimal solution. (Analogously, if* (D) *has a nondegenerate optimal solution, then* (P) *has a unique optimal solution.)*

Proof. Let X be a nondegenerate optimal solution of (P) and let $(y^{(1)}, Z_1)$, $(y^{(2)}, Z_2)$ be two dual optimal solutions. Complementary slackness implies that $y_j^{(1)} = y_j^{(2)} = 0$ holds for every $i \in J \setminus J_X$. Hence, $Z_1 - Z_2 \in \mathrm{lin}\{A_i : i \in I \cup J_X\}$. As there is no duality gap we have that $XZ_1 = XZ_2 = 0$ and then (6) implies that $Z_1 - Z_2 \in \mathcal{T}_X^{\perp}$. These two facts combined with the assumption that X is primal nondegenerate imply that $Z_1 = Z_2$. The other case is similar. □

The next lemma provides a characterization of the space of perturbations in terms of tangent spaces for a pair of strict complementary optimal solutions.

Lemma 2.5. *Assume that the optimal values of* (P) *and* (D) *are equal and that both are attained. Let* $X, (y, Z)$ *be a strict complementary pair of primal and dual optimal solutions for* (P) *and* (D)*, respectively. Then,*

$$\mathrm{Pert}_{\mathcal{P}}(X) = \mathrm{lin}\{A_i : i \in I \cup J_X\}^{\perp} \cap \mathcal{T}_Z^{\perp}, \tag{9}$$

$$\mathrm{Pert}_{\mathcal{D}}(Z) = \mathrm{lin}\{A_i : i \in I \cup J_Z\}^{\perp} \cap \mathcal{T}_X^{\perp}. \tag{10}$$

Proof. By assumption, $ZX = XZ = 0$, which implies that X and Z can be simultaneously diagonalized by the same orthogonal matrix Q. Let $r = \operatorname{rank} X$ and write $Q = (Q_1\ Q_2)$, where the columns of $Q_1 \in \mathbb{R}^{n \times r}$ form a basis of the range of X. As X and Z are strict complementary we obtain that

$$X = Q \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} Q^{\mathsf{T}} = Q_1 \Lambda_1 Q_1^{\mathsf{T}},\ Z = Q \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} Q^{\mathsf{T}} = Q_2 \Lambda_2 Q_2^{\mathsf{T}},$$

where Λ_1 and Λ_2 are diagonal matrices of sizes r and $n - r$, respectively. The claim follows easily using the form of $\mathcal{T}_X$ (and $\mathcal{T}_Z$) given in (5). □

The next theorem establishes the converse of Theorem 2.4, assuming strict complementarity.

Theorem 2.6. *[4] Assume that the optimal values of* (P) *and* (D) *are equal and that both are attained. Let* $X, (y, Z)$ *be a strict complementary pair of optimal solutions for* (P) *and* (D)*, respectively, and assume that* $J_X = J_Z$*. If* X *is the unique optimal solution of* (P) *then* (y, Z) *is dual nondegenerate. (Analogously, if* (y, Z) *is the unique optimal solution of* (D) *then* X *is primal nondegenerate.)*

Proof. By assumption, X is the unique optimal solution of (P). Hence X is an extreme point of the primal feasible region and thus, using (9), we obtain that $\mathcal{T}_Z + \mathrm{lin}\{A_i : i \in I \cup J_X\} = \mathcal{S}^n$. As $J_X = J_Z$, (8) holds and thus (y, Z) is dual nondegenerate. □

As an application we obtain the following characterization for the extreme points of $\mathcal{P}$, assuming strict complementarity.

Theorem 2.7. *Assume that the optimal values of* (P) *and* (D) *are equal and that both are attained. Let* $X, (y, Z)$ *be a pair of strict complementary optimal solutions of the primal and dual programs* (P) *and* (D)*, respectively, and assume that* $J_X = J_Z$*. The following assertions are equivalent:*

(i) X *is an extreme point of* $\mathcal{P}$*.*

(ii) X *is the unique primal optimal solution of* (P)*.*

(iii) Z *is a dual nondegenerate.*

Proof. The equivalence *(ii)* $\Longleftrightarrow$ *(iii)* follows directly from Theorems 2.4 and 2.6 and the equivalence *(i)* $\Longleftrightarrow$ *(iii)* follows by Lemma 2.5 and the definition of dual nondegeneracy from (8). □

Note that Theorems 2.6 and 2.7 still hold if we replace the condition $J_X = J_Z$ by the weaker condition:

$$\forall i \in J_X \setminus J_Z \quad A_i \in \mathcal{T}_Z + \text{lin}\{A_i : i \in I \cup J_X\}. \tag{11}$$

Note also that this condition is automatically satisfied in the case when $J = \emptyset$, i.e., when the semidefinite program (P) involves only linear equalities.

3. Uniqueness of positive semidefinite matrix completions

3.1. Basic definitions

Let $G = (V = [n], E)$ be a given graph. Recall that a vector $a \in \mathbb{R}^{V \cup E}$ is called a *G-partial psd matrix* if it admits at least one completion to a full psd matrix, i.e., if the semidefinite program (1) has at least one feasible solution. We denote by $\mathcal{S}_+(G)$ the set of all G-partial psd matrices. In other words, $\mathcal{S}_+(G)$ is equal to the projection of the positive semidefinite cone $\mathcal{S}^n_+$ onto the subspace $\mathbb{R}^{V \cup E}$ indexed by the nodes (corresponding to the diagonal entries) and the edges of G. We can reinterpret G-partial psd matrices in terms of Gram representations. Namely, $a \in \mathcal{S}_+(G)$ if and only if there exist vectors $p_1, \ldots, p_n \in \mathbb{R}^d$ (for some $d \geq 1$) such that $a_{ij} = p_i^\mathsf{T} p_j$ for all $\{i, j\} \in V \cup E$. This leads to the notion of frameworks which will make the link between the Gram (spherical) setting of this section and the Euclidean distance setting considered in the next section.

A *tensegrity graph* is a graph G whose edge set is partitioned into three sets: $E = B \cup C \cup S$, whose members are called *bars, cables* and *struts*, respectively. A *tensegrity framework* $G(\mathbf{p})$ consists of a tensegrity graph G together with an assignment of vectors $\mathbf{p} = \{p_1, \ldots, p_n\}$ to the nodes of G. A *bar framework* is a tensegrity framework where $C = S = \emptyset$.

Given a tensegrity framework $G(\mathbf{p})$ consider the following pair of primal and dual semidefinite programs:

$$\begin{aligned} \sup_X\{0 : X \succeq 0, \quad & \langle E_{ij}, X\rangle = p_i^\mathsf{T} p_j \ \text{ for } \{i,j\} \in V \cup B, \\ & \langle E_{ij}, X\rangle \leq p_i^\mathsf{T} p_j \ \text{ for } \{i,j\} \in C, \\ & \langle E_{ij}, X\rangle \geq p_i^\mathsf{T} p_j \ \text{ for } \{i,j\} \in S\} \end{aligned} \tag{$\mathcal{P}_G$}$$

and

$$\begin{aligned} \inf_{y,Z}\{\textstyle\sum_{ij \in V \cup E} y_{ij} p_i^\mathsf{T} p_j : \quad & \textstyle\sum_{ij \in V \cup E} y_{ij} E_{ij} = Z \succeq 0, \\ & y_{ij} \geq 0 \ \text{ for } \{i,j\} \in C, \\ & y_{ij} \leq 0 \ \text{ for } \{i,j\} \in S\}. \end{aligned} \tag{$\mathcal{D}_G$}$$

The next definition captures the analogue of the notion of universal rigidity for the Gram setting.

Definition 3.1. *A tensegrity framework* $G(\mathbf{p})$ *is called* universally completable *if the matrix* $\text{Gram}(p_1, \ldots, p_n)$ *is the unique solution of the semidefinite program* ($\mathcal{P}_G$).

In other words, a universally completable framework $G(\mathbf{p})$ corresponds to a G-partial psd matrix $a \in \mathcal{S}_+(G)$, where $a_{ij} = p_i^\mathsf{T} p_j$ for all $\{i,j\} \in V \cup E$, that admits a unique completion to a full psd matrix. Consequently, identifying sufficient conditions guaranteeing that a framework $G(\mathbf{p})$ is universally completable will allow us to construct G-partial matrices with a unique psd completion.

3.2. A sufficient condition for universal completability

In this section we derive a sufficient condition for determining the universal completability of tensegrity frameworks.

We use the following notation: For a graph $G = (V, E)$, $\overline{E}$ denotes the set of pairs $\{i,j\}$ with $i \neq j$ and $\{i,j\} \notin E$, corresponding to the non-edges of G.

Theorem 3.2. *Let $G = ([n], E)$ be a tensegrity graph with $E = B \cup C \cup S$ and consider a tensegrity framework $G(\mathbf{p})$ in $\mathbb{R}^d$ such that $p_1, \ldots, p_n$ span linearly $\mathbb{R}^d$. Assume there exists a matrix $Z \in \mathcal{S}^n$ satisfying the conditions (i)-(vi):*

(i) Z *is positive semidefinite.*

(ii) $Z_{ij} = 0$ *for all* $\{i,j\} \in \overline{E}$.

(iii) $Z_{ij} \geq 0$ *for all (cables)* $\{i,j\} \in C$ *and* $Z_{ij} \leq 0$ *for all (struts)* $\{i,j\} \in S$.

(iv) Z *has corank* d.

(v) $\sum_{j \in V} Z_{ij} p_j = 0$ *for all* $i \in [n]$.

(vi) For any matrix $R \in \mathcal{S}^d$ *the following holds:*

$$p_i^\mathsf{T} R \, p_j = 0 \ \forall \{i,j\} \in V \cup B \cup \{\{i,j\} \in C \cup S : Z_{ij} \neq 0\} \Longrightarrow R = 0. \quad (12)$$

Then the tensegrity framework $G(\mathbf{p})$ is universally completable.

Proof. Set $X = \mathrm{Gram}(p_1, \ldots, p_n)$. Assume that $Y \in \mathcal{S}^n_+$ is another matrix which is feasible for the program $(\mathcal{P}_G)$, say $Y = \mathrm{Gram}(q_1, \ldots, q_n)$ for some vectors $q_1, \ldots, q_n$. Our goal is to show that $Y = X$. By (v), $ZX = 0$ and thus $\mathrm{Ran}\, X \subseteq \mathrm{Ker}\, Z$. Moreover, $\dim \mathrm{Ker}\, Z = d$ by (iv), and $\mathrm{rank} X = d$ since $\mathrm{lin}\{p_1, \ldots, p_n\} = \mathbb{R}^d$. This implies that $\mathrm{Ker}\, X = \mathrm{Ran}\, Z$.

By (ii) we can write $Z = \sum_{\{i,j\} \in V \cup E} Z_{ij} E_{ij}$. Next notice that

$$0 \leq \langle Z, Y \rangle = \left\langle \sum_{\{i,j\} \in V \cup E} Z_{ij} E_{ij}, Y \right\rangle \leq \sum_{\{i,j\} \in V \cup E} Z_{ij} \langle E_{ij}, X \rangle = \langle Z, X \rangle = 0, \quad (13)$$

where the first (left most) inequality follows from the fact that $Y, Z \succeq 0$ and the second one from the feasibility of Y for $(\mathcal{P}_G)$ and the sign conditions (iii) on Z. This gives $\langle Z, Y \rangle = 0$, which implies that $\mathrm{Ker}\, Y \supseteq \mathrm{Ran}\, Z$ and thus $\mathrm{Ker}\, Y \supseteq \mathrm{Ker}\, X$.

Write $X = PP^\mathsf{T}$, where $P \in \mathbb{R}^{n \times d}$ has rows $p_1^\mathsf{T}, \ldots, p_n^\mathsf{T}$. From the inclusion $\mathrm{Ker}\,(Y - X) \supseteq \mathrm{Ker}\, X$, we deduce that $Y - X = PRP^\mathsf{T}$ for some matrix $R \in \mathcal{S}^d$.

As equality holds throughout in (13), we obtain that $\langle E_{ij}, Y - X\rangle = 0$ for all $\{i,j\} \in C \cup S$ with $Z_{ij} \neq 0$. Additionally, as X, Y are both feasible for $(\mathcal{P}_G)$, we have that $\langle E_{ij}, Y - X\rangle = 0$ for all $\{i,j\} \in V \cup B$. Substituting PRP^{T} for $Y - X$, we obtain that $p_i^{\mathsf{T}} R p_j = 0$ for all $\{i,j\} \in V \cup B$ and all $\{i,j\} \in C \cup S$ with $Z_{ij} \neq 0$. We can now apply (vi) and conclude that $R = 0$. This gives $Y = X$, which concludes the proof. □

Note that the conditions (i)-(iii) express that Z is feasible for the dual semidefinite program $(\mathcal{D}_G)$. In analogy to the Euclidean setting (see Section 4), such matrix Z is called a *spherical stress matrix* for the framework $G(\mathbf{p})$. Moreover, (v) says that Z is dual optimal and (iv) says that $X = \mathrm{Gram}(p_1, \ldots, p_n)$ and Z are strictly complementary solutions to the primal and dual semidefinite programs $(\mathcal{P}_G)$ and $(\mathcal{D}_G)$. Finally, in the case of bar frameworks (when $C = S = \emptyset$), condition (vi) means that Z is dual nondegenerate. Hence, for bar frameworks, Theorem 3.2 also follows as a direct application of Theorem 2.7.

As a last remark notice that the assumptions of Theorem 3.2 imply that $n \geq d$. Moreover, for $n = d$, the matrix Z is the zero matrix and in this case (12) reads: $p_i^{\mathsf{T}} R p_j = 0$ for all $\{i,j\} \in V \cup B$ then $R = 0$. Observe that this condition can be satisfied only when $G = K_n$ and $C = S = \emptyset$, so that Theorem 3.2 is useful only in the case when $d \leq n - 1$.

3.3. Applying the sufficient condition

In this section we use Theorem 3.2 to construct several instances of partial psd matrices admitting a unique psd completion. Most of these constructions have been considered in [14, 24, 25]. While the proofs there for unicity of the psd completion consisted of ad hoc arguments and case checking, Theorem 3.2 provides us with a unified and systematic approach. In all examples below we only deal with bar frameworks and hence we apply Theorem 3.2 with $C = S = \emptyset$. In particular, there are no sign conditions on the stress matrix Z and moreover condition (12) assumes the simpler form: If $p_i^{\mathsf{T}} R p_j = 0$ for all $\{i,j\} \in V \cup E$ then $R = 0$.

Example 1: The octahedral graph. Consider a framework for the octahedral graph $K_{2,2,2}$ defined as follows:

$$p_1 = e_1,\ p_2 = e_2,\ p_3 = e_1 + e_2,\ p_4 = e_3,\ p_5 = e_4,\ p_6 = e_5,$$

where e_i ($i \in [5]$) denote the standard unit vectors in $\mathbb{R}^5$ and the numbering of the nodes refers to Figure 1. In [24] it is shown that the corresponding $K_{2,2,2}$-partial matrix $a = (p_i^{\mathsf{T}} p_j) \in \mathcal{S}_+(K_{2,2,2})$ admits a unique psd completion. This result follows easily, using Theorem 3.2. Indeed it is easy to check that condition (12) holds. Moreover, the matrix $Z = (1,1,-1,0,0,0)(1,1,-1,0,0,0)^{\mathsf{T}}$ is psd with corank 5, it is supported by $K_{2,2,2}$, and satisfies $\langle Z, \mathrm{Gram}(p_1, \ldots, p_5)\rangle = 0$. Hence Theorem 3.2 applies and the claim follows.

Example 2: The family of graphs F_r. For an integer $r \geq 2$, we define a graph $F_r = (V_r, E_r)$ with $r + \binom{r}{2}$ nodes denoted as v_i (for $i \in [r]$) and v_{ij} (for $1 \leq i < j \leq r$). It consists of a central clique of size r based on the nodes

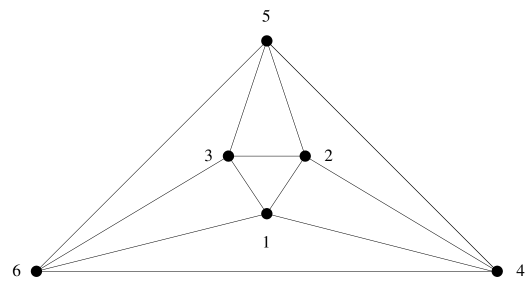

Figure 1: The graph $K_{2,2,2}$.

$v_1, \ldots, v_r$ together with the cliques C_{ij} on the nodes $\{v_i, v_j, v_{ij}\}$. The graphs F_3 and F_4 are shown in Figure 2 below. We construct a framework in $\mathbb{R}^r$ for

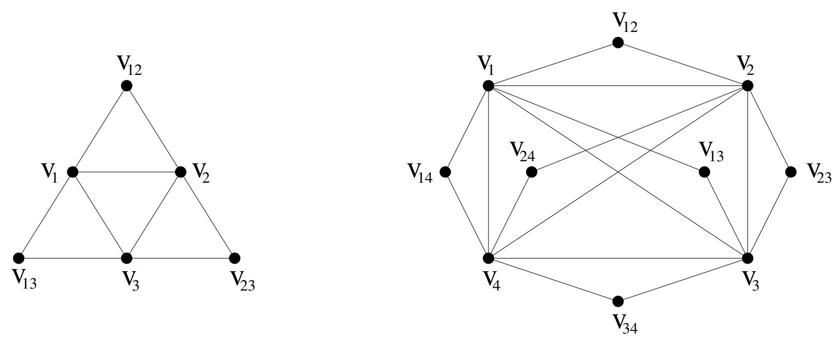

Figure 2: The graphs F_3 and F_4.

the graph F_r as follows:

$$p_{v_i} = e_i \text{ for } i \in [r] \text{ and } p_{v_{ij}} = e_i + e_j \text{ for } 1 \leq i < j \leq r.$$

In [14] it is shown that for any $r \geq 2$ the corresponding F_r-partial matrix admits a unique psd completion. We now show this result, using Theorem 3.2.

Fix $r \geq 2$. It is easy to check that (12) holds. Define the nonzero matrix $Z_r = \sum_{1 \leq i < j \leq r} u_{ij} {u_{ij}}^\mathsf{T}$, where the vectors $u_{ij} \in \mathbb{R}^{r+\binom{r}{2}}$ are defined as follows: For $1 \leq k \leq r$, $(u_{ij})_k = 1$ if $k \in \{i, j\}$ and 0 otherwise; for $1 \leq k < l \leq r$, $(u_{ij})_{kl} = -1$ if $\{k, l\} = \{i, j\}$ and 0 otherwise. By construction, Z_r is psd, it is supported by the graph F_r, $\langle Z_r, \mathrm{Gram}(p_v : v \in V_r) \rangle = 0$ and $\operatorname{corank} Z_r = r$. Thus Theorem 3.2 applies and the claim follows.

Example 3: The family of graphs G_r. This family of graphs has been considered in the study of the Colin de Verdière graph parameter [8]. For any integer $r \geq 2$ consider an equilateral triangle and subdivide each side into $r - 1$ equal segments. Through these points draw line segments parallel to the sides of the triangle. This construction creates a triangulation of the big triangle into $(r-1)^2$ congruent equilateral triangles. The graph $G_r = (V_r, E_r)$ corresponds to the edge graph of this triangulation. Clearly, the graph G_r has $\binom{r+1}{2}$ vertices, which we denote (i,l) for $l \in [r]$ and $i \in [r-l+1]$. For any fixed $l \in [r]$ we say that the vertices $(1,l),\dots,(r-l+1,l)$ are at level l. Note that $G_2 = K_3 = F_2$, $G_3 = F_3$, but $G_r \neq F_r$ for $r \geq 4$. The graph G_5 is illustrated in Figure 3.

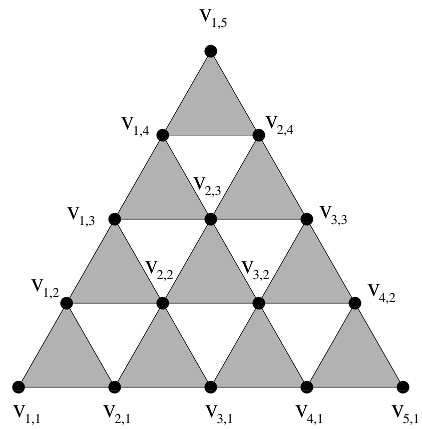

Figure 3: The graph G_5.

Fix an integer $r \geq 2$. We consider the following framework in $\mathbb{R}^r$ for the graph G_r:

$$p_{(i,1)} = e_i \;\forall i \in [r] \;\text{ and }\; p_{(i,l)} = p_{(i,l-1)} + p_{(i+1,l-1)} \;\forall l \geq 2 \text{ and } i \in [r-l+1]. \quad (14)$$

In [14] it is shown that for any $r \geq 2$ the partial G_r-partial matrix that corresponds to the framework defined in (14) has a unique psd completion. We now recover this result, using Theorem 3.2.

First we show that this framework satisfies (12). For this, consider a matrix $R \in \mathcal{S}^r$ such that $p_{(i,l)}^{\mathsf{T}} R\, p_{(i',l')} = 0$ for every $\{(i,l),(i',l')\} \in V_r \cup E_r$. Specializing this relation for $i' = i \in [r]$ and $l' = l = 1$ we get that $R_{ii} = 0$ for all $i \in [r]$ and for $i' = i+1$ and $l = l' = 1$ we get that $R_{i,i+1} = 0$ for $i \in [r-1]$. Similarly, for $i' = i+1$ and $l' = l \geq 2$ we get that $R_{i,i+l} = 0$ for all $i \in [r-l]$ and thus $R = 0$.

We call a triangle in G_r *black* if it is of the form $\{(i,l),(i+1,l),(i,l+1)\}$ and we denote by $\mathcal{B}_r$ the set of black triangles in G_r. The black triangles in G_5 are illustrated in Figure 3 as the shaded triangles. Let $Z_r = \sum_{t \in \mathcal{B}_r} u_t u_t^{\mathsf{T}}$ where the vector $u_t \in \mathbb{R}^{\binom{r+1}{2}}$ is defined as follows: If $t \in \mathcal{B}_r$ corresponds to the black triangle $\{(i,l),(i+1,l),(i,l+1)\}$ then $u_t(i,l) = u_t(i+1,l) = 1,\ u_t(i,l+1) = -1$ and 0 otherwise. Since $|\mathcal{B}_r| = \binom{r+1}{2} - r$ and the vectors $(u_t)_{t \in \mathcal{B}_r}$ are linearly independent we have that $\operatorname{corank} Z_r = r$. Moreover, as every edge of G_r belongs to exactly one black triangle we have that Z_r is supported by G_r. By construction of the framework we have that $\sum_{(i,l) \in V_r} p_{(i,l)} u_t = 0$ for all $t \in \mathcal{B}_r$ which implies that $\langle \text{Gram}(p_{(i,l)} : (i,l) \in V_r), Z_r \rangle = 0$. Thus Theorem 3.2 applies and the claim follows.

Example 4: Tensor products of graphs. This construction was considered in [29], where universally rigid frameworks were used as a tool to construct uniquely colorable graphs. The original construction was carried out in the Euclidean setting for a suspension bar framework. Here we present the construction in the spherical setting which, as we will see in Section 4.3, is equivalent.

Let $H = ([n], E)$ be a k-regular graph satisfying $\max_{2\leq i\leq n} |\lambda_i| < k/(r-1)$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of its adjacency matrix A_H. For $r \in \mathbb{N}$ we let $G_r = (V_r, E_r)$ denote the graph $K_r \times H$, obtained by taking the tensor product of the complete graph K_r and the graph H. By construction, the adjacency matrix of G_r is the tensor product of the adjacency matrices of K_r and H: $A_{G_r} = A_{K_r} \otimes A_H$. Let us denote the vertices of G_r by the pairs (i, h) where $i \in [r]$ and $h \in V(H)$.

Let $w_1, \dots, w_r \in \mathbb{R}^{r-1}$ be vectors that linearly span $\mathbb{R}^{r-1}$ and moreover satisfy $\sum_{i=1}^{r} w_i = 0$. We construct a framework for G_r in $\mathbb{R}^r$ by assigning to all nodes (i, h) for $h \in V(H)$ the vector $p_{(i,h)} = w_i$, for each $i \in [r]$. We now show, using Theorem 3.2, that the associated G_r-partial matrix admits a unique psd completion.

First we show that this framework satisfies (12). For this, consider a matrix $R \in \mathcal{S}^r$ satisfying $p_{(i,h)}^{\mathsf{T}} R p_{(i',h')} = 0$ for every $\{(i,h),(i',h')\} \in V_r \cup E_r$. This implies that $w_i^{\mathsf{T}} R w_j = 0$ for all $i, j \in [r]$ and as $\mathrm{lin}\{w_i w_j^{\mathsf{T}} + w_j w_i^{\mathsf{T}} : i, j \in [r]\} = \mathcal{S}^r$ it follows that $R = 0$.

Next consider the matrix $Z_k = I_{rn} + \frac{1}{k} A_{G_r} \in \mathcal{S}^{rn}$, where I_{rn} denotes the identity matrix of size rn. Notice that the matrix Z_r is by construction supported by G_r. One can verify directly that $\langle \mathrm{Gram}(p_{(i,h)} : (i,h) \in V_r), Z\rangle = 0$. The eigenvalues of A_{K_r} are $r-1$ with multiplicity one and -1 with multiplicity r. This fact combined with the assumption on the eigenvalues of H implies that Z_r is positive semidefinite with $\mathrm{corank}\, Z_r = r-1$. Thus Theorem 3.2 applies and the claim follows.

Example 5: The odd cycle C_5. The last example illustrates the fact that sometimes the sufficient conditions from Theorem 3.2 cannot be used to show existence of a unique psd completion. Here we consider the 5-cycle graph $G = C_5$ (although it is easy to generalize the example to arbitrary odd cycles).

First we consider the framework in $\mathbb{R}^2$ given by the vectors

$$p_i = (\cos(4(i-1)\pi/5), \sin(4(i-1)\pi/5))^{\mathsf{T}} \quad \text{for } 1 \leq i \leq 5.$$

The corresponding C_5-partial matrix has a unique psd completion, and this can be shown using Theorem 3.2.

It is easy to see that (12) holds. Let A_{C_5} denote the adjacency matrix of C_5 and recall that its eigenvalues are $2\cos\frac{2\pi}{5}$ and $-2\cos\frac{\pi}{5}$, both with multiplicity two and 2 with multiplicity one. Define $Z = 2\cos\frac{\pi}{5} I + A_{C_5}$ and notice that $Z \succeq 0$ and $\mathrm{corank}\, Z = 2$. Moreover, one can verify that $\sum_{j\in[5]} Z_{ij} p_j = 0$ for all $i \in [5]$ which implies that $\langle Z, \mathrm{Gram}(p_1, \dots, p_5)\rangle = 0$. Thus Theorem 3.2 applies and the claim follows.

Next we consider another framework for C_5 in $\mathbb{R}^2$ given by the vectors

$$q_1 = (1,0)^\mathsf{T}, q_2 = (-1/\sqrt{2}, 1/\sqrt{2})^\mathsf{T}, q_3 = (0,-1)^\mathsf{T}, q_4 = (1/\sqrt{2}, 1/\sqrt{2})^\mathsf{T},$$

$$q_5 = (-1/\sqrt{2}, -1/\sqrt{2})^\mathsf{T}.$$

We now show that the corresponding C_5-partial matrix admits a unique psd completion. This cannot be shown using Theorem 3.2 since there does not exist a nonzero matrix $Z \in \mathcal{S}^5$ supported by C_5 satisfying $\langle Z, \mathrm{Gram}(q_1, \ldots q_5)\rangle = 0$. Nevertheless one can prove that there exists a unique psd completion by using the following geometric argument.

Let $X \in \mathcal{S}^5_+$ be a psd completion of the partial matrix and set $\vartheta_{ij} = \arccos X_{ij} \in [0,\pi]$ for $1 \le i \le j \le 5$. Then, $\vartheta_{12} = \vartheta_{23} = \vartheta_{34} = \vartheta_{45} = 3\pi/4$ and $\vartheta_{15} = \pi$. Therefore, the following linear equality holds:

$$\sum_{i=1}^{5} \vartheta_{i,i+1} = 4\pi \tag{15}$$

(where indices are taken modulo 5). As we will see this implies that the remaining angles are uniquely determined by the relations:

$$\vartheta_{i,i+2} + \vartheta_{i,i+1} + \vartheta_{i+1,i+2} = 2\pi \quad \text{for } 1 \le i \le 5 \tag{16}$$

and thus that X is uniquely determined To see why the identities (16) hold, we use the well known fact that the angles ϑ_{ij} satisfy the (triangle) inequalities:

$$\vartheta_{12} + \vartheta_{23} + \vartheta_{13} \le 2\pi, \; -\vartheta_{13} - \vartheta_{14} + \vartheta_{34} \le 0, \; \vartheta_{14} + \vartheta_{45} + \vartheta_{15} \le 2\pi. \tag{17}$$

Summing up the three inequalities in (17) and combining with (15), we deduce that equality holds throughout in (17). This permits to derive the values of $\vartheta_{13} = \pi/2$ and $\vartheta_{14} = \pi/4$ and proceed analogously for the remaining angles. (For details on the parametrization of positive semidefinite matrices using the arccos map, see [5] or [23]).

4. Universal rigidity of tensegrity frameworks

Our goal in this section is to give a concise and self-contained treatment of some known results concerning the universal rigidity of tensegrity frameworks. In particular, building on ideas from the two previous sections we give a very short and elementary proof of Connelly's sufficient condition for universal rigidity for both generic and non-generic tensegrity frameworks. Lastly, we also investigate the relation of our sufficient condition from Theorem 3.2 (for the Gram setting) to Connelly's sufficient condition from Theorem 4.4 (for the Euclidean distance setting).

4.1. Connelly's characterization

The framework $G(\mathbf{p})$ is called *d-dimensional* if $p_1, \cdots, p_n \in \mathbb{R}^d$ and their affine span is $\mathbb{R}^d$. A d-dimensional framework is said to be in *general position* if every $d+1$ vectors are affinely independent. Given a framework $G(\mathbf{p})$ in $\mathbb{R}^d$, its *configuration matrix* is the $n \times d$ matrix P whose rows are the vectors $p_1^\mathsf{T}, \ldots, p_n^\mathsf{T}$, so that $PP^\mathsf{T} = \text{Gram}(p_1, \ldots, p_n)$. The framework $G(\mathbf{p})$ is said to be *generic* if the coordinates of the vectors $p_1, \ldots, p_n$ are algebraically independent over the rational numbers.

Definition 4.1. *Let $G = ([n], E)$ be a tensegrity graph with $E = B \cup C \cup S$. A tensegrity framework $G(\mathbf{p})$ is said to* dominate *a tensegrity framework $G(\mathbf{q})$ if the following conditions hold:*

(i) $\|p_i - p_j\| = \|q_i - q_j\|$ *for all (bars)* $\{i,j\} \in B$,

(ii) $\|p_i - p_j\| \geq \|q_i - q_j\|$ *for all (cables)* $\{i,j\} \in C$,

(iii) $\|p_i - p_j\| \leq \|q_i - q_j\|$ *for all (struts)* $\{i,j\} \in S$.

Two frameworks $G(\mathbf{p})$ and $G(\mathbf{q})$ are called *congruent* if

$$\|p_i - p_j\| = \|q_i - q_j\|, \ \forall i \neq j \in [n].$$

Equivalently, this means that $G(\mathbf{q})$ can be obtained by $G(\mathbf{p})$ by a rigid motion of the Euclidean space. In this section we will be concerned with tensegrity frameworks which, up to the group of rigid motions of the Euclidean space, admit a unique realization.

Definition 4.2. *A tensegrity framework $G(\mathbf{p})$ is called* universally rigid *if it is congruent to any tensegrity it dominates.*

An essential ingredient for characterizing universally rigid tensegrities is the notion of equilibrium stress matrix which we now introduce.

Definition 4.3. *A matrix $\Omega \in \mathcal{S}^n$ is called an* equilibrium stress matrix *for a tensegrity framework $G(\mathbf{p})$ if it satisfies:*

(i) $\Omega_{ij} = 0$ *for all* $\{i,j\} \in \overline{E}$.

(ii) $\Omega e = 0$ *and* $\Omega P = 0$, *i.e.,* $\sum_{j \in V} \Omega_{ij} p_j = 0$ *for all* $i \in V$.

(iii) $\Omega_{ij} \geq 0$ *for all (cables)* $\{i,j\} \in C$ *and* $\Omega_{ij} \leq 0$ *for all (struts)* $\{i,j\} \in S$.

Note that, by property (i) combined with the condition $\Omega e = 0$, any equilibrium stress matrix Ω can be written as $\Omega = \sum_{\{i,j\} \in E} \Omega_{ij} F_{ij}$, where we set $F_{ij} = (e_i - e_j)(e_i - e_j)^\mathsf{T}$.

The following result (Theorem 4.4), due to R. Connelly, establishes a *sufficient condition* for determining the universal rigidity of tensegrities. All the ingredients for its proof are already present in [9] although there is no explicit statement of the theorem there. An exact formulation and a proof of Theorem 4.4 can be found in the (unpublished) work [10]. We now give an elementary proof of Theorem 4.4 which relies only on basic properties of positive semidefinite matrices. Our proof goes along the same lines as the proof of Theorem 3.2 above and it is substantially shorter and simpler in comparison with Connelly's original proof.

Theorem 4.4. *Let $G = ([n], E)$ be a tensegrity graph with $E = B \cup C \cup S$ and let $G(\mathbf{p})$ be a tensegrity framework in $\mathbb{R}^d$ such that $p_1, \ldots, p_n$ affinely span $\mathbb{R}^d$. Assume there exists an equilibrium stress matrix Ω for $G(\mathbf{p})$ such that:*

(i) *Ω is positive semidefinite.*

(ii) *Ω has corank $d + 1$.*

(iii) *For any matrix $R \in \mathcal{S}^d$ the following holds:*

$$(p_i-p_j)^\mathsf{T} R\,(p_i-p_j) = 0\ \forall \{i,j\} \in B \cup \{\{i,j\} \in C \cup S : \Omega_{ij} \neq 0\} \Longrightarrow R = 0. \tag{18}$$

Then, $G(p)$ is universally rigid.

Proof. Assume that $G(\mathbf{p})$ dominates another framework $G(\mathbf{q})$, our goal is to show that $G(\mathbf{p})$ and $G(\mathbf{q})$ are congruent. Recall that P is the $n \times d$ matrix with the vectors $p_1, \cdots, p_n$ as rows and define the augmented $n \times (d+1)$ matrix $P_a = \begin{pmatrix} P & e \end{pmatrix}$ obtained by adding the all-ones vector as last column to P. Set $X = PP^T$ and $X_a = P_a P_a^\mathsf{T}$, so that $X_a = X + ee^\mathsf{T}$. As the tensegrity $G(\mathbf{p})$ is d-dimensional, we have that rank $X_a = d + 1$. We claim that $\operatorname{Ker} X_a = \operatorname{Ran} \Omega$. Indeed, as Ω is an equilibrium stress matrix for $G(\mathbf{p})$, we have that $\Omega P_a = 0$ and thus $\Omega X_a = 0$. This implies that $\operatorname{Ran} X_a \subseteq \operatorname{Ker} \Omega$ and, as $\operatorname{corank} \Omega = d + 1 = \operatorname{rank} X_a$, it follows that $\operatorname{Ker} X_a = \operatorname{Ran} \Omega$.

Let Y denote the Gram matrix of the vectors $q_1, \cdots, q_n$. We claim that $\operatorname{Ker} Y \supseteq \operatorname{Ker} X_a$. Indeed, we have that

$$0 \leq \langle \Omega, Y \rangle = \left\langle \sum_{\{i,j\} \in E} \Omega_{ij} F_{ij}, Y \right\rangle \leq \sum_{\{i,j\} \in E} \Omega_{ij} \langle F_{ij}, X_a \rangle = \langle \Omega, X_a \rangle = 0. \tag{19}$$

The first inequality follows from the fact that $\Omega, Y \succeq 0$; the second inequality holds since $\Omega_{ij}\langle F_{ij}, Y\rangle \leq \Omega_{ij}\langle F_{ij}, X\rangle = \Omega_{ij}\langle F_{ij}, X_a\rangle$ for all edges $\{i,j\} \in E$, using the fact that $G(\mathbf{p})$ dominates $G(\mathbf{q})$ and the sign conditions on Ω. Therefore equality holds throughout in (19). This gives $\langle \Omega, Y \rangle = 0$, implying $Y\Omega = 0$ (since $Y, \Omega \succeq 0$) and thus $\operatorname{Ker} Y \supseteq \operatorname{Ran} \Omega = \operatorname{Ker} X_a$.

As $\operatorname{Ker} Y \supseteq \operatorname{Ker} X_a$, we deduce that $\operatorname{Ker}(Y - X_a) \supseteq \operatorname{Ker} X$ and thus $Y - X_a$ can be written as

$$Y - X_a = P_a R P_a^\mathsf{T} \text{ for some matrix } R = \begin{pmatrix} A & b \\ b^\mathsf{T} & c \end{pmatrix} \in \mathcal{S}^{d+1}, \tag{20}$$

where $A \in \mathcal{S}^d$, $b \in \mathbb{R}^d$ and $c \in \mathbb{R}$.

As equality holds throughout in (19) holds, we obtain $\Omega_{ij}\langle F_{ij}, Y - X_a\rangle = 0$ for all $\{i,j\} \in C \cup S$. Therefore, $\langle F_{ij}, P_a R P_a^{\mathsf{T}}\rangle = (p_i - p_j)^{\mathsf{T}} A (p_i - p_j) = 0$ for all $\{i,j\} \in B$ and for all $\{i,j\} \in C \cup S$ with $\Omega_{ij} \neq 0$. Using condition (iii), this implies that $A = 0$. Now, using (20) and the fact that $A = 0$, we obtain that

$$(Y - X_a)_{ij} = b^{\mathsf{T}} p_i + b^{\mathsf{T}} p_j + c \quad \text{for all } i,j \in [n].$$

From this follows that

$$\|q_i - q_j\|^2 = Y_{ii} + Y_{jj} - 2Y_{ij} = (X_a)_{ii} + (X_a)_{jj} - 2(X_a)_{ij} = \|p_i - p_j\|^2$$

for all $i,j \in [n]$, thus showing that $G(\mathbf{p})$ and $G(\mathbf{q})$ are congruent. □

Notice that the assumptions of the theorem imply that $n \geq d+1$. Moreover, for $n = d+1$ we get that Ω is the zero matrix in which case (18) is satisfied only for $G = K_n$ and $C = S = \emptyset$. Hence Theorem 4.4 is useful only in the case when $n \geq d+2$.

There is a natural pair of primal and dual semidefinite programs attached to a given tensegrity framework $G(\mathbf{p})$:

$$\sup_X \{0 : X \succeq 0, \quad \begin{aligned} &\langle F_{ij}, X\rangle = \|p_i - p_j\|^2 \text{ for } \{i,j\} \in B, \\ &\langle F_{ij}, X\rangle \leq \|p_i - p_j\|^2 \text{ for } \{i,j\} \in C, \\ &\langle F_{ij}, X\rangle \geq \|p_i - p_j\|^2 \text{ for } \{i,j\} \in S\}, \end{aligned} \tag{21}$$

$$\inf_{y,Z} \{\textstyle\sum_{ij\in E} y_{ij}\|p_i - p_j\|^2 : \quad \begin{aligned} &Z = \textstyle\sum_{ij\in E} y_{ij} F_{ij} \succeq 0, \\ &y_{ij} \geq 0 \text{ for } \{i,j\} \in C, \\ &y_{ij} \leq 0 \text{ for } \{i,j\} \in S\}. \end{aligned} \tag{22}$$

The feasible (optimal) solutions of the primal program (21) correspond to the frameworks $G(\mathbf{q})$ that are dominated by $G(\mathbf{p})$, while the optimal solutions to the dual program (22) correspond to the positive semidefinite equilibrium stress matrices for the tensegrity framework $G(\mathbf{p})$.

Both matrices $X = PP^{\mathsf{T}}$ and $X_a = P_a P_a^{\mathsf{T}}$ (defined in the proof of Theorem 4.4) are primal optimal, with $\operatorname{rank} X = d$ and $\operatorname{rank} X_a = d+1$. Hence, a psd equilibrium stress matrix Ω satisfies the conditions (i) and (ii) of Theorem 4.4 precisely when the pair (X_a, Ω) is a strict complementary pair of primal and dual optimal solutions.

In the case of bar frameworks (i.e., $C = S = \emptyset$), the condition (iii) of Theorem 4.4 expresses the fact that the matrix $X = \operatorname{Gram}(p_1, \ldots, p_n)$ is an extreme point of the feasible region of (21). Moreover, X_a lies in its relative interior (since $\operatorname{Ker} Y \supseteq \operatorname{Ker} X_a$ for any primal feasible Y, as shown in the above proof of Theorem 4.4)).

Remark 4.1. *In the terminology of Connelly, the condition (18) says that* the edge directions $p_i - p_j$ *of* $G(\mathbf{p})$ *for all edges* $\{i,j\} \in B$ *and all edges* $\{i,j\} \in C \cup S$ *with nonzero stress* $\Omega_{ij} \neq 0$ do not lie on a conic at infinity.

Observe that this condition cannot be omitted in Theorem 4.4. This is illustrated by the following example, taken from [3]. Consider the graph G on 4 nodes with edges $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ and $\{2,4\}$, and the 2-dimensional bar framework $G(\mathbf{p})$ given by

$$p_1 = (-1,0)^\mathsf{T},\ p_2 = (0,0)^\mathsf{T},\ p_3 = (1,0)^\mathsf{T} \text{ and } p_4 = (0,1)^\mathsf{T}.$$

Clearly, the framework $G(\mathbf{p})$ is not universally rigid (as one can rotate p_4 and get a new framework, which is equivalent but not congruent to $G(\mathbf{p})$). On the other hand, the matrix $\Omega = (1,-2,1,0)(1,-2,1,0)^\mathsf{T}$ is the only equilibrium stress matrix for $G(\mathbf{p})$, it is positive semidefinite with corank 3. Observe however that the condition (18) does not hold (since the nonzero matrix $R = e_1e_2^\mathsf{T} + e_2e_1^\mathsf{T}$ satisfies $(p_i - p_j)^\mathsf{T} R(p_i - p_j) = 0$ for all $\{i,j\} \in E$).

4.2. Generic universally rigid frameworks

It is natural to ask for a converse of Theorem 4.4. This question has been settled recently in [16] in the affirmative for generic frameworks (cf. Theorem 4.8). First, we show that, for generic frameworks, the 'no conic at infinity' condition (18) can be omitted since it holds automatically. This result was obtained in [11] (Proposition 4.3), but for the sake of completeness we have included a different and more explicit argument.

We need some notation. Given a framework $G(\mathbf{p})$ in $\mathbb{R}^k$, we let $\mathcal{P}_\mathbf{p}$ denote the $\binom{k+1}{2} \times |E|$ matrix, whose ij-th column contains the entries of the upper triangular part of the matrix $(p_i - p_j)(p_i - p_j)^\mathsf{T} \in \mathcal{S}^k$. For a subset $I \subseteq E$, $\mathcal{P}_\mathbf{p}(I)$ denotes the $\binom{k+1}{2} \times |I|$ submatrix of $\mathcal{P}_\mathbf{p}$ whose columns are indexed by edges in I.

Lemma 4.5. *Let $k \in \mathbb{N}$ and let $G = ([n], E)$ be a graph on $n \geq k+1$ nodes and with minimum degree at least k. Define the polynomial $\pi_{k,G}$ in kn variables by*

$$\pi_{k,G}(\mathbf{p}) = \sum_{I \subseteq E, |I| = \binom{k+1}{2}} (\det \mathcal{P}_\mathbf{p}(I))^2$$

for $\mathbf{p} = \{p_1, \ldots, p_n\} \subseteq (\mathbb{R}^k)^n$. Then, the polynomial $\pi_{k,G}$ has integer coefficients and it is not identically zero.

Proof. Notice that for the specific choice of parameters we have that $|E| \geq \frac{nk}{2} \geq \frac{(k+1)k}{2}$. It is clear that $\pi_{k,G}$ has integer coefficients. We show by induction on $k \geq 2$ that for every graph $G = ([n], E)$ with $n \geq k+1$ nodes and minimum degree at least k the polynomial $\pi_{k,G}$ is not identically zero.

For $k = 2$, we distinguish two cases: (i) $n = 3$ and (ii) $n \geq 4$. In case (i), $G = K_3$ and, for the vectors $p_1 = (0,0)^\mathsf{T}, p_2 = (1,0)^\mathsf{T}, p_3 = (0,1)^\mathsf{T}$, we have that $\pi_{2,G}(\mathbf{p}) \neq 0$. In case (ii), we can now assume without loss of generality that the edge set contains the following subset $I = \{\{1,2\},\{1,3\},\{2,4\}\}$. For the vectors $p_1 = (0,0)^\mathsf{T}, p_2 = (1,0)^\mathsf{T}, p_3 = (0,1)^\mathsf{T}, p_4 = (2,1)^\mathsf{T}$, we have that $\det \mathcal{P}_\mathbf{p}(I) \neq 0$ and thus $\pi_{2,G}(\mathbf{p}) \neq 0$.

Let $k \geq 3$ and consider a graph $G = ([n], E)$ with $n \geq k+1$ and minimum degree at least k. Let $G \setminus n$ be the graph obtained from G by removing node n and all edges adjacent to it. Then, $G \setminus n$ has at least k nodes and minimum degree at least $k-1$. Hence, by the induction hypothesis, the polynomial $\pi_{k-1,G\setminus n}$ is not identically zero. Let $\mathbf{p} = \{p_1, \ldots, p_{n-1}\} \subseteq \mathbb{R}^{k-1}$ be a generic set of vectors and define $\tilde{\mathbf{p}} = \{\tilde{p}_1, \ldots, \tilde{p}_n\} \subseteq \mathbb{R}^k$, where $\tilde{p}_i = (p_i^\mathsf{T}, 0)^\mathsf{T} \in \mathbb{R}^k$ for $1 \leq i \leq n-1$ and $\tilde{p}_n = (\mathbf{0}, 1)^\mathsf{T} \in \mathbb{R}^k$. As $\mathbf{p}$ is generic, $\pi_{k-1,G\setminus n}(\mathbf{p}) \neq 0$ and thus $\det \mathcal{P}_\mathbf{p}(I) \neq 0$ for some subset $I \subseteq E(G \setminus n)$ with $|I| = \binom{k}{2}$. Say, node n is adjacent to the nodes $1, \ldots, k$ in G and define the edge subset $\tilde{I} = I \cup \{\{n,1\}, \ldots, \{n,k\}\} \subseteq E$. Then, the matrix $\mathcal{P}_{\tilde{\mathbf{p}}}(\tilde{I})$ has the block-form

$$\mathcal{P}_{\tilde{\mathbf{p}}}(\tilde{I}) = \begin{pmatrix} \overbrace{\begin{matrix} & \mathcal{P}_\mathbf{p}(I) & \\ \mathbf{0} & \ldots & \mathbf{0} \\ 0 & \ldots & 0 \end{matrix}}^{\binom{k}{2}} & \overbrace{\begin{matrix} * & \ldots & * \\ -p_1 & \ldots & -p_k \\ 1 & \ldots & 1 \end{matrix}}^{k} \end{pmatrix}.$$

As the vectors $p_1, \ldots, p_{n-1} \in \mathbb{R}^{k-1}$ were chosen to be generic, every k of them are affinely independent. This implies that the vectors $(-p_1^\mathsf{T}, 1)^\mathsf{T}, \ldots (-p_k^\mathsf{T}, 1)^\mathsf{T}$ are linearly independent. Hence, $\det \mathcal{P}_{\tilde{\mathbf{p}}}(\tilde{I}) \neq 0$ and thus $\pi_{k,G}(\tilde{\mathbf{p}}) \neq 0$. $\square$

Theorem 4.6. *[11] Let $G(\mathbf{p})$ be a generic d-dimensional framework and assume that G has minimum degree at least d. Then the edge directions of $G(\mathbf{p})$ do not lie on a conic at infinity; that is, the system $\{(p_i - p_j)(p_i - p_j)^\mathsf{T} : \{i,j\} \in E\} \subseteq \mathcal{S}^d$ has full rank $\binom{d+1}{2}$.*

Proof. As the framework $G(\mathbf{p})$ is d-dimensional, G must have at least $d+1$ nodes. By Lemma 4.5, the polynomial $\pi_{d,G}$ is not identically zero and thus, since $G(\mathbf{p})$ is generic, we have that $\pi_{d,G}(\mathbf{p}) \neq 0$. By definition of $\pi_{d,G}$ there exists $I \subseteq E$ with $|I| = \binom{d+1}{2}$ such that $\det \mathcal{P}_\mathbf{p}(I) \neq 0$. This implies that the system $\{(p_i - p_j)(p_i - p_j)^\mathsf{T} : \{i,j\} \in E\} \subseteq \mathcal{S}^d$ has full rank $\binom{d+1}{2}$. $\square$

Next we show that for *generic* frameworks Theorem 4.4 remains valid even when (18) is omitted.

Corollary 4.7. *[11] Let $G(\mathbf{p})$ be a generic d-dimensional tensegrity framework. Assume that there exists a positive semidefinite equilibrium stress matrix Ω with corank $d+1$. Then $G(p)$ is universally rigid.*

Proof. Set $E_0 = \{\{i,j\} \in E : \Omega_{ij} \neq 0\}$ and define the subgraph $G_0 = ([n], E_0)$ of G. First we show that G_0 has minimum degree at least d. For this, we use the equilibrium conditions: For all $i \in [n]$, $\sum_{j:\{i,j\} \in E_0} \Omega_{ij} p_j = 0$, which give an affine dependency among the vectors p_i and p_j for $\{i,j\} \in E_0$. By assumption, $\mathbf{p}$ is generic and thus in general position, which implies that any $d+1$ of the vectors $p_1, \ldots, p_n$ are affinely dependent. From this we deduce that each node $i \in [n]$ has degree at least d in G_0.

Hence we can apply Theorem 4.6 to the generic framework $G_0(\mathbf{p})$ and conclude that the system $\{(p_i - p_j)(p_i - p_j)^\mathsf{T} : \{i,j\} \in E_0\}$ has full rank $\binom{d+1}{2}$. This shows that the condition (18) holds. Now we can apply Theorem 4.4 to $G(\mathbf{p})$ and conclude that $G(\mathbf{p})$ is universally rigid. □

We note that for bar frameworks this fact has been also obtained independently by A. Alfakih using the related concepts of *dimensional rigidity* and *Gale matrices*. The notion of dimensional rigidity was introduced in [1] where a sufficient condition was obtained for showing that a framework is dimensionally rigid. In [2], using the concept of a Gale matrix, this condition was shown to be equivalent to the sufficient condition from Theorem 4.4 (for bar frameworks). Lastly, in [2] it is shown that for generic frameworks the notions of dimensional rigidity and universal rigidity coincide.

In the special case of bar frameworks, the converse of Corollary 4.7 was proved recently by S.J. Gortler and P. Thurston.

Theorem 4.8. *[16] Let $G(\mathbf{p})$ be a generic d-dimensional bar framework and assume that it is universally rigid. Then there exists a positive semidefinite equilibrium stress matrix Ω for $G(\mathbf{p})$ with corank $d+1$.*

4.3. Connections with unique completability

In this section we investigate the links between the two notions of universally completable and universally rigid tensegrity frameworks. We start the discussion with defining the suspension of a tensegrity framework.

Definition 4.9. *Let $G = (V = [n], E)$ be a tensegrity graph with $E = B \cup C \cup S$. We denote by $\nabla G = (V \cup \{0\}, E')$ its* suspension tensegrity graph*, with $E' = B' \cup C' \cup S'$ where $B' = B \cup \{\{0,i\} : i \in [n]\}$, $C' = S$ and $S' = C$. Given a tensegrity framework $G(\mathbf{p})$, we define the* extended tensegrity framework $\nabla G(\widehat{\mathbf{p}})$ *where $\widehat{p}_i = p_i$ for all $i \in [n]$ and $\widehat{p}_0 = \mathbf{0}$.*

Our first observation is a correspondence between the universal completability of a tensegrity framework $G(\mathbf{p})$ and the universal rigidity of its extended tensegrity framework $\nabla G(\widehat{\mathbf{p}})$. The analogous observation in the setting of global rigidity has been also made in [12] and [35].

Lemma 4.10. *Let $G(\mathbf{p})$ be a tensegrity framework and let $\nabla G(\widehat{p})$ be its extended tensegrity framework as defined in Definition 4.9. Then, the tensegrity framework $G(\mathbf{p})$ is universally completable if and only if the extended tensegrity framework $\nabla G(\widehat{\mathbf{p}})$ is universally rigid.*

Proof. Notice that for any family of vectors $q_1, \ldots, q_n$, their Gram matrix satisfies the conditions:

$$\begin{aligned}
\langle E_{ij}, X\rangle &= p_i^\mathsf{T} p_j \quad \text{for all } \{i,j\} \in V \cup B,\\
\langle E_{ij}, X\rangle &\le p_i^\mathsf{T} p_j \quad \text{for all } \{i,j\} \in C,\\
\langle E_{ij}, X\rangle &\ge p_i^\mathsf{T} p_j \quad \text{for all } \{i,j\} \in S,
\end{aligned}$$

if and only if the Gram matrix of $q_0 = \mathbf{0}, q_1, \ldots, q_n$ satisfies:

$$\begin{aligned}
\langle F_{ij}, X\rangle &= \|p_i - p_j\|^2 \text{ for all } \{i,j\} \in B',\\
\langle F_{ij}, X\rangle &\le \|p_i - p_j\|^2 \text{ for all } \{i,j\} \in C',\\
\langle F_{ij}, X\rangle &\ge \|p_i - p_j\|^2 \text{ for all } \{i,j\} \in S',
\end{aligned}$$

which implies the claim. □

In view of Lemma 4.10 it is reasonable to ask whether Theorem 3.2 can be derived from Theorem 4.4 applied to the tensegrity framework $\nabla G(\widehat{\mathbf{p}})$. We will show that this is the case for bar frameworks, i.e., when $C = S = \emptyset$. Indeed, for a bar framework, the condition (18) from Theorem 4.4 applied to the suspension tensegrity framework $\nabla G(\widehat{\mathbf{p}})$ becomes

$$R \in \mathcal{S}^d,\ (p_i - p_j)^{\mathsf{T}} R\, (p_i - p_j) = 0 \text{ for all } \{i,j\} \in E \cup \{\{0,i\} : i \in [n]\} \Longrightarrow R = 0,$$

and, as $\widehat{p}_0 = 0$, this coincides with the condition (12).

The following lemma shows that for bar frameworks there exists a one to one correspondence between equilibrium stress matrices for $\nabla G(\widehat{\mathbf{p}})$ and spherical stress matrices for $G(\mathbf{p})$. The crucial fact that we use here is that for bar frameworks there are no sign conditions for a spherical stress matrix for $G(\mathbf{p})$ or for an equilibrium stress matrix for $\nabla G(\widehat{\mathbf{p}})$.

Lemma 4.11. *Let $G(\mathbf{p})$ be a bar framework in $\mathbb{R}^d$ such that $p_1, \ldots, p_n$ span linearly $\mathbb{R}^d$. The following assertions are equivalent:*

- *(i) There exists an equilibrium stress matrix $\Omega \in \mathcal{S}^{n+1}_+$ for the framework $\nabla G(\widehat{\mathbf{p}})$ with* $\operatorname{corank}\Omega = d + 1$.
- *(ii) There exists a spherical stress matrix for $G(\mathbf{p})$.*

Proof. Let $P \in \mathbb{R}^{n\times d}$ be the configuration matrix of the framework $G(\mathbf{p})$ and let $\widehat{P}_a = \begin{pmatrix} \mathbf{0} & 1 \\ P & e \end{pmatrix}$. Write a matrix $\Omega \in \mathcal{S}^{n+1}_+$ in block-form as

$$\Omega = \begin{pmatrix} w_0 & w^{\mathsf{T}} \\ w & Z \end{pmatrix} \quad \text{where } Z \in \mathcal{S}^n_+, w \in \mathbb{R}^n, w_0 \in \mathbb{R}. \tag{23}$$

Notice that Ω is supported by ∇G precisely when Z is supported by G. The matrix Ω is a stress matrix for $\nabla G(\widehat{\mathbf{p}})$ if and only if $\Omega \widehat{P}_a = 0$ which is equivalent to

$$ZP = 0,\ w = -Ze,\ w_0 = -w^{\mathsf{T}} e. \tag{24}$$

Moreover, $\operatorname{Ker}\Omega = \operatorname{Ran}\widehat{P_a}$ if and only if $\operatorname{Ker} Z = \operatorname{Ran} P$, so that $\operatorname{corank}\Omega = d+1$ if and only if $\operatorname{corank} Z = d$. The lemma now follows easily: If Ω satisfies (i), then its principal submatrix Z satisfies (ii). Conversely, if Z satisfies (ii), then the matrix Ω defined by (23) and (24) satisfies (i). □

Summarizing, we have established that in the special case of a bar framework $G(\mathbf{p})$ (i.e., $C = S = \emptyset$), Theorem 3.2 is equivalent to Theorem 4.4 applied to the extended bar framework $\nabla G(\widehat{\mathbf{p}})$. It is not clear whether this equivalence remains valid for arbitrary tensegrity frameworks. To deal with such frameworks, Lemma 4.11 has to be generalized so as to accommodate the sign conditions for the sperical stress matrix and the equilibrium stress matrix for $G(\mathbf{p})$ and $\nabla G(\widehat{\mathbf{p}})$, respectively.

5. The Strong Arnold Property and graph parameters

In this section we revisit the Strong Arnold Property (SAP) and we show that matrices fulfilling the SAP posses some nice geometric properties. We also show that psd matrices fulfilling the SAP can be characterized as nondegenerate solutions of some appropriate semidefinite program. Additionally, we investigate the relation between the graph parameters $\nu^{=}(\cdot)$ and $\mathrm{gd}(\cdot)$, introduced in [18, 19] and [24, 25], respectively.

5.1. The Strong Arnold Property

For a graph $G = (V = [n], E)$ consider the linear space

$$\mathcal{C}(G) = \{X \in \mathcal{S}^n : \langle E_{ij}, X\rangle = 0 \;\; \forall \{i,j\} \in \overline{E}\}.$$

Definition 5.1. *For a graph $G = ([n], E)$, a matrix $M \in \mathcal{C}(G)$ is said to satisfy the* Strong Arnold Property (SAP) *if*

$$\mathcal{T}_M + \mathrm{lin}\{E_{ij} : \{i,j\} \in V \cup E\} = \mathcal{S}^n. \tag{25}$$

The SAP has received a significant amount of attention due to its connection to the Colin de Verdière graph parameter $\mu(\cdot)$, introduced and studied in [7]. The *Colin de Verdière number* $\mu(G)$ of a graph G is defined as the maximum corank of a matrix $M \in \mathcal{C}(G)$ satisfying: $\langle E_{ij}, M\rangle < 0$ for all $\{i,j\} \in E$, M has exactly one negative eigenvalue, and M satisfies the SAP. The graph parameter $\mu(\cdot)$ is minor monotone, and it turns out that the SAP plays a crucial role for showing this. The importance of the graph parameter $\mu(G)$ stems in particular from the fact that it permits to characterize several topological properties of graphs. For instance, it is known that $\mu(G) \leq 3$ if and only if G is planar [7] and $\mu(G) \leq 4$ if and only of G is linklessly embeddable [27] (more details can be found e.g. in [21] and further references therein).

By taking orthogonal complements in (25) and using (6), we arrive at the following equivalent expression for the SAP, that we will use in the sequel:

$$X \in \mathcal{S}^n,\; MX = 0,\; X_{ij} = 0 \text{ for all } \{i,j\} \in V \cup E \Longrightarrow X = 0. \tag{26}$$

Our next goal is to give a geometric characterization of matrices satisfying the SAP using the notion of null space representations. Consider a matrix $M \in \mathcal{S}^n$, fix an arbitrary basis for $\mathrm{Ker}\, M$ and form the $n \times \mathrm{corank}\, M$ matrix that has as columns the basis elements. The vectors corresponding to the rows of the resulting matrix form a *nullspace representation* of M. If we impose structure on M in terms of some graph G, nullspace representations of M exhibit intriguing geometric properties and have been extensively studied (see e.g. [28]).

The next theorem shows that null space representations of matrices satisfying the SAP enjoy some nice geometric properties. The equivalence between and the first and the third item item has been rediscovered independently by [20] (Theorem 4.2) and [15] (Lemma 3.1).

Theorem 5.2. *Consider a graph $G = ([n], E)$ and a matrix $M \in \mathcal{C}(G)$ with $\operatorname{corank} M = d$. Let $P \in \mathbb{R}^{n\times d}$ be a matrix whose columns form an orthonormal basis for $\operatorname{Ker} M$ and let $\{p_1, \ldots, p_n\}$ denote the row vectors of P. The following assertions are equivalent:*

(i) M satisfies the Strong Arnold Property.

(ii) PP^T is an extreme point of the spectrahedron

$$\{X \succeq 0 : \langle E_{ij}, X\rangle = p_i^\mathsf{T} p_j \text{ for } \{i, j\} \in V \cup E\}.$$

(iii) For any matrix $R \in \mathcal{S}^d$ the following holds:

$$p_i^\mathsf{T} R p_j = 0 \text{ for all } \{i, j\} \in V \cup E \Longrightarrow R = 0.$$

Proof. The equivalence $(ii) \Longleftrightarrow (iii)$ follows directly from Corollary 2.2.
$(i) \Longrightarrow (iii)$ Let $R \in \mathcal{S}^d$ such that $p_i^\mathsf{T} R p_j = 0$, i.e., $\langle PRP^\mathsf{T}, E_{ij}\rangle = 0$ for all $\{i, j\} \in V \cup E$. Thus the matrix $Y = PRP^\mathsf{T}$ belongs to $\operatorname{lin}\{E_{ij} : \{i, j\} \in V \cup E\}^\perp$ and satisfies $MY = 0$. By (6) we have that $Y \in \mathcal{T}_M^\perp$ and then (i) implies $Y = 0$ and thus $R = 0$ (since $P^\mathsf{T} P = I_r$).
$(iii) \Longrightarrow (i)$ Write $M = Q \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} Q^\mathsf{T}$, where $Q = (Q_1\ P)$ is orthogonal and the columns of Q_1 form a basis of the range of M. Consider a matrix $Y \in \mathcal{T}_M^\perp \cap \operatorname{lin}\{E_{ij} : \{i, j\} \in \overline{E}\}$. Then, by (5), $Y = PRP^\mathsf{T}$ for some matrix $R \in \mathcal{S}^d$. Moreover, $\langle Y, E_{ij}\rangle = \langle PRP^\mathsf{T}, E_{ij}\rangle = 0$ for all $\{i, j\} \in V \cup E$, which by (iii) implies that $R = 0$ and thus $Y = 0$. □

Our final observation in this section is that a psd matrix having the SAP can be also understood as a nondegenerate solution of some semidefinite program.

Theorem 5.3. *Consider a graph $G = ([n], E)$ and let $M \in \mathcal{C}(G) \cap \mathcal{S}_n^+$. The following assertions are equivalent:*

(i) M satisfies the Strong Arnold Property.

(ii) M is a primal nondegenerate solution for the semidefinite program:

$$\sup_X \{\langle C, X\rangle : \langle E_{ij}, X\rangle = 0 \text{ for } \{i, j\} \in \overline{E},\ X \succeq 0\},$$

for any $C \in \mathcal{S}^n$.

(iii) M *is a dual nondegenerate solution for the dual of the semidefinite program:*

$$\sup_X \{0 : \langle E_{ij}, X\rangle = a_{ij} \text{ for } \{i,j\} \in V \cup E,\ X \succeq 0\}, \tag{27}$$

for any $a \in \mathcal{S}_+(G)$.

Proof. Taking orthogonal complements in (25) we see that M satisfies the SAP if and only if ${\mathcal{T}_M}^{\perp} \cap \text{lin}\{E_{ij} : \{i,j\} \in \overline{E}\} = \{0\}$. Moreover, observe that the feasible region of the dual of the semidefinite program (27) is equal to $\mathcal{S}^n_+ \cap \mathcal{C}(G)$. Now, using (7), we obtain the equivalence of $(i), (ii)$ and (iii).

5.2. Graph parameters

In this section we explore the relation between the two graph parameters $\text{gd}(\cdot)$ and $\nu^{=}(\cdot)$ using the machinery developed in the previous sections. Recall that $\mathcal{S}_+(G)$ denotes the set of G-partial psd matrices.

Definition 5.4. *Given a graph* $G = (V, E)$*, a vector* $a \in \mathcal{S}_+(G)$ *and an integer* $k \geq 1$*, a* Gram representation *of* a *in* $\mathbb{R}^k$ *consists of a set of vectors* $p_1, \dots, p_n \in \mathbb{R}^k$ *such that*

$$p_i^{\mathsf{T}} p_j = a_{ij} \quad \text{for all } \{i,j\} \in V \cup E.$$

The Gram dimension of $a \in \mathcal{S}_+(G)$*, denoted as* $\text{gd}(G, a)$*, is the smallest integer* $k \geq 1$ *for which* a *has a Gram representation in* $\mathbb{R}^k$*.*

Definition 5.5. *The* Gram dimension *of a graph* G *is defined as*

$$\text{gd}(G) = \max_{a \in \mathcal{S}_+(G)} \text{gd}(G, a). \tag{28}$$

This graph parameter was introduced and studied in [24, 25], motivated by its relevance to the low rank positive semidefinite matrix completion problem. Indeed, if G is a graph satisfying $\text{gd}(G) \leq k$, then every G-partial psd matrix also has a psd completion of rank at most k. In [24, 25] the graph parameter $\text{gd}(\cdot)$ is shown to be minor monotone and the graphs with small Gram dimension are characterized: $\text{gd}(G) \leq 2 \iff G$ is a forest (no K_3 minor), $\text{gd}(G) \leq 3 \iff G$ is series-parallel (no K_4 minor), $\text{gd}(G) \leq 4 \iff G$ has no K_5 and $K_{2,2,2}$ minors.

Next we recall the definition of the graph parameter $\nu^{=}(\cdot)$.

Definition 5.6. *[18, 19] Given a graph* $G = ([n], E)$ *the parameter* $\nu^{=}(G)$ *is defined as the maximum corank of a matrix* $M \in \mathcal{C}(G) \cap \mathcal{S}^n_+$ *satisfying the SAP.*

The study of the parameter $\nu^{=}(\cdot)$ is motivated by its relation to the Colin de Verdière graph parameter $\mu(\cdot)$ mentioned above; for instance, $\mu(G) \leq \nu^{=}(G)$ for any graph. In [18, 19] it is shown that $\nu^{=}(\cdot)$ is minor monotone and the graphs with small value of $\nu^{=}(\cdot)$ are characterized: $\nu^{=}(G) \leq 2 \iff G$ is a forest (no K_3 minor), $\nu^{=}(G) \leq 3 \iff G$ is series-parallel (no K_4 minor), $\nu^{=} \leq 4 \iff G$ has no K_5 and $K_{2,2,2}$ minors.

In view of the above two characterizations it is natural to try to identify the exact relation between these two graph parameters. The following theorem is a first result in this direction.

Theorem 5.7. *[24] For any graph G,* $\mathrm{gd}(G) \geq \nu^{=}(G)$.

It is not known whether the two graph parameters coincide or not. We now derive a new characterization of the parameter $\nu^{=}(\cdot)$ in terms of the maximum Gram dimension of certain G-partial psd matrices satisfying some nondegeneracy property, which could be helpful to clarify the links between the two parameters. Recall that with a vector $a \in \mathcal{S}_+(G)$ we can associate the following pair of primal and dual semidefinite programs:

$$\sup_X \{0 : \langle E_{ij}, X\rangle = a_{ij} \;\text{ for } \{i,j\} \in V \cup E, \text{ and } X \succeq 0\}, \tag{P_a}$$

$$\inf_{y,Z} \{ \sum_{\{i,j\}\in V\cup E} y_{ij} a_{ij} : \sum_{\{i,j\}\in V\cup E} y_{ij} E_{ij} = Z \succeq 0\}. \tag{D_a}$$

Notice that, for any $a \in \mathcal{S}_+(G)$, the primal program (P_a) is feasible and the dual program (D_a) is strictly feasible. Thus there is no duality gap.

Definition 5.8. *Given a graph G, let $\mathcal{D}(G)$ denote the set of partial matrices $a \in \mathcal{S}_+(G)$ for which the semidefinite program (D_a) has a nondegenerate optimal solution.*

We can now reformulate the parameter $\nu^{=}(G)$ as the maximum Gram dimension of a partial matrix in $\mathcal{D}(G)$.

Theorem 5.9. *For any graph G we have that*

$$\nu^{=}(G) = \max_{a\in\mathcal{D}(G)} \mathrm{gd}(G,a).$$

Proof. Suppose that $\max_{a\in\mathcal{D}(G)} \mathrm{gd}(G,a) = \mathrm{gd}(G,a^*)$. As $a^* \in \mathcal{D}(G)$ it follows that (D_{a^*}) has a nondegenerate optimal solution which we denote by M. Then, Theorem 2.4 implies that (P_{a^*}) has a unique solution which we denote by A. Notice that the matrix A is the unique psd completion of the partial matrix $a^* \in \mathcal{S}_+(G)$ which implies that $\mathrm{gd}(G,a^*) = \operatorname{rank} A$. Moreover, as A and Z are a pair of primal dual optimal solutions we have that $AM = 0$ which implies that $\operatorname{corank} M \geq \operatorname{rank} A$. As the matrix M is feasible for $\nu^{=}(G)$ (recall Definition 5.6 and Theorem 5.3) it follows that $\nu^{=}(G) \geq \max_{a\in\mathcal{D}(G)} \mathrm{gd}(G,a)$.

For the other direction, assume $\nu^{=}(G) = \operatorname{corank} M = d$ where $M \in \mathcal{C}(G) \cap \mathcal{S}^n_+$ and M satisfies the SAP. Let $P \in \mathbb{R}^{n\times d}$ be a matrix whose columns form a basis for $\operatorname{Ker} M$ and consider the partial matrix $a \in \mathcal{S}_+(G)$ defined as $a_{ij} = (PP^{\mathsf{T}})_{ij}$ for every $\{i,j\} \in V \cup E$. As $\langle M, PP^{\mathsf{T}}\rangle = 0$ it follows that M is a dual nondegenerate optimal solution for (D_a) and thus $a \in \mathcal{D}(G)$. Additionally, as $\operatorname{corank} M = \operatorname{rank} PP^{\mathsf{T}}$ we have that M and PP^{T} are a pair of strict complementary optimal solutions for (P_a) and (D_a), respectively. Then Theorem 2.6 implies that the matrix PP^{T} is the unique optimal solution of (P_a) and thus $\mathrm{gd}(G,a) = \operatorname{rank} PP^{\mathsf{T}} = \operatorname{corank} M = \nu^{=}(G)$. □

Corollary 5.10. *For any graph G, we have that* $\mathrm{gd}(G) \geq \nu^{=}(G)$*. Moreover, equality* $\mathrm{gd}(G) = \nu^{=}(G)$ *holds if and only if there exists some* $a \in \mathcal{D}(G)$ *for which* $\mathrm{gd}(G) = \mathrm{gd}(G, a)$*.*

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