Computing the Grothendieck constant of some graph classes

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Abstract

Given a graph $G = ([n], E)$ and $w \in \mathbb{R}^E$, consider the integer program $\max_{x \in \{\pm 1\}^n} \sum_{ij \in E} w_{ij} x_i x_j$ and its canonical semidefinite programming relaxation $\max \sum_{ij \in E} w_{ij} v_i^T v_j$, where the maximum is taken over all unit vectors $v_i \in \mathbb{R}^n$. The integrality gap of this relaxation is known as the Grothendieck constant $\kappa(G)$ of G. We present a closed-form formula for the Grothendieck constant of K_5-minor free graphs and derive that it is at most $3/2$. Moreover, we show that $\kappa(G) \leq \kappa(K_k)$ if the cut polytope of G is defined by inequalities supported by at most k points. Lastly, since the Grothendieck constant of K_n grows as $\Theta(\log n)$, it is interesting to identify instances with large gap. However this is not the case for the clique-web inequalities, a wide class of valid inequalities for the cut polytope, whose integrality ratio is shown to be bounded by 3.

Keywords: Grothendieck constant, elliptope, cut polytope, clique-web inequality

1. Introduction

Let $G = ([n], E)$ be a (simple loopless) graph and $w = (w_{ij}) \in \mathbb{R}^E$. Consider the integer quadratic program over the hypercube

$$\text{ip}(G, w) := \max_{x \in \{\pm 1\}^n} \sum_{ij \in E} w_{ij} x_i x_j, \qquad (1)$$

and its canonical semidefinite programming relaxation

$$\text{sdp}(G, w) := \max_{u_1, \ldots, u_n \in \mathbb{R}^n, \; \|u_i\| = 1} \sum_{ij \in E} w_{ij} u_i^T u_j. \qquad (2)$$

Let $\kappa(G)$ denote the integrality gap of relaxation (2), defined by

$$\kappa(G) = \sup_{w \in \mathbb{R}^E} \frac{\text{sdp}(G, w)}{\text{ip}(G, w)}. \qquad (3)$$

In other words, $\kappa(G)$ is the smallest constant $K > 0$ for which $\text{sdp}(G, w) \leq K \cdot \text{ip}(G, w)$, $\forall w \in \mathbb{R}^E$. Alon et al. [1] call this graph parameter the *Grothendieck constant* of G and prove that

$$\Omega(\log \omega(G)) = \kappa(G) = O(\log \vartheta(\bar{G})). \qquad (4)$$

Here $\omega(G)$ denotes the maximum size of a clique in G and $\vartheta(\bar{G})$ the Lovász theta function of the complementary graph $\bar{G}$, for which it is known that $\omega(G) \leq \vartheta(\bar{G}) \leq \chi(G)$ [16]. Hence, for the complete graph $G = K_n$, $\kappa(K_n) = \Theta(\log n)$.

The name of the constant goes back to Grothendieck [12], who considered the case of bipartite graphs and showed the existence of a constant $K > 0$ for which $\text{sdp}(K_{m,n}, w) \leq K \cdot \text{ip}(K_{m,n}, w)$ for all $m, n \in \mathbb{N}$ and $w \in \mathbb{R}^{mn}$. The smallest such constant is known as *Grothendieck's constant* and is denoted by K_G. It is a long standing open problem to compute the exact value of K_G. It is known that $K_G < \pi[2 \ln(1 + \sqrt{2})]^{-1} \sim 1.782$ [13, 7], and that $K_G \geq 1.6769...$ [20]. Recently Briët et al. [8] show that $\kappa(G) \leq \frac{\pi}{2 \, \text{arcsinh}((\vartheta(\bar{G}) - 1)^{-1})}$, which gives the above bound 1.782 for bipartite graphs and improves the upper bound in (4) when $\vartheta(\bar{G})$ is small.

In recent years, Grothendieck type inequalities have received a significant amount of attention due to their various applications, most notably in the design of approximation algorithms and quantum information theory (see, e.g., [1, 2, 7, 10, 17, 22]).

The paper is organized as follows. In Section 2 we collect basic properties of $\kappa(G)$. In Section 3 we establish a closed-form formula for the Grothendieck constant of K_5-minor free graphs in terms of their girth and bound $\kappa(G)$ in terms of the size of the

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supports of the facets of the cut polytope. In Section 4 we show that the integrality gap achieved by the clique-web inequalities, a wide class of valid inequalities for the cut-polytope, is bounded by 3.

2. Basic properties

We first introduce some notation. Throughout $[n] = \{1, \ldots, n\}$. Let $\mathcal{S}_n^+$ denote the cone of positive semidefinite matrices; the notation $A \succeq 0$ means that $A \in \mathcal{S}_n^+$. For matrices A, B, $\langle A, B \rangle = \sum_{i,j} A_{ij} B_{ij}$ stands for the usual trace inner product. Let e denote the all ones vector and $J = ee^T$ the all ones matrix, of the appropriate dimension.

Let $\mathcal{E}_n := \{X \in \mathcal{S}_n^+ \mid X_{ii} = 1 \ \forall i \in [n]\}$ and $\mathrm{CUT}_n := \mathrm{conv}(X \in \mathcal{E}_n \mid \mathrm{rank}\, X = 1)$. Moreover, define $\mathcal{E}(G) := \pi_E(\mathcal{E}_n)$, $\mathrm{CUT}(G) := \pi_E(\mathrm{CUT}_n)$, where π_E denotes the projection from $\mathbb{R}^{n \times n}$ onto the subspace $\mathbb{R}^E$ indexed by the edge set of G. They are known, respectively, as the *elliptope* and the *cut polytope* of G and satisfy $\mathrm{CUT}(G) \subseteq \mathcal{E}(G)$. We refer, e.g., to [9] and further references therein for a detailed study of these geometric objects.

For $w \in \mathbb{R}^E$, let $\kappa(G, w) = \mathrm{sdp}(G, w)/\mathrm{ip}(G, w)$.

2.1. A geometric reformulation for $\kappa(G)$

Clearly, the Grothendieck constant $\kappa(G)$ is the smallest dilation of $\mathrm{CUT}(G)$ containing $\mathcal{E}(G)$.

Lemma 2.1. *For any graph G,*

$$\kappa(G) = \min\{K \mid \mathcal{E}(G) \subseteq K \cdot \mathrm{CUT}(G)\}.$$

PROOF. Directly, since $\mathrm{ip}(G, w) = \max\limits_{x \in \mathrm{CUT}(G)} w^T x$ and $\mathrm{sdp}(G, w) = \max\limits_{x \in \mathcal{E}(G)} w^T x$. $\square$

As the origin lies in the interior of $\mathrm{CUT}(G)$, the polytope $\mathrm{CUT}(G)$ has a linear inequality description consisting of finitely many facet-defining inequalities of the form $w^T x \leq 1$. Let us recall the following *switching operation:* Given $w \in \mathbb{R}^E$, its switching by $S \subseteq [n]$ is the vector $w^{(S)} \in \mathbb{R}^E$ whose (i, j)-th entry is $-w_{ij}$ if the edge ij is cut by the partition $(S, [n] \setminus S)$ and w_{ij} otherwise. It is well known that the switching operation preserves valid inequalities and facet defining inequalities of the cut polytope [5, 9]. Moreover, $\mathrm{sdp}(G, w) = \mathrm{sdp}(G, w^{(S)})$ and $\mathrm{ip}(G, w) = \mathrm{ip}(G, w^{(S)})$. This implies the next lemma which gives a useful reformulation for $\kappa(G)$.

Lemma 2.2. *For any graph G,*

$$\kappa(G) = \sup_{w \in \mathbb{R}^E} \kappa(G, w),$$

where the supremum ranges over all facet defining inequalities of $\mathrm{CUT}(G)$, distinct up to switching.

2.2. Connections with max-cut

The study of the cut polytope $\mathrm{CUT}(G)$ and of the elliptope $\mathcal{E}(G)$ is largely motivated by their relevance to the maximum cut problem in combinatorial optimization. Given $G = ([n], E)$ and $w \in \mathbb{R}^E$, the max-cut problem asks for a cut of maximum weight. Thus we want to compute $\mathrm{mc}(G, w) = \max_{x \in \{\pm 1\}^n} \frac{1}{2} \sum_{ij \in E} w_{ij}(1 - x_i x_j) = \max_{x \in \{\pm 1\}^n} \frac{1}{4} x^T L_{G,w} x$. Here, $L_{G,w}$ is the Laplacian matrix, with (i, i)-th entry $\sum_j w_{ij}$ and with (i, j)-th entry $-w_{ij}$ if $ij \in E$ and 0 otherwise. The canonical semidefinite programming relaxation of max-cut (considered e.g. in [11]) is $\mathrm{sdp}_{\mathrm{GW}}(G, w) = \max_{X \in \mathcal{E}_n} \frac{1}{4} \langle L_{G,w}, X \rangle$. Hence the quadratic integer problem (1) and the max-cut problem are affine transforms of each other, and the same for their canonical semidefinite relaxations; namely, $\mathrm{mc}(G, w) = \frac{1}{2}(w(E) + \mathrm{ip}(G, -w))$ and $\mathrm{sdp}_{\mathrm{GW}}(G, w) = \frac{1}{2}(w(E) + \mathrm{sdp}(G, -w))$.

In particular, this implies that, given $w \in \mathbb{Q}^E$, deciding whether $\mathrm{ip}(G, w) = \mathrm{sdp}(G, w)$ is an NP-complete problem [18].

The following lemma is easy to verify.

Lemma 2.3. *Let $A \in \mathcal{S}_n^+$ and $B = \begin{pmatrix} 0 & A/2 \\ A/2 & 0 \end{pmatrix}$. Then, $\max\limits_{Z \in \mathcal{E}_{2n}} \langle B, Z \rangle = \max\limits_{X \in \mathcal{E}_n} \langle A, X \rangle$, and $\max\limits_{z \in \{\pm 1\}^{2n}} z^T B z = \max\limits_{x \in \{\pm 1\}^n} x^T A x$.*

When $w \geq 0$, $L_{G,w} \succeq 0$ and thus Lemma 2.3 implies that $\mathrm{sdp}_{\mathrm{GW}}(G, w) = \max\limits_{Z \in \mathcal{E}_{2n}} \langle B, Z \rangle$, where B is as in the lemma with $A/2 := L_{G,w}/8$. By the definition of the Grothendieck constant K_G, this implies that $\mathrm{sdp}_{\mathrm{GW}}(G, w) \leq K_G \cdot \mathrm{mc}(G, w)$. However, this approximation guarantee is not interesting since we know by [11] that $\mathrm{sdp}_{\mathrm{GW}}(G, w) \leq 1.138 \cdot \mathrm{mc}(G, w)$, while $K_G \geq 1.6$.

On the other hand, the Grothendieck constant $\kappa(G)$ bounds the semidefinite approximation for max-cut for edge weights satisfying $w(E) \geq 0$.

Lemma 2.4. *Let $G = (V, E)$ be a graph and $w \in \mathbb{R}^E$ with $w(E) \geq 0$ and $\mathrm{mc}(G, w) > 0$. Then, $\mathrm{sdp}_{GW}(G, w) \leq \kappa(G) \cdot \mathrm{mc}(G, w)$.*

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PROOF. Indeed, $\mathrm{sdp}(G, -w) \leq \kappa(G) \cdot \mathrm{ip}(G, -w)$ and $w(E) \leq \kappa(G) \cdot w(E)$ imply

$$\frac{\mathrm{sdp}_{\mathrm{GW}}(G, w)}{\mathrm{mc}(G, w)} = \frac{w(E) + \mathrm{sdp}(G, -w)}{w(E) + \mathrm{ip}(G, -w)} \leq \kappa(G).$$

□

2.3. Behaviour under graph operations

It follows immediately from the definition that the graph parameter $\kappa(\cdot)$ is monotone nonincreasing with respect to deleting edges. That is,

Lemma 2.5. *If $H \subseteq G$ then $\kappa(H) \leq \kappa(G)$.*

This is not true for the operation of contracting an edge. For instance, $\kappa(K_2) = 1 < \kappa(C_3) = 3/2$, while $\kappa(C_4) < \kappa(C_3) = 3/2$ (cf. Theorem 3.3). So $\kappa(G)$ and $\kappa(G/e)$ are not comparable in general.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ for which $V_1 \cap V_2$ is a clique in both G_1 and G_2, the graph $G = (V_1 \cup V_2, E_1 \cup E_2)$ is called their *clique sum*, or their *clique k-sum* when $|V_1 \cap V_2| = k$.

Lemma 2.6. *Assume G is the clique k-sum of G_1 and G_2, $k \leq 3$. Then, $\kappa(G) = \max(\kappa(G_1), \kappa(G_2))$.*

PROOF. Let $\lambda := \max(\kappa(G_1), \kappa(G_2))$ and $n = |V|$. The inequality $\kappa(G) \geq \lambda$ follows from Lemma 2.5.

For the other direction, let $x \in \mathcal{E}(G)$ and $X \in \mathcal{E}_n$ such that $x = \pi_E(X)$; we have to show that $x \in \lambda \cdot \mathrm{CUT}(G)$. Let X_i denote the principal submatrix of X indexed by V_i, for $i = 1, 2$. As $\mathcal{E}(G_i) \subseteq \kappa(G_i) \cdot \mathrm{CUT}(G_i) \subseteq \lambda \cdot \mathrm{CUT}(G_i)$, we deduce that $\pi_{E_i}(X_i) \in \lambda \cdot \mathrm{CUT}(G_i)$. Since the linear inequality description of $\mathrm{CUT}(G)$ is obtained by juxtaposing the linear inequality descriptions of $\mathrm{CUT}(G_1)$ and $\mathrm{CUT}(G_2)$ and identifying the variables corresponding to edges contained in $V_1 \cap V_2$ [4], the claim follows. □

3. Computing the Grothendieck constant for some graph classes

We start this section by introducing the main objects and some fundamental results associated with them, that form the basic ingredients of our approach.

A graph H is called a *minor* of a graph G, denoted by $H \preccurlyeq G$, if H can be obtained from G, through a series of edge deletions and edge contractions.

Given a graph $G = ([n], E)$, consider the *metric polytope* $\mathrm{MET}(G) \subseteq \mathbb{R}^E$ defined by the inequalities $-1 \leq x_e \leq 1$ for $e \in E$, and

$$x(C \setminus F) - x(F) \leq |C| - 2,$$

for every circuit C in G and $F \subseteq C$ with $|F|$ odd [9]. Additionally, define $\mathrm{MET}^{01}(G) := f(\mathrm{MET}(G))$, where $f(x) = e - 2x$ for $x \in \mathbb{R}^E$. The cut and metric polytopes are related as follows.

Theorem 3.1. *[5] For any graph G, $\mathrm{CUT}(G) \subseteq \mathrm{MET}(G)$, with equality if and only if $K_5 \npreccurlyeq G$.*

Any matrix $X \in \mathcal{E}_n$ has its diagonal entries all equal to 1. Hence all its entries lie in $[-1, 1]$ and can thus be parametrized as $x_{ij} = \cos(\pi y_{ij})$ where $y_{ij} \in [0, 1]$. Let $\cos(\pi \mathrm{MET}^{01}(G)) =$

$$\{(\cos(\pi y_e))_{e \in E} \mid y \in \mathrm{MET}^{01}(G)\}.$$

Theorem 3.2. *[14] $\mathcal{E}(G) \subseteq \cos(\pi \mathrm{MET}^{01}(G))$, with equality if and only if $K_4 \npreccurlyeq G$.*

Thus, equality holds when $G = C_n$. Moreover,

Lemma 3.1. *[6] For p even, we have $ce^T \in \mathcal{E}(C_p)$ for all $c \in [-1, 1]$. For p odd, we have $ce^T \in \mathcal{E}(C_p)$ if and only if $-\cos \frac{\pi}{p} \leq c \leq 1$.*

3.1. The case of circuits

Using the parametrizations of $\mathrm{MET}(C_n)$ and $\mathcal{E}(C_n)$ given by Theorems 3.1 and 3.2, respectively, we are able to compute $\kappa(C_n)$. Specifically,

Theorem 3.3. *The Grothendieck constant of a circuit C_n of length $n \geq 3$ is equal to*

$$\kappa(C_n) = \frac{n}{n-2} \cos\left(\frac{\pi}{n}\right).$$

PROOF. By Lemma 2.2 it suffices to compute $\kappa(C_n, w)$ for facet defining inequalities of $\mathrm{CUT}(C_n)$. By Theorem 3.1, they correspond to the circuit inequalities and, since they are all switching equivalent, it suffices to consider one of them; for instance, we can choose $w^T x = -x(E)$ for odd n, and $w^T x = x_e - x(E \setminus \{e\})$ for even n. In both cases, we find that $\mathrm{ip}(C_n, w) = n - 2$. Thus it now suffices to show that $\mathrm{sdp}(C_n, w) = n \cos(\pi/n)$ as this will give the desired value for $\kappa(C_n, w)$.

For n odd, it is known that $\mathrm{sdp}_{\mathrm{GW}}(C_n, e) = \frac{n}{4}\left(2 + 2\cos\frac{\pi}{n}\right)$ (see [19]), which implies that $\mathrm{sdp}(C_n, -e) = 2\,\mathrm{sdp}_{\mathrm{GW}}(C_n, e) - n = n\cos(\pi/n)$.

This can also be easily verified using the parametrization of $\mathcal{E}(C_n)$ from Theorem 3.2.

One can also compute $\mathrm{sdp}(C_n, w)$ for n even and $w = (-1, 1, \ldots, 1)$ using Theorem 3.2; it turns out that this has also been computed in [22] in the context of quantum information theory. $\qquad\square$

3.2. The case of K_5-minor free graphs

Since K_5-minor free graphs are 4-colorable [21], we deduce from (4) that their Grothendieck constant $\kappa(G)$ is bounded. Here we give a closed-form formula for $\kappa(G)$ in terms of the girth of G.

Theorem 3.4. *If G is a graph with no K_5 minor (and G is not a forest), then*

$$\kappa(G) = \frac{g}{g-2} \cos\left(\frac{\pi}{g}\right),$$

where g is the minimum length of a circuit in G.

PROOF. Directly from Theorem 3.3 using the facts that all facets of G are supported by circuits (Theorem 3.1) and that the function $\frac{n}{n-2} \cos(\frac{\pi}{n})$ is monotone nonincreasing in n. $\qquad\square$

As a direct application we recover the values $\kappa(K_{2,n}) = \kappa(K_{3,n}) = \sqrt{2}$, for $n \geq 3$ [10].

3.3. Graphs whose cut polytope is defined by inequalities supported by at most k points

We show here that the Grothendieck constant can be bounded in terms of the size of the supports of the inequalities defining facets of the cut polytope. The *support graph* of an inequality $w^T x \leq 1$ is the graph $H = (W, F)$, where $F = \{ij \in E \mid w_{ij} \neq 0\}$ and W is the set of nodes covered by F. We say that $w^T x \leq 1$ is *supported by at most k points* when $|W| \leq k$. For instance, a triangle inequality is supported by three points.

Fix an integer $k \geq 2$. Let $\mathcal{R}_k(K_n) \subseteq \mathbb{R}^{E_n}$ be the polyhedron defined by all valid inequalities for CUT_n supported by at most k points. For $G = ([n], E)$, let $\mathcal{R}_k(G) := \pi_E(\mathcal{R}_k(K_n))$. For instance, $\mathcal{R}_3(K_n) = \mathrm{MET}(K_n)$, and thus $\mathcal{R}_3(G) = \mathrm{MET}(G)$.

Clearly, $\mathrm{CUT}(G) \subseteq \mathcal{R}_k(G)$. Define the class $\mathcal{G}_k$ of all graphs G for which $\mathrm{CUT}(G) = \mathcal{R}_k(G)$. For instance, $\mathcal{G}_2$ consists of all forests (thus the K_3-minor free graphs) and $\mathcal{G}_3$ of the K_5-minor free graphs. Thus both are closed under taking minors; this holds for any $\mathcal{G}_k$.

Theorem 3.5. *The class $\mathcal{G}_k$ is closed under taking minors.*

PROOF. It follows directly from the definition that $\mathcal{G}_k$ is closed under edge deletion. It remains to verify that it is closed under edge contraction. Let $G = (V, E)$ and $G' := G/e = (V', E')$, where $e = (1, 2)$ and $V' = \{2, \ldots, n\}$.

Given $y \in \mathbb{R}^{E'}$, define its extension $\tilde{y} \in \mathbb{R}^E$ by $\tilde{y}_{12} = 1$, $\tilde{y}_{1i} = y_{2i}$ if $1i \in E$ with $i \geq 3$, $\tilde{y}_{2i} = y_{2i}$ if $2i \in E$ with $i \geq 3$, and $\tilde{y}_{ij} = y_{ij}$ if $ij \in E$ with $i, j \geq 3$. One can easily verify that $\tilde{y} \in \mathrm{CUT}(G)$ iff $y \in \mathrm{CUT}(G/e)$.

We now verify that if $y \in \mathcal{R}_k(K_{n-1})$, then $\tilde{y} \in \mathcal{R}_k(K_n)$. Indeed, say $w^T x \leq 1$ is a valid inequality for CUT_n supported by at most k points. Define the inequality on $x = (x_{ij})_{2 \leq i < j \leq n}$:

$$b^T x := \sum_{i=3}^{n}(w_{1i}+w_{2i})x_{2i} + \sum_{3 \leq i < j \leq n} w_{ij}x_{ij} \leq 1 - w_{12}.$$

Obviously it is supported by at most k points and it is valid for CUT_{n-1}. Hence $b^T y \leq 1 - w_{12}$, which implies $w^T \tilde{y} \leq 1$.

Assume $G \in \mathcal{G}_k$, i.e., $\mathrm{CUT}(G) = \pi_E(\mathcal{R}_k(K_n))$ and let $z \in \pi_{E'}(\mathcal{R}_k(K_{n-1}))$; we show that $z \in \mathrm{CUT}(G/e)$. Say $z = \pi_{E'}(y)$ where $y \in \mathcal{R}_k(K_{n-1})$. By the discussion above, the extension $\tilde{y}$ of y belongs to $\mathcal{R}_k(K_n)$ and thus $\pi_E(y) \in \pi_E(\mathcal{R}_k(K_n)) = \mathrm{CUT}(G)$. This in turn implies that $z \in \mathrm{CUT}(G/e)$ since $\pi_E(y)$ is the extension of z. $\qquad\square$

Clearly, for $G \in \mathcal{G}_2$, $\kappa(G) = \kappa(K_2) = 1$. Moreover, Theorem 3.4 implies that $\kappa(G) \leq \kappa(K_3) = 3/2$ for $G \in \mathcal{G}_3$. This pattern extends to any k.

Theorem 3.6. *If $G \in \mathcal{G}_k$ then $\kappa(G) \leq \kappa(K_k)$. Moreover, this bound is tight since $K_k \in \mathcal{G}_k$.*

PROOF. It is enough to show that for any graph G, $\mathcal{E}(G) \subseteq \kappa(K_k) \cdot \mathcal{R}_k(G)$. Moreover, it suffices to consider only $G = K_n$, as the general result follows by taking projections.

Let $y \in \mathcal{E}(K_n)$ and let $w^T x \leq 1$ be a valid inequality for CUT_n with support $H = (W, F)$ where $|W| \leq k$. Then, $w^T y = \pi_F(w)^T \pi_F(y) \leq \mathrm{sdp}(H, \pi_F(w)) \leq \kappa(H) \cdot \mathrm{ip}(H, \pi_F(w)) \leq \kappa(K_k)$, where we use the facts that $\kappa(H) \leq \kappa(K_k)$ and $\mathrm{ip}(H, \pi_F(w)) \leq 1$ for the right most inequality. $\qquad\square$

One can verify that $\kappa(K_7) = 3/2$ (see [15]). Hence, $\kappa(G) \leq 3/2$ for all $G \in \mathcal{G}_7$.

4. Integrality gap of clique-web inequalities

We have already seen that $\kappa(K_n) = \Theta(\log n)$ and it is an interesting question to identify explicit instances that achieve this integrality gap. This was

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posed as an open question in [1] and instances with large gap are given in [3]. In this section we show that the gap is bounded by 3 for clique-web inequalities, a wide class of valid inequalities for CUT_n.

Given integers p and r with $p \geq 2r + 3$, the *anti-web graph* AW_p^r is the graph with vertex set $[p]$, and with edges $(i, i+1), \ldots, (i, i+r)$ for $i \in [p]$, where the indices are taken modulo p. The *web graph* W_p^r is defined as the complement of AW_p^r in K_p. Call the set of edges $(i, i+s)$ for $i \in [p]$ (indices taken modulo p) the *s-th band*, so that AW_p^r consists of the first r bands and W_p^r consists of the last $\lceil q/2 \rceil$ bands in K_p.

Let p, q, r, n be integers satisfying $p - q = 2r + 1$, $q \geq 2, n = p + q$. The (pure) *clique-web inequality* with parameters n, p, q, r is the inequality

$$-x(K_q) - \sum_{\substack{1 \leq i \leq q \\ q+1 \leq j \leq n}} x_{ij} - x(\text{W}_p^r) \leq q(r+1). \quad (5)$$

The support graph of (5), denoted by CW_p^r, consists of a clique on the first q nodes, a web on the last p ones, and a complete bipartite graph between them. It is known that clique-web inequalities define facets of CUT_n. Note that hypermetric and bicycle odd wheel inequalities arise as special cases of (5), for $r = 0$ and $r = \frac{n-5}{2}$, respectively (see [9]).

Since the left-hand side of (5) is equal to $-x(K_{p+q}) + x(\text{AW}_p^r)$, one easily obtains that $\text{sdp}(\text{CW}_p^r, -e) \leq (p+q)/2 + pr = q(r+1) + (2r+1)^2/2$, which implies that $\kappa(\text{CW}_p^r, -e) \leq 1 + \frac{(2r+1)^2}{q(2r+2)}$. This directly implies the following:

Lemma 4.1. *The integrality gap of a clique-web inequality with $q \geq 2r + 1$ is upper bounded by 2.*

We now consider the case when $q \leq 2r$.

Theorem 4.1. *The integrality gap of a clique-web inequality with $q \leq 2r$ is upper bounded by 3.*

PROOF. We can rewrite $\text{sdp}(\text{CW}_p^r, -e)$ as

$$\max_{X \in \mathcal{E}_n} - \sum_{ij \in K_q} X_{ij} - \sum_{\substack{1 \leq i \leq q \\ q+1 \leq j \leq n}} X_{ij} - \sum_{ij \in \text{W}_p^r} X_{ij}. \quad (6)$$

Notice that the program (6) is invariant under the action of the full symmetric group S_q acting on the row/column indices in $[q]$. Moreover, (6) is invariant under the action of the group of cyclic permutations in S_p acting on the row/column indices in $\{q+1, .., n\}$. Thus, we can restrict without loss of generality the matrix X in (6) to satisfy the following invariance conditions:

$$
\begin{aligned}
X_{ij} &= a && \text{for } 1 \leq i \neq j \leq q, \\
X_{ij} &= b && \text{for } 1 \leq i \leq q < j \leq n, \\
X_{ij} &= c_{|j-i \mod p|} && \text{for } q+1 \leq i \neq j \leq n
\end{aligned}
$$

for some scalars $a, b, c_1, \ldots, c_{\lceil q/2 \rceil}$. Hence X has the form $X := \begin{pmatrix} aJ_{q,q} + (1-a)I_q & bJ_{q,p} \\ bJ_{p,q} & X_p \end{pmatrix}$, where X_p denotes the principal submatrix of X indexed by $\{q+1, \ldots, n\}$. One can easily verify that $X \succeq 0$ if and only if $Y := \begin{pmatrix} \beta & be^T \\ be & X_p \end{pmatrix} \succeq 0$, after setting $\beta := \frac{(q-1)a+1}{q}$.

Consider first the case when q is even; so p is odd, all bands in W_p^r have size p, and the objective function in (6) reads

$$\frac{q}{2}(1 - q\beta) - pqb - p(c_1 + \ldots + c_{q/2}). \quad (7)$$

If $\beta = 0$, then $b = 0$ and Lemma 3.1 implies that $c_s \geq -\cos(\pi/p)$ for all s. Indeed each band of W_p^r is a circuit or a disjoint union of circuits (e.g. the first band of W_9^2 is a union of three triangles). As p is odd, at least one of these circuits is an odd circuit of size $p' \leq p$, so that Lemma 3.1 implies that the entries on the band are at least $-\cos(\pi/p') \geq -\cos(\pi/p)$. Now the objective value is equal to $\frac{q}{2} - p(c_1 + \ldots + c_{q/2}) \leq \frac{q}{2}(p\gamma + 1) \leq \frac{q}{2}(p+1) = \frac{q}{2}(q + 2r + 2) \leq 2\,q(r+1)$ (as $q \leq 2r$), setting $\gamma := \cos(\pi/p)$.

Assume now $\beta > 0$. Taking the Schur complement in the above matrix Y with respect to the entry β, we can rewrite the condition $Y \succeq 0$ as $X_p - \frac{b^2}{\beta} J \succeq 0$. If $\beta = b^2$, then $c_s = 1$ for all s and the maximum of $\frac{q}{2}(1 - qb^2) - pqb - pq/2$ for $b \in [-1, 1]$ is easily verified to be equal to $q(r+1)$, attained at $b = -1$. Now let $\beta > b^2$ and $X \succeq 0$ is equivalent to $Z := \frac{\beta}{\beta - b^2} X_p - \frac{b^2}{\beta - b^2} J \in \mathcal{E}_p$. As above, Lemma 3.1 permits to bound the entries of Z as follows: $\frac{\beta}{\beta - b^2} c_s - \frac{b^2}{\beta - b^2} \geq -\gamma$ for $1 \leq s \leq q/2$. Therefore, the program (6) is upper bounded by

$$
\begin{aligned}
\max_{b,c,\beta} \quad & \frac{q}{2}(1 - q\beta) - pqb - cpq/2 \\
\text{s.t.} \quad & \beta(c + \gamma) \geq b^2(\gamma + 1) \\
& b^2 < \beta \leq 1, \ -1 \leq b, c \leq 1.
\end{aligned} \quad (8)
$$

At optimality, equality $\beta(c + \gamma) = b^2(\gamma + 1)$ holds. This permits to express c in terms of b, β and to rewrite the objective function of (8) as $\frac{q}{2}(1 - q\beta) -$

$\frac{pq}{2}(b^2\frac{\gamma+1}{\beta} + 2b - \gamma)$. For fixed β, the maximum of this quadratic function in b is attained at $b = -\frac{\beta}{\gamma+1} \in [-1, 1]$ and is equal to $\frac{q}{2}(1-q\beta) + \frac{pq}{2}(\frac{\beta}{\gamma+1} + \gamma) = \frac{q}{2}(\beta(\frac{p}{\gamma+1} - q) + p\gamma + 1)$. As $q \le 2r$, we have $\frac{p}{\gamma+1} \le q$ and thus the latter quantity is maximized when $\beta = 1$, so that the maximum of (8) is equal to $\frac{pq}{2}(\gamma + \frac{1}{\gamma+1}) - \frac{q(q-1)}{2}$. Hence, using $q \le 2r$ and $\gamma + \frac{1}{\gamma+1} \le \frac{3}{2}$, we deduce that this maximum is upper bounded by $3q(r+1)$. This concludes the proof that the integrality gap of the clique-web inequality is at most 3 when q is even.

Consider now the case when q is odd. Then p is even and W_p^r consists of $(q-1)/2$ bands of size p and one band of size $p/2$. The treatment is analogous to the case q even, except we must replace the objective function in (7) by

$$\frac{q}{2}(1 - q\beta) - pqb - p(c_1 + \ldots + c_{\frac{q-1}{2}}) - \frac{p}{2}c_{\frac{q+1}{2}} \quad (9)$$

and, as p is even, the values on the bands can only be claimed to lie in $[-1, 1]$ by Lemma 3.1 (which amounts to setting $\gamma = 1$ in the above argument). Specifically, if $\beta = 0$ we can upper bound the objective function (9) by $2q(r + 1)$ and, if $\beta = b^2$, we can upper bound (9) by $q(r + 1)$. Finally, if $\beta > b^2$, as above we do a Schur complement and obtain $\frac{\beta}{\beta - b^2}c_s - \frac{b^2}{\beta - b^2} \ge -1$, so that (9) is upper bounded by the program (8) setting there $\gamma = 1$. Hence the integrality gap of the clique-web inequality is also bounded by 3 for q odd. $\square$

We conclude with several remarks.
- We just showed: (i) $\kappa(\mathrm{CW}_r^p, -e) \le 1 + \frac{(2r+1)^2}{q(2r+2)}$, and (ii) $\kappa(\mathrm{CW}_r^p, -e) \le \frac{q+6r+5}{4(r+1)}$ when $q \le 2r$. Therefore, asymptotically, the integrality gap tends to 1 as $q \to \infty$ and r is fixed (by (i)), and it tends to $3/2$ as $r \to \infty$ and q is fixed (by (ii)).
- Our analysis is tight for $q = 2$, the case of bicycle odd wheel inequalities (since then the program (8) is equivalent to (6)); that is, $\mathrm{sdp}(\mathrm{CW}_{2r+3}^r, -e) = -1 + p(\cos(\pi/p) + \frac{1}{\cos(\pi/p)+1})$ (as mentioned in [19]). This explains why in the proof of Theorem 4.1 we use the precise estimate $-\cos\frac{\pi}{p} \le c$, as opposed to the trivial bound $-1 \le c$, which was equally good for the our asymptotic analysis.
- Pitowsky [17] shows the asymptotic lower bound $4/\pi \sim 1.27$ for the integrality gap of the clique-web inequality with $q = 2r$ and $r \to \infty$.
- Given $b \in \mathbb{Z}^n$ with $\sum_i b_i = 1$, the *hypermetric inequality* $w^T x := -\sum_{1 \le i < j \le n} b_i b_j x_{ij} \le (\sum_i b_i^2 - 1)/2$

satisfies $\mathrm{sdp}(K_n, w) \le \sum_i b_i^2/2$; thus its integrality gap is at most $3/2$, with equality if and only if $b = (1, 1, -1, 0, \ldots, 0)$ (the case of triangle inequalities).

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