Bounding the Inefficiency of Altruism Through Social Contribution Games

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Abstract. We introduce a new class of games, called *social contribution games (SCGs)*, where each player's individual cost is equal to the cost he induces on society because of his presence. Our results reveal that SCGs constitute useful abstractions of altruistic games when it comes to the analysis of the robust price of anarchy. We first show that SCGs are *altruism-independently smooth*, i.e., the robust price of anarchy of these games remains the same under arbitrary altruistic extensions. We then devise a general reduction technique that enables us to reduce the problem of establishing smoothness for an altruistic extension of a base game to a corresponding SCG. Our reduction applies whenever the base game relates to a canonical SCG by satisfying a simple *social contribution boundedness* property. As it turns out, several well-known games satisfy this property and are thus amenable to our reduction technique. Examples include min-sum scheduling games, congestion games, second-price auctions and valid utility games. Using our technique, we derive mostly tight bounds on the robust price of anarchy of their altruistic extensions. For the majority of the mentioned game classes, the results extend to the more differentiated friendship setting. As we show, our reduction technique covers this model if the base game satisfies three additional natural properties.

1 Introduction

The study of the inefficiency of equilibria in strategic games has been one of main research streams in algorithmic game theory in the last decade and contributed to the explanation of several phenomena observed in real life. More recently, researchers have also started to incorporate more complex social relationships among the players in such studies, accounting for the fact that players cannot always be regarded as isolated entities that merely act on their own behalf (see also [12]). In particular, the extent by which other-regarding preferences such as *altruism* and *spite* impact the inefficiency of equilibria has been studied intensively; see, e.g., [1, 4–7, 11, 15, 14, 16].

In this context, some counterintuitive results have been shown that are still not well-understood. For example, in a series of papers [4, 5, 7] it was observed that for congestion games the inefficiency of equilibria gets worse as players become more altruistic, therefore suggesting that altruistic behavior can actually be harmful for society. On the other hand, valid utility games turn out to be unaffected by altruism as their inefficiency remains unaltered under altruistic behavior [7]. These discrepancies triggered our interest in the research conducted in this paper. The basic question that we are asking here is: What is it that impacts the inefficiency of equilibria of games with altruistic players?

To this aim, we consider two different models that have previously been studied in the literature: the *altruism model* [7] and the *friendship model* [1]. In both models, one starts from a strategic game (called the *base game*) specifying the *direct cost* of each player and then extends this game by defining the *perceived cost* of each player as a function of his neighbors' direct costs. In the altruism model, player i's perceived cost is a convex combination of his direct cost and the overall social cost. In the more general friendship model, player i's perceived cost is a linear combination of his direct cost and his friends' costs.

In order to quantify the inefficiency of equilibria in our games we resort to the concept of the *price of anarchy (PoA)* [18], which is defined as the worst-case relative gap between the cost of a Nash equilibrium and a social optimum (over all instances of the game). By now, a standard approach to prove upper bounds on the PoA is through the use of the *smoothness framework* introduced by Roughgarden [19]. Basically, this framework allows us to derive bounds on the *robust price of anarchy* by showing that the underlying game satisfies a certain (λ, μ)-*smoothness property* for some parameters λ and μ. The robust PoA holds for various solution concepts, ranging from pure Nash equilibria to coarse correlated equilibria (see, e.g., Young [24]).

The original smoothness framework [19] has been extended to both the altruism and the friendship model in [7] and [1], respectively. Applying these adapted smoothness frameworks to bound the robust PoA is often technically involved because of the altruistic terms that need to be taken into account additionally (see also the analyses in [1, 7]).

Instead, we take a different approach here. As we will show, there is a natural class of games, which we term *social contribution games (SCGs)*, that is intimately connected with our altruism and friendship games. We establish a general reduction technique that enables us to reduce the problem of establishing smoothness for our altruism or friendship game to the problem of proving smoothness for a corresponding SCG. The latter is usually much simpler than proving smoothness for the altruism or friendship game directly. This also opens up the possibility to derive better bounds on the robust PoA of these games through the usage of our new reduction technique.

Our Contributions. Our main contributions are as follows:

- We introduce a new class of games, which we term *social contribution games (SCGs)*, where each player's individual cost is defined as the cost he incurs on society because of his presence. Said differently, player i's cost is equal to the difference in social cost if player i is present/absent in the game. We show that SCGs are *altruism-independently smooth*, i.e., if the SCG is (λ, μ)-smooth then every altruistic extension is (λ, μ)-smooth as well.

Table 1. Robust PoA bounds derived in this paper for the friendship model.

Games	Robust PoA (our results)	Robust PoA (previous best)	Remarks
$R\|\|\sum_j w_j C_j$	$= 4^\star$	$\leq 23.31^\S$ [1]	RPoA = 4 (selfish players) [10]
$P\|\|\sum_j C_j$	≤ 2		RPoA $= \frac{3}{2} - \frac{1}{2m}$ (selfish players)
linear congestion games	$= \frac{17}{3}$	≤ 7 [1]	$5 \leq$ PoA $\leq \frac{17}{3}$ (special case) [2]
p-poly. congestion games	$\leq (1+p)\gamma(p)^\dagger$		PoA $= \gamma(p)^\dagger$ (selfish players) [8]
second-price auctions	$= 2$		RPoA = 2 (selfish players) [22]
valid utility games	$= 2^\ddagger$	$= 2^\ddagger$ [7]	RPoA = 2 (selfish players) [19]

* holds only if a certain weight condition is satisfied
§ for the special case $R\|\|\sum_j C_j$ only
† $\gamma(p) = p^{p(1-o(1))}$
‡ for the altruism model only

- We derive a general reduction technique to bound the robust PoA of both altruism and friendship games. Basically, the reduction can be applied whenever the underlying base game is *social contribution bounded*, meaning that the direct cost of each player is bounded by his respective cost in the corresponding SCG (for the friendship model a slightly stronger condition needs to hold). It is worth mentioning that this reduction preserves the (λ, μ)-smoothness parameters, i.e., the altruism or friendship game inherits the (λ, μ)-smoothness parameters of the SCG.
- We generalize smoothness for friendship extensions to *weight-bounded* social cost functions. In previous papers, the used techniques usually required sum-boundedness, which is a stronger condition [1]. Applying this definition to scheduling games with weighted sum as social cost, we derive a nice characterization of those scheduling games whose robust PoA does not grow for friendship extensions.
- We show that social contribution boundedness is satisfied by several well-known games, like min-sum scheduling games, congestion games, second-price auctions and valid utility games. Using our reduction technique, we then derive upper bounds on the robust PoA of their friendship/altruism extensions. In most cases we prove matching lower bounds. The results are summarized in Table 1.

Even though we focus on the complete information setting in this paper, our results extend to the incomplete information setting in which players are uncertain about the friendship levels of the other players. More details will be given in the full version of the paper.

Related Work. Several articles propose models of altruism and spite [2, 4–7, 11, 14–16]. Among these articles, the inefficiency of equilibria in the presence of altruism and spite was studied for various games in [2, 4–7, 11]. After its introduction in [19], the smoothness framework has been extended to incomplete information settings [20, 22] and altruism/spite settings [1, 7].

The robust PoA for minsum scheduling (not taking altruism or friendship into account) was studied in various papers. In [17] the authors show that it does not exceed 2 for $Q||\sum_j C_j$ (here we improve this bound to $\frac{3}{2} - \frac{1}{2m}$ for the special case $P||\sum_j C_j$). A robust PoA of 4 for $R||\sum_j w_j C_j$ has been proven in [9]. Our work on linear congestion games generalizes a result in [2]. They show that the pure price of anarchy does not exceed 17/3 in a restricted friendship setting ($\alpha_{ij} \in \{0, 1\}$).

As indicated above, most related to our work are the articles [1, 7]. We significantly improve the bounds on the robust price of anarchy for congestion games and unrelated machine scheduling games in [1] and at the same time simplify the analysis by using our reduction technique.

2 Preliminaries

Let $G = (N, \{\Sigma_i\}_{i\in N}, \{C_i\}_{i\in N})$ be a *cost-minimization game*, where $N = [n]$ is the set of players, Σ_i is player i's strategy space, $\Sigma = \prod_{i\in N} \Sigma_i$ is the set of strategy profiles, and $C_i : \Sigma \to \mathbb{R}$ denotes the cost player i must pay for a given strategy profile. We assume that each player seeks to minimize his cost. A *social cost function* $C : \Sigma \to \mathbb{R}$ assigns a social cost to each strategy profile. We usually require C to be *sum-bounded*, i.e., $C(s) \le \sum_{i\in N} C_i(s)$ for all $s \in \Sigma$.

We denote *payoff-maximization games* as $G = (N, \{\Sigma_i\}_{i\in N}, \{\Pi_i\}_{i\in N})$ with *social welfare* $\Pi : \Sigma \to \mathbb{R}$. In this case, each player i tries to maximize his *utility (or payoff)* Π_i. Again, we usually assume that Π is *sum-bounded*, i.e. $\Pi(s) \ge \sum_{i\in N} \Pi_i(s)$ for all $s \in \Sigma$.

Subsequently, we state most of the definitions and theorems only for cost-minimization games. The payoff-maximization case works similarly by reversing all inequalities. So, unless stated otherwise, G denotes a cost-minimization game with social cost function C.

Definition 1. *A* coarse equilibrium *is a probability distribution σ over Σ such that the following holds: If s is a random variable with distribution σ, then for all players i and all strategies $s_i^* \in \Sigma_i$, $\mathbf{E}_{s\sim\sigma}[C_i(s)] \le \mathbf{E}_{s_{-i}\sim\sigma_{-i}}[C_i(s_i^*, s_{-i})]$, where σ_{-i} is the projection of σ on $\Sigma_{-i} = \prod_{j\neq i} \Sigma_j$. A* mixed Nash equilibrium *is a coarse equilibrium σ that is the product of independent probability distributions σ_i on Σ_i. A* (pure) Nash equilibrium (NE) *is a strategy profile $s \in \Sigma$ such that for all $s^* \in \Sigma$, $C_i(s) \le C_i(s_i^*, s_{-i})$, where $s_{-i} = s|_{\Sigma_{-i}}$.*

The coarse (resp. correlated, mixed, pure) price of anarchy (PoA) *is defined as* $\sup_s C(s)/C(s^*)$*, where s^* minimizes C and s runs over the coarse (resp. correlated, mixed, pure) Nash equilibria of G.*[3] *The coarse (resp. correlated, mixed, pure) PoA of a class $\mathcal{G}$ of games is defined as the supremum of the respective PoA values of games in $\mathcal{G}$.*

[3] Similarly, we define the respective types of PoA for a payoff-maximization game as $\sup \Pi(s^*)/\Pi(s)$, where s and s^* are as above.

Note that pure Nash equilibria constitute a subset of mixed Nash equilibria which constitute a subset of coarse equilibria. This implies that the respective prices of anarchy are non-decreasing (in this order).

Due to lack of space, several proofs were omitted from this extended abstract and will be given in the full version of the paper.

2.1 The Altruism Model

Definition 2 ([7]). *Let* $\alpha \in [0,1]^N$*. The* α-altruistic extension *of* G *is defined as the cost-minimization game* $G^\alpha = (N, \{\Sigma_i\}_{i \in N}, \{C_i^\alpha\}_{i \in N})$*, where for any* $i \in N$ *the* perceived cost *is the convex combination* $C_i^\alpha = (1-\alpha_i)C_i + \alpha_i C$*. We call* G *the* base game*. The social cost function of* G^α *is again* C*, i.e., the cost of the base game.*

The higher the 'altruism level' α_i, the more i cares about the society in general.

Definition 3. *Let* G *have sum-bounded social cost and let* $\alpha \in [0,1]^N$*. Define* $C_{-i} := C - C_i$*.* G^α *is* (λ, μ)-smooth *if there exists an optimal strategy* s^* *such that for any strategy* $s \in \Sigma$*,*

$$\sum_{i \in N} \big(C_i(s_i^*, s_{-i}) + \alpha_i(C_{-i}(s_i^*, s_{-i}) - C_{-i}(s))\big) \leq \lambda C(s^*) + \mu C(s),$$

The robust PoA *of* G^α *is defined as* $\inf\{\frac{\lambda}{1-\mu} \mid G^\alpha$ *is* (λ,μ)*-smooth,* $\mu < 1\}$.

Theorem 1 ([7]). *Let* G^α *be an* α*-altruistic extension of* G*. Then the coarse (and thus the correlated, mixed and pure) PoA of* G^α *is bounded from above by the robust PoA of* G^α*.*

2.2 The Friendship Model

Definition 4 ([1]). *Let* $\alpha \in [0,1]^{N \times N}$ *such that* $\alpha_{ii} = 1$ *for all* $i \in N$*. The* α-friendship extension *of* G *is defined as* $G^\alpha = (N, \{\Sigma_i\}_{i \in N}, \{C_i^\alpha\}_{i \in N})$*, where for any* $i \in N$ *the* perceived cost *is defined as* $C_i^\alpha = \sum_j \alpha_{ij} C_j$*. Like in the altruism model, we consider* C*, the social cost function of the base game, as the social cost for* G^α*.*

For players i and j, α_{ij} can be interpreted as the level of affection i feels towards j. Note that if $C = \sum_j C_j$, then the altruism model is a special case of the friendship model because in this case, $C_i^\alpha = C_i + \sum_{j \neq i} \alpha_i C_j$ (for $\alpha \in [0,1]^N$).

Next we adapt the smoothness definition in [1] for the friendship model to the weighted player case.

Definition 5. *Let* G^α *be friendship extension of a cost-minimization game with a* weight-bounded *social cost function, i.e.,* $C \leq \sum_i w_i C_i$ *for some* $w \in \mathbb{R}_+^N$*.* G^α

is (λ, μ)-smooth *if there exists a (possibly randomized) strategy profile* $\bar{s}$ *such that for all strategy profiles* s *and all optima* s^*,

$$\sum_{i \in N} w_i\big(C_i(\bar{s}_i, s_{-i}) + \sum_{j \neq i} \alpha_{ij}(C_j(\bar{s}_i, s_{-i}) - C_j(s))\big) \leq \lambda C(s^*) + \mu C(s).$$

We define the robust PoA *of* G^α *as* $\inf\{\frac{\lambda}{1-\mu} \mid G^\alpha$ *is* (λ, μ)*-smooth,* $\mu < 1\}$.

Theorem 2. *Let* G^α *be a friendship extension of a cost-minimization game with weight-bounded social cost function* C*. If* G^α *is* (λ, μ)*-smooth with* $\mu < 1$*, then the coarse PoA of* G^α *is at most* $\frac{\lambda}{1-\mu}$.

In both models, we can replace the deterministic factor α by a stochastic variable that is distributed with respect to some probability distribution over $[0,1]^N$ (in the altruism model) or $[0,1]^{N \times N}$ (in the friendship model). Thus, we can incorporate *incomplete information* into our model, reflecting the fact that often players are uncertain about other players' feelings. The bounds on the PoA continue to hold in this case. We defer the details to the full version of the paper.

3 Social Contribution Games

Definition 6. *We call* G *a* (cost-minimization) social contribution game (SCG) *if for all players* i *there exists a default strategy* $\emptyset_i$ *such that for all* $s \in \Sigma$, $C_i(s) = C(s) - C(\emptyset_i, s_{-i})$.

The strategy $\emptyset_i$ is often interpreted as 'refusing to participate in the game'. In that sense, i pays exactly the social cost he causes by choosing to play; in the payoff-maximization case, he gets exactly what he contributes to the social welfare. So social contribution games are 'fair' in some sense.

Basic utility games [23] satisfy the definition of an SCG (see also Section 7). In particular, the competitive facility location game (which is a basic utility game by [23]) is an SCG.

We now show that social contribution games satisfy the following invariance property with respect to their α-altruistic extensions.

Lemma 1. *Any social contribution game is* altruism-independently smooth*, i.e., for all* $\alpha = (\alpha_i)_{i \in N}$ *and corresponding altruistic extensions* G^α *of* G*, the robust price of anarchy in* G *and* G^α *is the same.*

Proof. For all players i, $C_{-i}(s) = C(s) - C_i(s)$ is independent of s_i since $C(s) - C_i(s) = C(\emptyset_i, s_{-i})$. Thus for all strategy profiles s, s^*, and all $\alpha \in \mathbb{R}^N$,

$$\sum_i \big(C_i(s_i^*, s_{-i}) + \alpha_i(C_{-i}(s_i^*, s_{-i}) - C_{-i}(s))\big) = \sum_i C_i(s_i^*, s_{-i}).$$

It follows that for all $(\lambda, \mu) \in \mathbb{R}^2$, G^α is (λ, μ)-smooth iff G is. □

The notions of α-altruistic extensions and α-independent smoothness can be easily extended to $\alpha \in \mathbb{R}^N$. The above lemma continues to hold in this case. So even if a player wants to *hurt* society, the robust PoA stays the same for SCGs.

3.1 Social Contribution Bounded Games

Definition 7. *Assume C is sum-bounded. We call G* social contribution bounded (SC-bounded) *if for all players i there exists a default strategy $\emptyset_i$ such that for all $s \in \Sigma$, $C_i(s) \leq C(s) - C(\emptyset_i, s_{-i})$. In this case, we define* the corresponding social contribution game $\bar{G} = (N, \{\Sigma_i\}_{i \in N}, \{\bar{C}_i\}_{i \in N})$ *by setting* $\bar{C}_i(s) = C(s) - C(\emptyset_i, s_{-i})$.

As before, we think of $\emptyset_i$ as the option that i does not participate.[4]

The following theorem shows that if we want to get a bound on the PoA of α-altruistic extensions of an SC-bounded game, we might as well consider the corresponding SCG *regardless of* α.

Theorem 3. *Let G be SC-bounded and suppose that the robust PoA of the corresponding SCG $\bar{G}$ is ξ. Then for all altruistic extensions G^α of G, the robust PoA is at most ξ.*

In order to be able to derive our results for the *friendship* extensions, we need a slightly stronger definition.

Definition 8. *A cost minimization game G with weight-bounded social cost is* strongly SC-bounded *if for all $s \in \Sigma$ and every player i:*

1. $C_i(\emptyset_i, s_{-i}) = 0$ *(if i does not participate, he pays nothing)*
2. $\forall j \neq i : \; C_j(\emptyset_i, s_{-i}) \leq C_j(s)$ *(other players' costs can only increase if i participates)*
3. $w_i \sum_j (C_j(s) - C_j(\emptyset_i, s_{-i})) \leq C(s) - C(\emptyset_i, s_{-i})$ *(the weighted impact of i's participation on the players' costs is bounded by his impact on the social cost)*

If all weights are 1, then assumption (3) easily follows from
3b. $C(s) = \sum_j C_j(s)$ *(social cost is sum of individual costs).*

Theorem 4. *Let G be strongly SC-bounded. Suppose the robust PoA of $\bar{G}$ is ξ. Then for all friendship extensions G^α, the robust PoA is at most ξ.*

Proof. We have for every player i,

$$w_i\big(C_i(\bar{s}_i, s_{-i}) + \sum_{j \neq i} \alpha_{ij}(C_j(\bar{s}_i, s_{-i}) - C_j(s))\big)$$
$$\overset{(2)}{\leq} w_i\big(C_i(\bar{s}_i, s_{-i}) + \sum_{j \neq i} \alpha_{ij}(C_j(\bar{s}_i, s_{-i}) - C_j(\emptyset_i, s_{-i}))\big)$$

[4] Note that $\emptyset_i$ need not actually be an element of Σ_i. In many games (such as scheduling or congestion games) it is not an option to not participate. So, formally we should require that there exists a function $\mathfrak{C} : \prod_i (\Sigma_i \cup \{\emptyset_i\}) \to \mathbb{R}$ such that $\mathfrak{C}|_\Sigma = C$ and $C_i(s) \leq \mathfrak{C}(s) - \mathfrak{C}(\emptyset_i, s_{-i})$ for all i and s. However, there is a natural way to extend C (and C_i) on $\prod_i (\Sigma_i \cup \{\emptyset_i\})$, as we will see later. For notational convenience, we write C instead of $\mathfrak{C}$.

$$\overset{(2)}{\leq} w_i\big(C_i(\bar{s}_i, s_{-i}) + \sum_{j \neq i}(C_j(\bar{s}_i, s_{-i}) - C_j(\emptyset_i, s_{-i}))\big)$$

$$\overset{(1)}{=} w_i \sum_j (C_j(\bar{s}_i, s_{-i}) - C_j(\emptyset_i, s_{-i})) \overset{(3)}{\leq} C(\bar{s}_i, s_{-i}) - C(\emptyset_i, s_{-i}) = \bar{C}_i(\bar{s}_i, s_{-i}).$$

Summing over all i, it follows that if $\bar{G}$ is (λ, μ)-smooth[5], then so is G^α. □

If all weights are 1, then SC-boundedness follows from strong SC-boundedness. To see this, consider the case where $\alpha = \mathbf{0}$ and carry out the proof of Theorem 4 for s instead of $(\bar{s}_i, s_{-i})$.

4 Minsum Machine Scheduling

A *scheduling game* $G = (m, n, (p_{ij})_{i \in M, j \in N}, (w_j)_{j \in N})$ consists of a set of jobs (players) $[n] = \{1, \dots, n\}$ and a set of machines $[m] = \{1, \dots, m\}$. For each machine i and job j, $p_{ij} \in \mathbb{R}_+$ denotes the *processing time* of j on i. Furthermore, w_j is the *weight* of job j. The strategy space Σ_i of a job j is simply the set of machines. By $\emptyset_i = \emptyset$ we mean the strategy where i uses no machine.

Let x be a strategy profile. For a machine i, we denote by X_i the set of jobs that are scheduled on i. Furthermore, x_j denotes the machine j is assigned to. Following the notation by Cole et al. [9], we define $\rho_{ij} = p_{ij}/w_j$. We assume that the jobs on a machine are scheduled in increasing order of ρ_{ij}, which is known as *Smith's rule* [21]; if two jobs on a machine have the same time-to-weight ratio, we use a tie-breaking rule. The *cost* C_j of job j which it seeks to minimize is simply its completion time. In the following, we assume for simplicity that the ρ_{ij} are pairwise distinct (but the results continue to hold without this assumption). Then we can write $C_j(x) = \sum_{k \in X_i : \rho_{ik} \leq \rho_{ij}} p_{ik}$. The social cost C we consider is the weighted sum of the players' completion times, i.e., $C = \sum_j w_j C_j$.

In the following, we use the three-field notation by Graham et al [13]. In this notation, the problem we described is denoted by $R||\sum_j w_j C_j$. If all weights are 1, we write $\sum_j C_j$ instead of $\sum_j w_j C_j$. Furthermore, if there are *speeds* s_i for each machine i and *fixed* processing times p_j for each job such that $p_{ij} = p_j/s_i$, we write Q instead of R. Finally, if we have in addition identical speeds $s_i = 1$ for all machines i, the problem is denoted by P.

4.1 $R||\sum_j w_j C_j$

Lemma 2 ([9]). *For all strategy profiles x and x^*,*

$$\sum_{i \in [m]} \sum_{j \in X_i^*} w_j p_{ij} + \sum_{i \in [m]} \sum_{j \in X_i^*} \sum_{k \in X_i} w_j w_k \min\{\rho_{ij}, \rho_{ik}\} \leq 2C(x^*) + \frac{1}{2}C(x),$$

where X_i^* is defined similarly to X_i as $X_i^* = \{j \in J \mid x_j^* = i\}$.

[5] in the sense that there exist $\bar{s} \in \Sigma$ and an optimal $s^* \in \Sigma$ such that for all $s \in \Sigma$ it holds that $\sum_i C_i(\bar{s}_i, s_{-i}) \leq \lambda C(s) + \mu C(s^*)$, generalizing Roughgarden's definition of smoothness [19].

Proof. The claim is shown in the proof of [9, Theorem 3.2]. □

Theorem 5. *Let G be an instance of $R||\sum_j w_j C_j$ that satisfies the following condition for all jobs j, k and all machines i: $\rho_{ij} \le \rho_{ik}$ implies $w_j \le w_k$ (i.e., if k gets scheduled after j on i, then it is because of its processing time, not its weight). Then the robust PoA of all friendship extensions G^α of G is at most 4.*

For jobs j and k, α_{jk} has an influence on j's strategy in an equilibrium only if there is a machine i such that k gets scheduled after j on i because j cannot influence k's costs otherwise. Hence the weight condition tells us that the only jobs that could potentially have an influence on j are in fact the jobs that are at least equally important as j. Hence j cannot 'misplace his affections' and care too much about unimportant jobs.

Proof. First we show that G is strongly SC-bounded. Clearly, (1) and (2) are satisfied. For (3), note that for all jobs j, strategy profiles x, and $i = x_j$,

$$\begin{aligned} w_j \sum_k (C_k(x) - C_k(\emptyset, x_{-i})) &= w_j \Big(C_j(x) + \sum_{k \in X_i \colon \rho_{ik} > \rho_{ij}} p_{ij} \Big) \\ &\le w_j C_j(x) + \sum_{k \in X_i \colon \rho_{ik} > \rho_{ij}} w_k p_{ij} = \bar{C}_j(x), \end{aligned}$$

where the inequality follows from the condition on the weights. We calculate

$$\begin{aligned} \bar{C}_j(x_j^*, x_{-j}) &= w_j C_j(x_j^*, x_{-j}) + \sum_{k \in X_i \colon \rho_{ik} > \rho_{ij}} w_k p_{ij} \\ &= w_j p_{ij} + \sum_{k \in X_i \colon \rho_{ik} < \rho_{ij}} w_k w_j \rho_{ik} + \sum_{k \in X_i \colon \rho_{ik} > \rho_{ij}} w_k w_j \rho_{ij} \\ &\le w_j p_{ij} + \sum_{k \in X_i} w_j w_k \min\{\rho_{ij}, \rho_{ik}\}. \end{aligned}$$

Summing over all machines i and $j \in X_i^*$, this is the same expression as in Lemma 2. Hence $\sum_j \bar{C}_j(x_j^*, x_{-j}) \le 2C(x^*) + \frac{1}{2} C(x)$ and $\bar{G}$ is $(2, \frac{1}{2})$-smooth. It follows by Theorem 4 that the robust PoA in G^α is at most 4. □

This bound is tight and the weight condition is necessary. In fact, if we drop it, the pure PoA is unbounded even for $P||\sum_j w_j C_j$ instances with unit-size jobs. We defer these results to the full version.

4.2 $P||\sum_j C_j$

Fix an ordering of the jobs such that $p_j > p_{j'}$ implies $j > j'$. We use the same notation as in [17]: For a schedule x, a job j and a machine i, let $h_i^x(j) = |\{j' > j | x_{j'} = x_j\}|$. This is the number of jobs that are scheduled after j on i. Using this notation, we can write $\bar{C}_j(x) = C_j(x) + h_{x_j}^x(j) \cdot p_j$ for instances with unit speeds. Throughout this section, let $\bar{x}$ denote the randomized schedule that assigns each job to each machine with probability $\frac{1}{m}$.

The following theorem will be helpful to establish an upper bound on the robust PoA for the friendship model and might be of independent interest.

Theorem 6. *For any schedule x and any optimal x^*, $\sum_j C_j(\bar{x}_j, x_{-j}) \leq C(x^*) + (\frac{1}{2} - \frac{1}{2m}) \sum_j p_j$. In particular, the robust price of anarchy of $P||\sum_j C_j$ is at most $\frac{3}{2} - \frac{1}{2m}$. This bound is tight.*

Theorem 7. *Let G be an instance of $P||\sum_j C_j$. Then the robust PoA for any friendship extension G^α is at most 2.*

Proof. Let x be arbitrary. Then by linearity of expectation,

$$\mathbf{E}\Big[\sum_j \bar{C}_j(\bar{x}_j, x_{-j})\Big] = \sum_j \mathbf{E}[C_j(\bar{x}_j, x_{-j})] + \sum_j \mathbf{E}[h^x_{\bar{x}_j}(j)] \cdot p_j.$$

We know that

$$\mathbf{E}[h^x_{\bar{x}_j}(j)] = \frac{1}{m} \sum_i h^x_i(j) = \frac{1}{m} |\{j' \in J \mid j' > j\}| = \mathbf{E}[h^{\bar{x}}_{x_j}(j)].$$

Hence the second term evaluates as

$$\sum_j \mathbf{E}[h^x_{\bar{x}_j}(j)] \cdot p_j = \sum_j \mathbf{E}[h^{\bar{x}}_{x_j}(j)] \cdot p_j = \sum_j \mathbf{E}[C_j(\bar{x}_j, x_j)] - \sum_j p_j.$$

We know by Theorem 6 that $\sum_j \mathbf{E}[C_j(\bar{x}_j, x_{-j})] \leq C(x^*) + (\frac{1}{2} - \frac{1}{2m}) \sum_j p_j$. Hence

$$\sum_j \mathbf{E}[\bar{C}_j(\bar{x}_j, x_{-j})] = 2 \sum_j \mathbf{E}[C_j(\bar{x}_j, x_{-j})] - \sum_j p_j \leq 2C(x^*) - \frac{1}{m} \sum_j p_j \leq 2C(x^*),$$

for any schedule x^*. Hence the robust PoA for the friendship extension is at most 2. ☐

5 Congestion Games

An *atomic congestion game* $G = (N, E, \{\Sigma_i\}_{i \in N}, (d_e)_{e \in E})$ is given by a set E of *resources* together with *delay functions* $d_e : \mathbb{N} \to \mathbb{R}_+$ indicating the delay on e for a given number of players using e. Each player's strategy set consists of subsets of E; $\Sigma_i \subseteq \mathcal{P}(E)$ for all i. For $s \in \Sigma$, let $x_e(s) = |\{i \in N \mid e \in s_i\}|$. The cost of each player i under s is given by $C_i(s) = \sum_{e \in s_i} d_e(x_e(s))$. If all delay functions are linear, we say that G is *linear*. Further, if all delay functions are polynomials of maximum degree p with non-negative coefficients, we say that G is *p-polynomial*. The social cost C is simply the sum over all individual cost. By $\emptyset_i = \emptyset$ we mean the strategy where player i uses no machine.

It is known that we can without loss of generality assume that all latency functions are of the form $l_e(x) = x$. This was first mentioned in [8]; for a proof see [7]. The following lemma is shown in the proof of [8, Theorem 1].

Lemma 3 ([8]). *Let G be a linear congestion game and $s, s^* \in \Sigma$. Then $\sum_i C_i(s^*_i, s_{-i}) \leq \sum_e x_e(s^*)(x_e(s) + 1)$.*

Lemma 4 ([2]). *For any pair* $\alpha, \beta \in \mathbb{N}$*, it holds that* $\frac{2}{5}\alpha^2 + \frac{17}{5}\beta^2 \geq \beta(2\alpha + 1)$.

Bilò et al. show in their paper [2] that the *pure* PoA lies between 5 and 17/3 for a restricted friendship setting, where $\alpha_{ij} \in \{0, 1\}$ for all i, j. We generalize their result to the *robust* PoA for arbitrary $\alpha_{ij} \in [0, 1]$ and show tightness.

Theorem 8. *Let* G *be a linear congestion game. Then the robust PoA of all friendship extensions* G^α *is bounded by* $\frac{17}{3} \approx 5.67$*. This bound is tight.*

Proof. We have

$$\bar{C}_i(s) = C_i(s) + \sum_{e \in s_i} |\{j \neq i \mid e \in s_j\}| = C_i(s) + \sum_{e \in s_i} x_e(\emptyset, s_{-i}) \geq C_i(s),$$

so G is SC-bounded. Also G is strongly SC-bounded: If i does not use any resource, he experiences no cost; the other's costs can only increase if another player enters; and finally, $C = \sum_j C_j$.

Let $s, s^* \in \Sigma$. We abbreviate $x_e(s)$ and $x_e(s^*)$ by x_e and x_e^*, respectively. The calculation of the robust PoA for $\bar{G}$ yields

$$\sum_i \bar{C}_i(s_i^*, s_{-i}) = \sum_i C_i(s_i^*, s_{-i}) + \sum_i \sum_{e \in s_i^*} x_e(\emptyset, s_{-i}).$$

The first term is at most $\sum_e x_e^*(x_e+1)$ by Lemma 3. The second term is bounded by $\sum_i \sum_{e \in s_i^*} x_e(s) = \sum_{e \in E} x_e x_e^*$. Hence we get in total by Lemma 4

$$\sum_e x_e^*(2x_e + 1) \leq \sum_e \left(\frac{17}{5}(x_e^*)^2 + \frac{2}{5}x_e^2 \right) = \frac{17}{5}C(s^*) + \frac{2}{5}C(s).$$

It follows that the robust PoA of $\bar{G}$ is at most $\frac{17}{5}/(1 - \frac{2}{5}) = \frac{17}{3}$.

We show now that the bound of $\frac{17}{3}$ is asymptotically tight. Let $n \geq 0$. Consider an instance with $n+3$ blocks of players $B_0, \ldots, B_{n+2}$ consisting of three players each: $B_k = \{a_k, b_k, c_k\}$. We construct a NE s and an optimal strategy profile s^* as follows. For all resources e, we set $l_e(x) = x$. For $0 \leq k \leq n$, the pattern of strategies repeats (see Figure 1). Here player $i = a_k$ has two strategies $s_i = \{3k, 3k+1, 3k+2\}$ and $s_i^* = \{3k+6\}$. Player $i = b_k$ has two strategies $s_i = \{3k+2, 3k+3\}$ and $s_i^* = \{3k+7\}$. Player $i = c_k$ has two strategies $s_i = \{3k+3, 3k+4\}$ and $s_i^* = \{3k+8\}$.

The strategies s_i of players in the final blocks B_{n+1} and B_{n+2} are defined as above. However, we need to change the definition of s_i^* because otherwise, s is not a Nash equilibrium. So for each $i \in B_{n+1} \cup B_{n+2}$, we insert sets of new, previously unused resources s_i^* such that $C_i(s_i) = |s_i^*|$.

We define $\alpha_{ij} = 1$ for the following pairs of players: (a_k, b_{k+1}), (a_k, c_{k+1}), (a_k, a_{k+2}) as well as $(b_k, c_{k+1}), (b_k, a_{k+2})$ and $(c_k, a_{k+2}), (c_k, b_{k+2})$, where $0 \leq k \leq n$. All other α_{ij} are zero. Hence $\alpha_{ij} = 1$ iff s_i^* intersects s_j. Note that if $s_i \cap s_j \neq \emptyset$, then $\alpha_{ij} = 0$.

Now, we claim that s is a NE. In fact, for all $0 \leq k \leq n$ and $i = a_k$, $C^\alpha(s) = C(s) + \sum_{j \neq i} \alpha_{ij} C_j(s) = 7 + 5 + 5 + 7 = 24$, which equals $C_i^\alpha(s^*, s_{-i}) =$

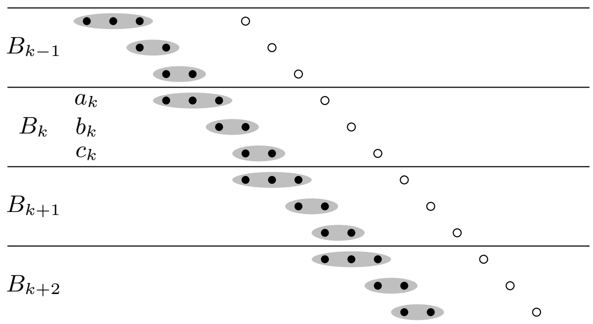

Fig. 1. The strategy profiles s (grey) and s^* (white). Columns correspond to resources.

$4+6+6+8$. A similar calculation shows $C_i(s) = C_i(s_i, s_{-i})$ for $i = b_k, c_k$. Observe that for $k = n+1, n+2$, and $i \in B_k$, $C_i^\alpha(s) = C(s) = |s_i^*| = C(s_i^*, s_{-i})$ by our construction of s_i^*. Hence s is indeed a NE.

For $k = 1, \ldots, n$, block B_k has the same cost: $C(B_k) := \sum_{i \in B_k} C_i(s) = 17$ and $C^*(B_k) := \sum_{i \in B_k} C_i(s^*) = 3$. Let $X = C(B_0) + C(B_{n+1}) + C(B_{n+2})$ and $X^* = C^*(B_0) + C^*(B_{n+1}) + C^*(B_{n+2})$ and observe that these are constants independent of n. It follows that

$$\frac{C(s)}{C(s^*)} = \frac{17n + X}{3n + X^*} = \frac{17 + o(n)}{3 + o(n)}. \qquad \square$$

We obtain the following result for friendship extensions of p-polynomial congestion games. Note that the pure PoA of the base game is $\gamma(p) := p^{p(1-o(1))}$ [8]. That is, altruism increases the PoA by at most a factor of $(1+p)$ in this case.

Theorem 9. *Let G be a p-polynomial congestion game. Then the robust PoA of all friendship extensions G^α is bounded by $(1+p) \cdot p^{p(1-o(1))}$.*

6 Second-Price Auctions

A single-item auction G consists of an *allocation rule* $a : \Sigma \to N$ which determines which bidder gets the item and a *pricing rule* $p : \Sigma \to \mathbb{R}^N$ indicating how much each player should pay. Each bidder i is assumed to have a certain *valuation* $v_i \in \mathbb{R}_+$ for the item. For a given bidding profile $b \in \mathbb{R}_+^N$, the social welfare is $\Pi(b) = v_{a(b)}$. Player i's utility is given by $\Pi_i(b) = v_i - p_i(b)$ if he gets the object and $-p_i(b)$ otherwise. In a *second-price auction*, the highest bidder gets the item and pays the second highest bid, while everybody else pays nothing.

We do not allow *overbidding*, i.e., for all bidders i, $b_i \leq v_i$. This is a standard assumption because overbidding is a dominated strategy. We denote by $\beta(b, i)$ the name of the player who places the i-th highest bid in b. We write $\beta(i)$ instead of $\beta(b, i)$ if the bidding profile is clear from the context. $\emptyset_i = 0$ denotes the strategy where bidder i bids nothing.

Note that here the friendship model is *not* a generalization of the altruism model because $\Pi \neq \sum_i \Pi_i$. We summarize our results in the following theorem.

Theorem 10. *Let G be a second-price auction. Then the robust PoA of all altruism extensions G^α is at most 2. Further, the coarse PoA of the class of friendship extensions of G is exactly 2.*

7 Valid Utility Games

A *valid utility game* [23] is defined as a payoff-maximization game $G = (N, E, \{\Sigma_i\}_{i \in N}, \{\Pi\}_{i \in N}, V)$, where E is a ground set of resources, $\Sigma_i \subseteq \mathcal{P}(E)$ and V is a submodular and non-negative function on E. The social welfare Π is given by $\Pi(s) = V(\bigcup_{i \in N} s_i)$ and is assumed to be sum-bounded. Furthermore, we require G to satisfy $\Pi_i(s) \geq \Pi(s) - \Pi(\emptyset, s_{-i})$ for all $s \in \Sigma$. If G additionally satisfies the last inequation with equality, it is called *basic utility game* [23]. For all players i, set $\emptyset_i = \emptyset$.

Theorem 11 ([19]). *The robust PoA of valid utility games with non-decreasing*[6] *set function V is bounded by 2.*

An example for valid utility games with non-decreasing set functions are competitive facility location games without fixed costs [23].

The following theorem has already been proven in [7] and tightness of this bound has been shown in [3] for the base game. We now use our framework to provide a shorter proof that illustrates why the robust PoA does not increase for altruistic extensions: The corresponding SCG falls into the same category of games.

Theorem 12. *Let G be a valid utility game with non-decreasing V. Then the robust price of anarchy of every altruistic extension G^α of G is bounded by 2.*

Proof. It follows directly from the definition that G is SC-bounded. It is easy to verify that the corresponding SCG $\bar{G} = (N, E, \{\Sigma_i\}_{i \in N}, \{\bar{\Pi}\}_{i \in N}, V)$ is again a valid utility game: $\sum_i \bar{\Pi}_i(s) \leq \sum_i \Pi_i(s) \leq \Pi(s)$ and $\bar{\Pi}_i(s) = \Pi(s) - \Pi(\emptyset, s_{-i})$. So the robust PoA of $\bar{G}$ is at most 2. Our claim follows by Theorem 3. □

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[6] where non-decreasing means that for all $A \subseteq B \subseteq E$ it holds that $V(A) \leq V(B)$.

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