Cross-Monotonic Cost-Sharing Methods for Connected Facility Location Games

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ABSTRACT

We devise cost sharing methods for connected facility location games that are cross-monotonic, competitive and recover a constant fraction of the optimal cost. The novelty of this work is that we use randomized algorithms and that we share the *expected cost* among the participating users. We also provide a primal-dual cost sharing method for the connected facility location game with opening costs.

Categories and Subject Descriptors

F.2 [**Analysis of Algorithms and Problem Complexity**]: General; G.0 [**Mathematics of Computing**]: General

General Terms

algorithms, design, theory

Keywords

cost sharing, mechanism design, connected facility location

1. INTRODUCTION

The problem of achieving truth-revealing or strategyproof mechanisms for sharing the cost of deploying a network infrastructure has recently received a growing attention. In this work, we are interested in designing cost sharing mechanisms that would encourage agents to cooperate to share the cost of the network facility and to reveal their true values for receiving the service, i.e., *group-strategyproof* mechanisms for which truthfulness is a dominant strategy for every user or coalition of users.

We are given a set U of (potential) users. Each user $j \in U$ has a *utility* u_j for receiving the service; if j is asked to pay more than u_j, he prefers to drop the offer. For a subset $Q \subseteq U$, let $C(Q)$ denote the cost of servicing all users in Q. A *cost sharing mechanism* determines (i) a set $Q \subseteq U$ of participating users that receive the service, and (ii) how to distribute the servicing cost $C(Q)$ among the users in Q such that each user $j \in Q$ is willing to pay his cost share, p_j. The *benefit* of a user j is $u_j - p_j$ if $j \in Q$, and zero otherwise. We assume that each user is selfish and hence may misreport his utility so as to maximize his benefit. A cost sharing mechanism is *strategyproof* if each user has no incentive to misreport his utility; it is said to be *group-strategyproof* if the same holds even if users collude.

Given a set Q of participating users, a *cost sharing method* ξ computes a cost share $\xi_j(Q)$ for each user $j \in Q$. A cost sharing method is *cross-monotonic* if it satisfies

$$\forall Q' \subseteq Q \subseteq U,\ \forall j \in Q', \quad \xi_j(Q') \geq \xi_j(Q).$$

Moulin and Shenker [5] proved that, given a cross-monotonic cost sharing method ξ, one can devise a cost sharing mechanism that is group-strategyproof.

We are interested in cost sharing methods that satisfy *competitiveness* and *approximate cost recovery*. That is, (i) the participating users in Q are charged at most the optimal cost, $C^*(Q)$, i.e., $\sum_{j \in Q} \xi_j(Q) \leq C^*(Q)$, and (ii) at least a constant fraction $1/\lambda$ of the cost $C(Q)$ of the constructed solution is recovered, i.e., $\sum_{j \in Q} \xi_j(Q) \geq C(Q)/\lambda$. We call such a method a λ-*approximate cost sharing method*.

We devise approximate cross-monotonic cost sharing methods for connected facility location games (*CFL*). We are given an undirected graph $G = (V, E)$ with non-negative edge costs c_e, a set $\mathcal{F} \subseteq V$ of facilities with opening cost f_i for each facility $i \in \mathcal{F}$, a set $U \subseteq V$ of users and a parameter $M > 1$. The goal is to open a subset $F \subseteq \mathcal{F}$ of facilities, to connect each user $j \in U$ to the closest open facility $i(j) \in F$ and to build a Steiner tree T connecting all open facilities in F. The objective is to minimize

$$\sum_{i \in F} f_i + \sum_{j \in U} c(j, i(j)) + M \cdot c(T),$$

where $c(\cdot, \cdot)$ is the shortest path distance with respect to c, and $c(T)$ is the cost of the edges in the Steiner tree T. We assume without loss of generality that a root node $r \in \mathcal{F}$, which is open in some optimal solution, is known in advance.

In *rent-or-buy* network design problems an edge e can either be *bought* at cost $M \cdot c_e$, or *rented* at cost c_e; a bought edge can be used by an arbitrary number of paths, while a rented edge e costs c_e for each path that uses it. The *single-source rent-or-buy problem* (*SSRB*) is a special case of *CFL*, where a facility can be opened at any node and all opening costs are zero, i.e., $\mathcal{F} = V$ and $f_i = 0$ for each $i \in \mathcal{F}$.

2. RELATED WORK

Moulin and Shenker [5] developed cross-monotonic cost sharing methods if the optimal cost function is a submodular function of the set U. However, this is not the case for

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several network design problems such as Steiner tree, facility location or rent-or-buy network design. Jain and Vazirani [4] presented a cross-monotonic cost sharing method for the minimum spanning tree game and therefore a 2-approximate cost sharing method for the Steiner tree game. Devanur, Mihail and Vazirani [1] proposed strategyproof mechanisms for vertex cover and facility location games based on primal-dual algorithms. However, their algorithms are not group-strategyproof. Very recently, Pál and Tardos [6] proposed cross-monotonic cost sharing methods for facility location and *SSRB*. They present a 3-approximate cost sharing method for facility location and a 15-approximate cost sharing method for *SSRB*.

3. SINGLE-SOURCE RENT-OR-BUY

Our method is based on the randomized algorithm proposed by Gupta, Kumar and Roughgarden [2], leading to a randomized $(2+\rho_{\mathrm{ST}})$-approximation algorithm for *SSRB*, where ρ_{ST} denotes the approximation ratio of the Steiner tree algorithm used in Step 2 below. For a given set $Q \subseteq U$ of users, the algorithm works as follows:

1. Mark each user $j \in Q$ with probability $1/M$. Let $Q' \subseteq Q$ denote the set of marked users.
2. Construct a ρ_{ST}-approximate Steiner tree T on $F = Q' \cup \{r\}$.
3. Connect each user $j \notin Q'$ to its closest facility in F.

In the following we outline how this algorithm can be turned into an approximate cross-monotonic cost sharing method.

Jain and Vazirani [4] gave a 2-approximate cross-monotonic cost sharing method, ξ^{ST}, for the Steiner tree game, which we use in Step 2. We define a *random cost share* $\alpha_j(Q)$ for each $j \in Q$ as follows.

$$\alpha_j(Q) := \begin{cases} M \cdot \xi_j^{\mathrm{ST}}(F) & \text{if } j \in F \text{, and} \\ c(j,F) & \text{oterwise.} \end{cases}$$

Here, $c(j,F)$ denotes the shortest path distance from j to a facility in F. Note that both $\xi_j^{\mathrm{ST}}(F)$ and $c(j,F)$ are random variables.

THEOREM 1. *The cost sharing method ξ, which for each $Q \subseteq U$, $j \in Q$ is defined as $\xi_j(Q) := \frac{1}{4}\mathbf{E}[\alpha_j(Q)]$, is cross-monotonic, competitive and with high probability recovers at least a $\frac{1}{4}(1+\varepsilon)^{-1}$-fraction of the cost of the constructed solution, for any constant $\varepsilon > 0$.*

Unfortunately, to compute the expected cost shares in polynomial time, one needs to derandomize the algorithm of Gupta et al. Besides some effort, we were not able to do so. However, we believe that the idea of sharing the expected cost will lead to attractive approximation ratios for cost sharing methods in the future. A similar approach has also been used in a recent independent work of Gupta, Srinivasan and Tardos [3] to obtain a polynomial time 4.5-approximate cross-monotonic cost sharing method for *SSRB*.

4. CONNECTED FACILITY LOCATION

In a recent work Pál and Tardos [6] gave a 15-approximate cross-monotonic cost sharing method for *SSRB*. Their idea was to consider two processes: The cost shares are determined by a "ghost process", which is designed such that the cost shares are trivially cross-monotonic, and the actual solution is constructed by a "real process". We extend their result to *CFL*. Most of the details of the two processes are very similar to those given in [6]. For *CFL*, however, we additionally need to define a cost share that accounts for the opening costs of the facilities and decide which facilities are eventually opened.

Ghost Process. We make the simplifying assumption that the edges of G consist of a continuum of points, which we call *locations*. We associate a notion of time with this process. For each user j, we grow a ghost ball uniformly around j as time progresses. If M or more balls intersect a location p, we open p. At any time t, all open locations form a set $\mathcal{C}(t)$ of connected components. The evolution of $\mathcal{C}(t)$ mimics the standard primal-dual algorithm for Steiner trees, except that new components may appear over time. As in [6], we use this process to define two cost shares: $\alpha_j(Q)$, which accounts for j's contribution towards building the Steiner tree, and $\alpha'_j(Q)$, which accounts for j's connection cost. Additionally, we define a third cost share, α''_j, accounting for the opening costs. At time t, j contributes $\max(0, t - c(j,i))$ to the opening cost of a facility i. If the total contribution towards i equals the opening cost f_i, we open i. Let t_i be the time when i is opened, and let Q_i be the set of users that contribute to the opening of i at time t_i. We define

$$\alpha''_j(Q) := \min(\min_{i: j \in Q_i} t_i, \min_{i: j \notin Q_i} c(j,i)).$$

Real Process. We run the ghost process but open a location p at time t_p only if there is no other open location q with $c(p,q) \leq 2t_p$. We use the same rule for opening facilities. All locations that are opened are called *centers*. For each center p we determine a facility, $i(p)$, which is open and closest to p. We assign all users in the cluster around p to facility $i(p)$. Users that are not contained in any open cluster are assigned to the facility of their closest center. Finally, to make sure that open facilities are connected, we build a Steiner tree on the centers and, for each center p, buy the shortest path from p to $i(p)$.

THEOREM 2. *The cost sharing method ξ, which for each $Q \subseteq U$, $j \in Q$ is defined as $\xi_j(Q) := \frac{1}{5}\alpha_j(Q) + \frac{3}{10}\alpha'_j(Q) + \frac{3}{10}\alpha''_j(Q)$, is cross-monotonic, competitive and recovers at least a $\frac{1}{30}$-fraction of the cost of the constructed solution.*

5. REFERENCES

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