Introduction to Modern Cryptography, Exercise # 7

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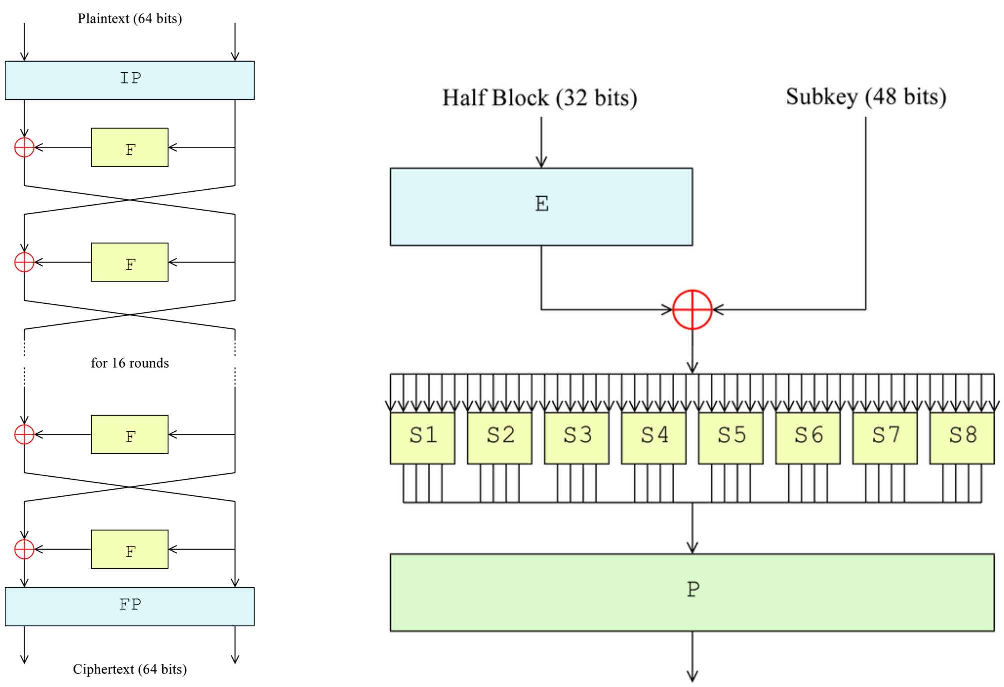
Complementarity Property of DES

In this exercise, we show that DES has the complementarity property, i.e., that $DES_k(x) = \overline{DES_{\overline{k}}(\overline{x})}$ for every key k and input x (where $\overline{z}$ denotes the bitwise complement of z) and how we can exploit that property.

1. Let f be the DES mangler function. Show that for every subkey k and message x, it holds that $f(k, x) = f(\overline{k}, \overline{x})$.

2. Use the above property to conclude that after every round i in the Feistel network, $L_i(x, k) = \overline{L_i(\overline{x}, \overline{k})}$ and $R_i(x, k) = \overline{R_i(\overline{x}, \overline{k})}$. Conclude that $DES_k(x) = \overline{DES_{\overline{k}}(\overline{x})}$ for every key k and input x. (Note that for all "permutations" P in DES, $P(\overline{x}) = \overline{P(x)}$.)

3. Use a chosen-plaintext attack with two messages x and $\overline{x}$ to argue that it is possible to find the secret key in DES (with probability 1) using 2^{55} local computations of DES.



Feistel Network and mangler function of DES
Image credit: `wikimedia.org`.

Group and Number Theory

[Thanks to Boaz Barak for his kind permission to use his exercises.] The following exercises introduce some group and number theory in order to prepare you for the treatment of public-key cryptography after the break.

As mathematicians, we expect you to be able to solve the group theory exercises 1.-4. with ease. **Exercises 1.-4. are optional**: we will correct them (if you decide to hand in solutions), but not grade them. Anyone who is not completely confident in his/her abilities should do them, though. **Exercises 5. and 6. are not optional** and will be graded.

The exercises are self-contained, so you can solve them without reading outside sources. If you want to brush up your knowledge, the following are recommended references: **(1)** [KL], Chapter 7 and Appendix B, **(2)** Victor Shoup's book *"A Computational Introduction to Number Theory and Algebra"* (also available online at `http://www.shoup.net/ntb/`) and **(3)** The mathematical background appendix of the *"Computational Complexity"* book by Sanjeev Arora and Boaz Barak also contains some basic number theory background.

A group $(S, \circ)$ is a set S with a binary operation $\circ$ defined on S for which the following properties hold:

1. **Closure:** For all $a, b \in S$ it holds that $a \circ b \in S$.
2. **Identity:** There is an element $e \in S$ such that $e \circ a = a \circ e = a$ for all $a \in S$.
3. **Associativity:** $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in S$.
4. **Inverses:** For each $a \in S$ there exists an element $b \in S$ such that $a \circ b = b \circ a = e$.

The order of a group, denoted by $|S|$, is the number of elements in S. If the order of a group is a finite number, the group is said to be a *finite group*. If a group $(S, \circ)$ satisfies the commutative law $a \circ b = b \circ a$ for all $a, b \in S$ then it is called an *Abelian group*.

1. (Optional) Let $+_n$ denote addition modulo n (e.g., $5 +_3 6 = [5 + 6 \mod 3] = 2$). Let $Z_n = \{0, 1, 2, \ldots, n-1\}$. Prove that $(Z_n, +_n)$ is a finite Abelian group for every natural number n.

2. (Optional) Prove that for every group:

   (a) The identity element e in the group is *unique*.

   (b) Every element a has a *single* inverse.

3. (Optional) Let a be an element in a group and let a^{-1} denote the (unique) inverse of a. Then, for every integer k we define:

$$a^k := \begin{cases} \underbrace{a \circ a \circ \ldots \circ a}_{k} & \text{if } k > 0; \\ e & \text{if } k = 0; \\ (a^{-1})^{-k} & \text{if } k < 0. \end{cases}$$

   Prove that for any integers m, n (not necessarily positive) it holds that:

(a) $a^m \circ a^n = a^{m+n}$.

(b) $(a^m)^n = a^{mn}$.

4. (Optional) Let $(S, \circ)$ be a group and let $S' \subseteq S$. If $(S', \circ)$ is also a group, then $(S', \circ)$ is called a *subgroup* of $(S, \circ)$. Prove that:

   (a) If $(S, \circ)$ is a finite group and $a \in S$ then there exists $m \geq 1$ such that $a^m = a^{-1}$.

   (b) If $(S, \circ)$ is a finite group and S' is a subset of S such that $a \circ b \in S'$ for every $a, b \in S'$, then $(S', \circ)$ is a subgroup of $(S, \circ)$.

5. Let a and b be two positive integers. We denote by $\gcd(a, b)$ the greatest common divisor of a and b; i.e, $d = \gcd(a, b)$ if d is the largest integer that divides both a and b. The *Euclidean algorithm* computes the gcd as follows:

$$\begin{array}{l} \text{input: } a > b > 0 \\ r_{-1} \leftarrow a \\ r_0 \leftarrow b \\ \text{for } i = 1, 2, \ldots \text{ till } r_i = 0 \\ \qquad r_i \leftarrow [r_{i-2} \mod r_{i-1}] \\ \text{output } r_{i-1} \end{array}$$

   (a) Prove that this algorithm indeed outputs the gcd of a and b.

   (b) Prove that if d is the gcd of a and b, then there exist (not necessarily positive) integers x, y such that $d = xa + yb$. How can you compute these numbers?

6. Let $\times_n$ denote multiplication modulo n (i.e., $5 \times_7 3 = [15 \mod 7] = 1$).

   (a) Prove that for every n, the set $\mathbb{Z}_n^* = \{k \in \{1, \ldots, n-1\} \ ; \ \gcd(k, n) = 1\}$ with the operation $\times_n$ is an Abelian group.

   (b) Give an algorithm that on input $a \in \mathbb{Z}_n^*$, computes a^{-1} (with respect to the group operation $\times_n$). Can you find an algorithm that runs in time polynomial in $|n|$?

   (c) If n is a prime number, how many elements exist in $\mathbb{Z}_n^*$?

   (d) If $n = p \cdot q$ is the product of two different prime numbers p and q, how many elements exist in $\mathbb{Z}_n^*$?

Fun Stuff

Read and enjoy the paper "New Directions in Cryptography" by Whitfield Diffie and Martin Hellman from November 1976, available from the course webpage (see course schedule, midterm break).