Cooperative Games with a Permission Structure: axiomatization and computation of solutions

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Cooperative Games with Transferable Utility

Cooperative TU-games describe situations in which a set of players can earn certain payoffs by cooperation (i.e. making binding agreements).

A cooperative game with transferable utility (TU-game) is a pair (N, v) with

N: set of **players** (finite)

$v\colon 2^N \to \mathbb{R}$: **characteristic function** satisfying $v(\emptyset) = 0$

The **worth** $v(S) \in \mathbb{R}$ is what the players in **coalition** $S \subseteq N$ can earn by cooperation.

$\mathcal{G}^N$: collection of all TU-games on N.

Two main questions:

1. Which coalitions will form?
2. How to distribute the earnings over the players?

A **solution** is a function f that assigns to every game (N, v) a payoff distribution $f(N, v) \in \mathbb{R}^N$ such that $f_i(N, v)$ is the payoff for player $i \in N$ in game (N, v).

In a TU-game it is assumed that all coalitions $S \subseteq N$ are feasible.
Usually we encounter restrictions in coalition formation, for example communication or hierarchical restrictions.

Restricted cooperation
$\mathcal{F} \subseteq 2^N$: set of feasible coalitions

Two examples:
Communication and hierarchy restrictions

1. Communication
(Myerson 1977)
Only connected coalitions in an undirected (communication) graph are feasible:
$\mathcal{F}$ is the set of connected coalitions in a communication graph.

2. Hierarchies

Several models of hierarchies

2A. Games with a permission structure

Gilles, Owen and van den Brink (1992)
van den Brink and Gilles (1996)
Gilles and Owen (1994)
van den Brink (1997, 1999, 2010)

A game with a permission structure on N describes a situation where some players in a TU-game need permission from other players before they are allowed to cooperate within a coalition.

A permission structure is described by a **digraph** (N, D) with

$N = \{1, ..., n\}$ a finite set of nodes (players)

$D \subseteq N \times N$ a binary relation on N

$\mathcal{D}^N$: collection of all digraphs on N

A tuple (N, v, D) is a **game with a permission structure**.

For permission structure $D \in \mathcal{D}^N$ and $i \in N$ we denote:

$S_D(i) = \{j \in N \mid (i,j) \in D\}$: **successors** of i in D

$P_D(i) = \{j \in N \mid (j,i) \in D\}$: **predecessors** of i

$\widehat{S}_D(i)$: set of successors of i in the **transitive closure** of D
i.e., $j \in \widehat{S}_D(i)$ if and only if there exists a sequence of players $(h_1, \ldots, h_t)$ such that $h_1 = i$, $h_{k+1} \in S_D(h_k)$ for all $1 \leq k \leq t-1$, and $h_t = j$.

$D \in \mathcal{D}^N$ is **acyclic** if $i \notin \widehat{S}_D(i)$ for all $i \in N$.

$\mathcal{D}^N_A$: collection of all acyclic digraphs on N

$T_D = \{i \in N \mid P_D(i) = \emptyset\}$: set of **top nodes** in D.

Note that $T_D \neq \emptyset$ if D is acyclic.

Conjunctive approach

Each player needs permission from *all* its predecessors

Disjunctive approach (for acyclic permission structures)

Each player needs permission from *at least one* of its predecessors

Conjunctive feasible coalitions in D

$$\Phi_D^c = \{E \subseteq N \,|\, P_D(i) \subseteq E \text{ for all } i \in E\}$$

Disjunctive feasible coalitions in D

$$\Phi_D^d = \{E \subseteq N \,|\, P_D(i) \cap E \neq \emptyset \text{ for all } i \in E \setminus T_D\}$$

Conjunctive sovereign part of $E \subseteq N$ in D

is the largest feasible subset of E in Φ_D^c, i.e.

$$\overline{\sigma}_D^c(E) = \cup\{F \in \Phi_D^c \mid F \subseteq E\}$$

$$= E \setminus \widehat{S}(N \setminus E)$$

Disjunctive sovereign part of $E \subseteq N$ in D

is the largest feasible subset of E in Φ_D^d, i.e.

$$\overline{\sigma}_D^d(E) = \cup\{F \in \Phi_D^d \mid F \subseteq E\}$$

Conjunctive restriction of v on D

$$\overline{r}^c_{v,D}(E) = v(\overline{\sigma}^c_D(E))$$

Disjunctive restriction of v on D

$$\overline{r}^d_{v,D}(E) = v(\overline{\sigma}^d_D(E))$$

Conjunctive (Shapley) permission value

$$\varphi^c(N, v, D) = Sh(\overline{r}^c_{N,v,D})$$

Disjunctive (Shapley) permission value

$$\varphi^d(N, v, D) = Sh(\overline{r}^d_{N,v,D})$$

Example

$N = \{1,2,3,4\}$

$$v(E) = \begin{cases} 1 & \text{if } E \ni 4 \\ 0 & \text{else,} \end{cases}$$

$$D = \{(1,2),(1,3),(2,4),(3,4)\}$$

Then

$$\overline{r}^c_{v,D}(E) = \begin{cases} 1 & \text{if } E = \{1,2,3,4\} \\ 0 & \text{else} \end{cases}$$

$$\varphi^c(N,v,D) = (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$$

$$\overline{r}^d_{v,D}(E) = \begin{cases} 1 & \text{if } E \in \{\{1,2,4\},\ \{1,3,4\},\ N\} \\ 0 & \text{else} \end{cases}$$

$$\varphi^d(N,v,D) = (\frac{5}{12},\frac{1}{12},\frac{1}{12},\frac{5}{12})$$

Results on
Game properties

Harsanyi dividends

Axiomatizations of solutions

Remark: Communication between hierarchies. Example, a network of hierarchically structured firms

2B. Games on antimatroids

Algaba, Bilbao, van den Brink and Jiménez-Losada (2003, 2004)

Definition A set of feasible coalitions $\mathcal{A} \subseteq 2^N$ is an **antimatroid** on N if it satisfies

1. $\emptyset \in \mathcal{A}$
2. (Closed under union) If $E, F \in \mathcal{A}$ then $E \cup F \in \mathcal{A}$
3. (Accessibility) If $E \in \mathcal{A}$, $E \neq \emptyset$, then there exists an $i \in E$ such that $E \setminus \{i\} \in \mathcal{A}$.

An antimatroid $\mathcal{A}$ is *normal* if for every $i \in N$ there is an $E \in \mathcal{A}$ such that $i \in E$.

Theorem

If S is an acyclic permission structure on N then Φ^d_D and Φ^c_D are antimatroids on N.

Definition

An antimatroid $\mathcal{A}$ is a **poset antimatroid** if it is closed under intersection
(i.e. $E, F \in \mathcal{A}$ implies that $E \cap F \in \mathcal{A}$).

Theorem

Let $\mathcal{A}$ be an antimatroid. Then there is a $D \in \mathcal{D}_A^N$ such that $\mathcal{A} = \Phi_D^c$ if and only if $\mathcal{A}$ is a poset antimatroid.

Remark: Algaba, Bilbao, van den Brink and Jiménez-Losada (2004) also characterize the class of antimatroids that can be a collection of disjunctive feasible sets of some $D \in \mathcal{D}_A^N$.

Comparison between communication and hierarchy

Theorem

A set of feasible coalitions $\mathcal{F} \subseteq 2^N$ is the set of connected coalitions in some undirected (communication) graph if and only if it satisfies

1. $\emptyset \in \mathcal{F}$
2. (Union stability) If $E, F \in \mathcal{F}$ with $E \cap F \neq \emptyset$ then $E \cup F \in \mathcal{F}$
3. (2-Accessibility) If $E \in \mathcal{F}$, $E \neq \emptyset$, then there exist an $i, j \in E$, $i \neq j$, such that $E \setminus \{i\}, E \setminus \{j\} \in \mathcal{F}$
4. (Normality) For every $i \in N$ there is an $E \in \mathcal{F}$ such that $i \in E$.

2C. Games on union closed systems

van den Brink, Katsev and van der Laan (2010)

A set of feasible coalitions $\Omega \subseteq 2^N$ is **union closed** if

1. $\emptyset \in \Omega$
2. If $E, \ F \in \Omega$ then $E \cup F \in \Omega$.

For a system $\Omega \in \mathcal{C}^N$, define

$$\sigma_\Omega(S) = \bigcup \{U \in \Omega \mid U \subseteq S\}$$

i.e. $\sigma_\Omega(S)$ is the largest feasible subset of S, and for the pair (v, Ω),

$$r_{v,\Omega}(S) = v(\sigma_\Omega(S))$$

is the restricted game that assigns to each coalition the worth of its largest feasible subset.

2D. Games on union stable systems

Algaba, Bilbao, Borm and López (2000, 2001)

Definition

A collection $\Omega \subseteq 2^N$ is **union stable** if

1. $\emptyset \in \Omega$
2. If $E,\ F \in \Omega$ with $E \cap F \neq \emptyset$ then $E \cup F \in \Omega$.

Upto now we discussed generalizations of games with a permission structure.

Permission structure
$\Rightarrow$ Antimatroid
$\Rightarrow$ Union closed system
$\Rightarrow$ Union stable system

Now, we go to special classes of games with a permission structure.

2E. Peer group games

Branzei, Fragnelli and Tijs (2002)

A game with a permission structure (N, v, D) is a **peer group situation** if

(N, v) is an inessential (or additive) game, and
(N, D) is a rooted tree.

Examples: Auction games, Airport games, Polluted river games.

A polynomial time algorithm to compute the nucleolus for these games is given by Branzei, Solymosi and Tijs (2005).

Nucleolus (Schmeidler (1959))

The excess $e(S, x)$ of a coalition $S \subseteq N$ in payoff vector $x \in \mathbb{R}^n$ is

$$e(S, x) = v(S) - x(S).$$

Let $E(x)$ be the $(2^n - 2)$-component vector that is composed of the excesses of all coalitions $S \subset N,\ S \neq \emptyset$, in a non-increasing order, so

$$E_1(x) \geq E_2(x) \geq \ldots \geq E_{2^n-2}(x).$$

Then the nucleolus $Nuc(N, v)$ of the game (N, v) is the unique imputation which lexicographically minimizes the vector-valued function $E(\cdot)$ over the imputation set:

$$Nuc(N, v) = x \in I(N, v)$$

such that $E(x) \preceq_L E(y)$ for all $y \in I(N, v)$, where

$$I(N, v) = \{x \in \mathbb{R}^n | x(N) = v(N) \text{ and } x_i \geq v(i),\ i \in N\}$$

is the imputation set of (N, v).

Results:

1. $I(N, v)$ is convex, closed and bounded.

2. For given game (N, v), the nucleolus selects a **unique** payoff vector from the imputation set: the set $\mathrm{Nuc}(N, v)$ contains precisely one element.

3. The payoff vector in $\mathrm{Nuc}(N, v)$ minimizes the 'dissatisfaction' of the most dissatisfied coalition.

4. The function $f^{Nuc}: \mathcal{G} \to \mathbb{R}^n$ such that $f^{Nuc}(N, v) = x$ with $\{x\} = \mathrm{Nuc}(N, v)$ is a value function that assigns to any game (N, v) the unique element x in the nucleolus as its outcome (payoff vector). Usually, $f^{Nuc}(N, v)$ is called the nucleolus of the game.

5. If the Core is non-empty, $f^{Nuc}(N, v) \in C(N, v)$: in some sense it is in the middle of the Core.

6. The Nucleolus belongs to the Kernel (set valued solution). In case $n = 3$, the Nucleolus is **equal** to the Kernel.

2F. Two special classes of (disjunctive) games with a permission structure that both contain the class of peer group games:

2F1.

A game with permission structure (N, v, D) satisfies **weak digraph monotonicity** if

$$S \in \Phi_D^d \Rightarrow v(S) \leq v(N).$$

A game with permission structure (N, v, D) satisfies **weak digraph concavity** if

$$[S \cup T = N \text{ and } S, T \in \Phi_D^d] \Rightarrow$$

$$v(S) + v(T) \geq v(S \cap T) + v(N).$$

van den Brink, Katsev and van der Laan (2008) provide a polynomial time algortihm to compute the nucleolus if

(N, v, D) is weak digraph monotone and weak digraph concave,
and
(N, D) is acyclic and quasi-strongly connected.

Algorithm

Step 1 Set $k = 0$, $U_0 = N$, $v_0 = v$, $D_0 = D$ and $r_0 = r$. Go to Step 2.

Step 2 Find $U_{k+1} \subset U_k$ satisfying

$$\tau(U_{k+1}, r_k) = \tau^*(r_k)$$

and

$$|U_{k+1}| = \max_{\{U \in \Omega^{D_k} | \tau(U, r_k) = \tau^*(r_k)\}} |U|,$$

where $\tau^*(r_k) = \min_{U \in \Omega^{D_k}} \tau(U, r_k)$ with $\tau(U, r_k) = \frac{r_k(U_k) - r_k(U)}{|U_k \backslash U| + 1}$.

Assign $y_j = \tau^*(r_k)$ to every player $j \in U_k \setminus U_{k+1}$.

Go to Step 3.

Step 3 If $U_{k+1} = \{1\}$ then Go to Step 4. If $U_{k+1} \neq \{1\}$, let i_{k+1} be the unique top-player of the subgraph $(U_k \setminus U_{k+1}, D_k(U_k \setminus U_{k+1})$ of the digraph (U_k, D_k) restricted to $U_k \setminus U_{k+1}$. Define game (U_{k+1}, v_{k+1}) by

$$v_{k+1}(U) = \begin{cases} v_k(U) \\ v_k(U \cup (U_k \setminus U_{k+1})) - \tau(U_{k+1}, r_k)|U_k \setminus U_k \end{cases}$$

let digraph (U_{k+1}, D_{k+1}) be given by

$$(i, j) \in D_{k+1} \text{ if } \begin{cases} (i, j) \in D_k \text{ or} \\ i \in P_{D_k}(i_{k+1}) \text{ and } j \in S_{D_k}(U_k \setminus U_{k+1} \end{cases}$$

and let r_{k+1} be the restricted game of $(U_{k+1}, v_{k+1},$
Set $k = k + 1$. Go to Step 2.

Step 4 Assign $y_1 = v(N) - \sum_{j \in N \setminus \{1\}} y_j$. Stop.

Complexity of the algorithm: $\mathcal{O}(n^4)$.

2F2.

van den Brink, Katsev and van der Laan (2010) provide a polynomial time algortihm to compute the nucleolus if

(N, v) is an inessential (or additive) game,
and
(N, D) is acyclic.

Let (N, D) be an acyclic permission structure, $t \in T_D$ be one of the top players and $K = N \setminus U^t$. Then define $D^K \in \mathcal{D}^K$ on the set of players K by

$$(i, j) \in D^K \text{ if and only if } (i, j) \in D \text{ and } P_D(j) \cap U^t = \emptyset$$

Algorithm

Step 1 Set $k = 1$, $N_1 = N$, $D_1 = D$ and $t_1 = 1$. Go to Step 2.

Step 2 Consider the non-negative additive game with acyclic, quasi-strongly connected permission structure $(U^{t_k}, v_k, D_k(U^{t_k}))$ with

$$v_k(U) = v(U) \text{ for all } U \subseteq U^{t_k}.$$

Let r_k be the restricted game of $(U^{t_k}, v_k, D_k(U^{t_k}))$ Go to Step 3.

Step 3 Apply the (polynomial time) algorithm of van den Brink *et al.* (2008) to find the nucleolus of the restricted game (U^{t_k}, r_k). Assign $y_i = Nuc_i(U^{t_k}, r_k)$ to every $i \in U^{t_k}$. Go to Step 4.

Step 4 If $U^{t_k} = N_k$ then Stop. Otherwise, go to Step 5.

Step 5 Define $N_{k+1} = N_k \setminus U^{t_k}$ and $D_{k+1} \in \mathcal{D}^{N_{k+1}}$ by $D_{k+1} = D_k^{N_{k+1}}$.

Define $t_{k+1} \in T_{D_{k+1}}$ as the top player in D_{k+1} with the lowest label ($t_{k+1} \leq h$ for every $h \in T_{D_{k+1}}$). Consider the set $U^{t_{k+1}}$ consisting of t_{k+1} and all its complete subordinates in the graph (N_{k+1}, D_{k+1}). Set $k = k + 1$ and return to step 2.

Complexity of the algorithm: $\mathcal{O}(n^4)$.

Applications of graph games

Line-graph games

Water distribution problems

Sequencing games

Bipartite graph games

Assignment games

Digraph games

Peer group games: Auction games, Airport games,

Polluted river games

Hierarchically structured firms

Concluding remark

After initial results on game properties, Harsanyi dividends and axiomatizations of solutions, attention now shifts to computation of solutions on (classes of) games with a permission structure and other models of restricted cooperation.