Heavy Tails: Performance Models and Scheduling Disciplines

Sindo Núñez-Queija

Parts III & IV – Delay asymptotics, class-based scheduling and flow-based scheduling

Part III: Delay Asymptotics

- Service disciplines and tail asymptotics
- Processor Sharing model for TCP-like traffic
- Service disciplines and tail asymptotics
- Tail asymptotics via conditional moments
- Other disciplines: FBPS, SRPT and LCFS-NP

Service disciplines and tail asymptotics

M/G/1 queue:

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(i) First-come first-served

$$\mathbf{P}\{S > x\} \sim \frac{\rho}{1-\rho}\mathbf{P}\{B^r > x\}, \quad x \to \infty$$

Service disciplines and tail asymptotics

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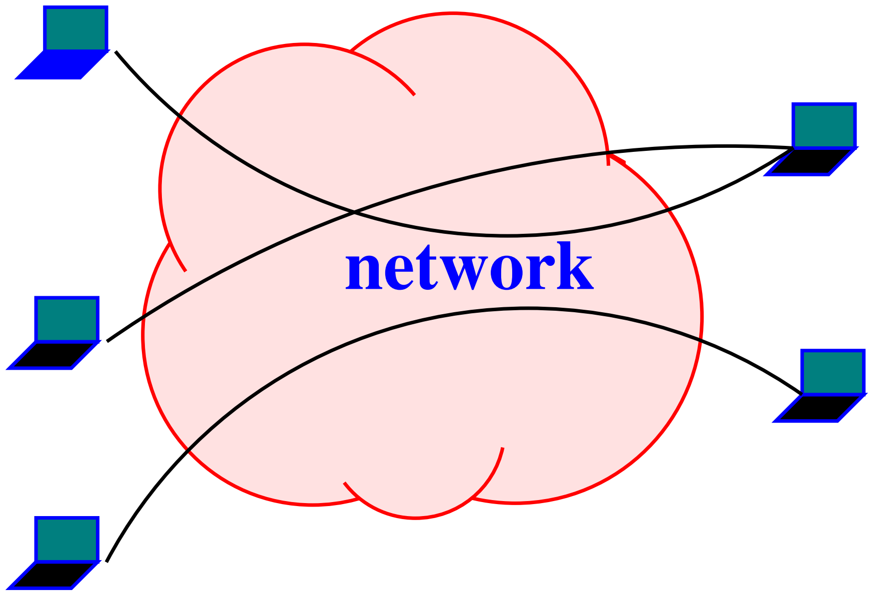
(iv) The $M/G/1$ LCFS-NP queue

$$\mathbf{P}\{W_{LNP} > x\} \sim \rho\mathbf{P}\{B^r > (1-\rho)x\}, \quad x \to \infty$$

Distributed Congestion Control in Data Networks: TCP



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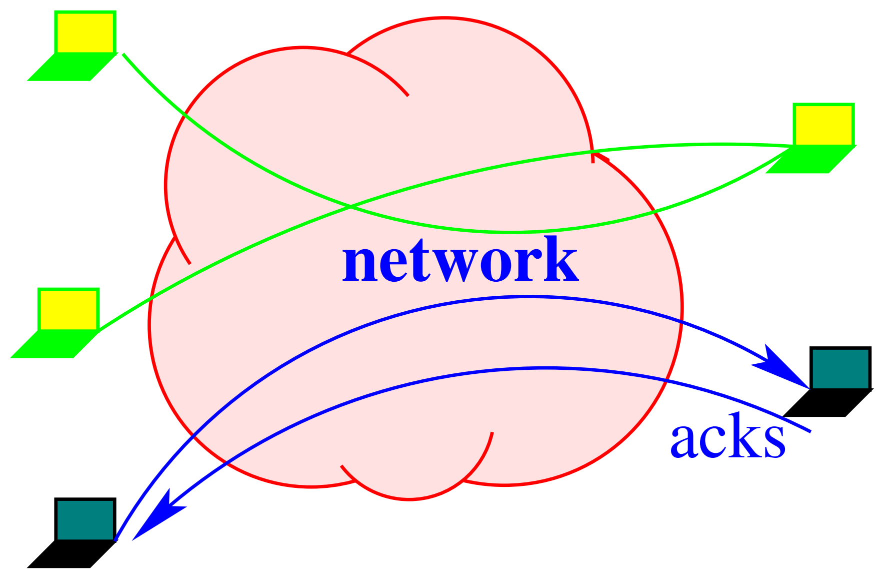

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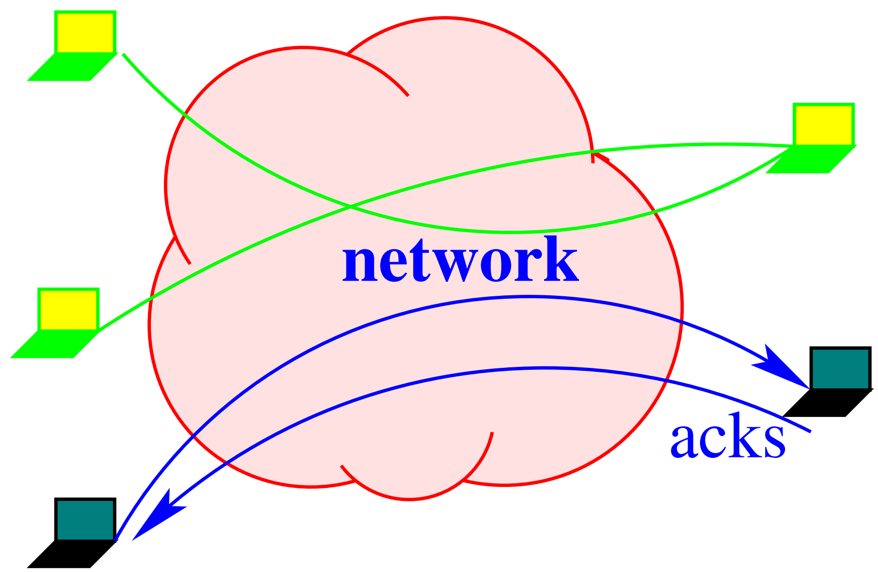
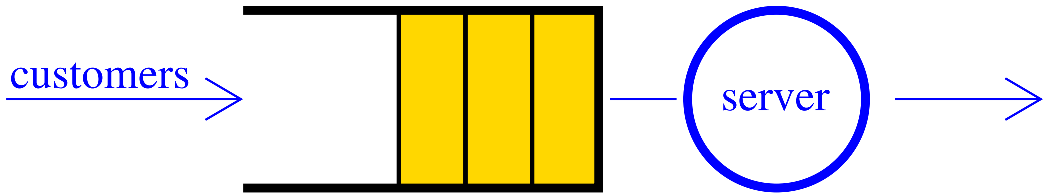


- separation of time scales

- simultaneous resource sharing

M/G/1 Processor Sharing

- flows are "initiated" according to a Poisson process of rate λ
- arbitrary flow size distribution



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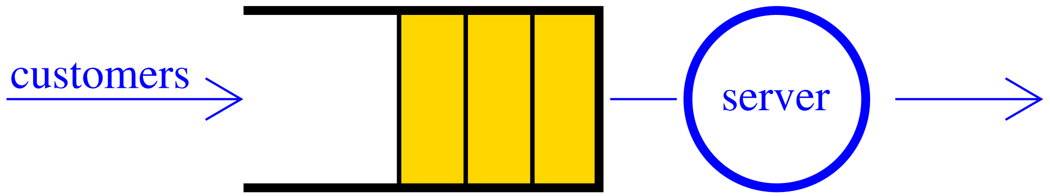

- throughput of each flow: $\frac{1}{N(t)}$

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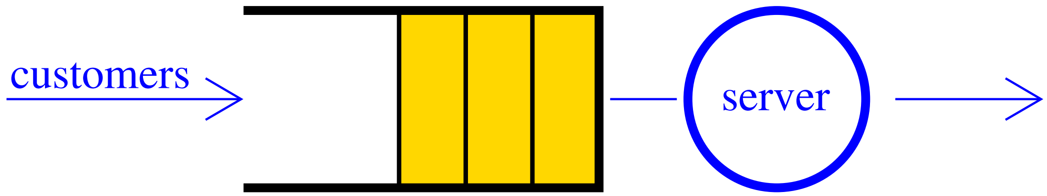

- throughput of each flow: $\frac{1}{N(t)}$
- "workload" in the queue does not correspond to workload in a buffer
- $\mathbf{P}\{N = n\} = (1 - \rho)\rho^n$ with $\rho = \lambda \mathbf{E}[F]$
- $\mathbf{E}[S \mid F = \tau] = \frac{\tau}{1-\rho}$ $\Rightarrow \mathbf{E}[S] = \frac{\mathbf{E}[F]}{1-\rho}$
- $P(S > x)$ delay asymptotics, $x \to \infty$

Intuition

$$\lim_{x\to\infty} \frac{\mathbf{P}\left\{S > \frac{x}{c-\rho}\right\}}{\mathbf{P}\left\{B > x\right\}} \;=\; 1$$

$c =$ **average service capacity**

$\rho =$ **traffic load (**$\rho < c$**)**

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"For large x, the probability that a customer's sojourn time exceeds the value $\frac{x}{c-\rho}$ is asymptotically equal to the probability that a customer's service requirement exceeds the value x"

We show that this is true because (asymptotically) the two events can only occur simultaneously

intuition . . .

$S(\tau) =$ **sojourn time of a customer with service requirement** τ
(in steady state)

When $S(\tau)$ **is "large"**

$$\frac{\tau}{S(\tau)} \approx c - \rho$$

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We show something stronger: a one-to-one correspondence between "large" sojourn times and "large" service requirements

$$\frac{S(\tau)}{\tau} \xrightarrow{\mathsf{P}} \frac{1}{c - \rho}$$

"fast enough" as $\tau \to \infty$.

Proof of tail equivalence

Service requirement distribution $B(x) := \mathrm{P}\{B \leq x\}$, $x \geq 0$
Tail distribution $\overline{B}(x) := 1 - B(x) = \mathrm{P}\{B > x\}$

Assumption (Service requirements)
$\overline{B}(x) \in \mathcal{IR}$

$$\liminf_{\varepsilon \downarrow 0} \liminf_{x \to \infty} \frac{\overline{B}(x(1+\varepsilon))}{\overline{B}(x)} = 1$$

$\Longrightarrow$ $\overline{B}(x)$ **"dominates" a Pareto tail:**

$$\frac{\overline{B}(x_2)}{\overline{B}(x_1)} \geq \eta \left(\frac{x_2}{x_1}\right)^{-\zeta}$$

proof of tail equivalence . . .

If $\overline{B}(x) \in \mathcal{IR}$ then the sojourn times have the following properties

- $S(\tau)$: **stochastically increasing in** $\tau \geq 0$

- $\mathbf{E}[S(\tau)] = \frac{\tau}{c-\rho} + o(\tau)$

- $\mathbf{P}\{S(\tau) - \mathbf{E}\{S(\tau)\} > t\} \leq \frac{o(\tau^{\kappa-\delta})}{t^{\kappa}}$, **for some** $\kappa > \zeta$, $\delta > 0$

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  Alternatives

  ◇ $\mathbf{E}\left[\left|S(\tau) - \mathbf{E}[S(\tau)]\right|^{\kappa}\right] = o(\tau^{\kappa-\delta})$, $\kappa > \zeta, \delta > 0$

  ◇

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$\diamond$ $\mathbf{P}\{S(\tau) > t\} \leq \frac{\mathbf{E}\{S(\tau)^{\kappa}\} - (\mathbf{E}\{S(\tau)\})^{\kappa}}{(t - \mathbf{E}\{S(\tau)\})^{\kappa}}$, $\tau \geq 0$, $t > \mathbf{E}\{S(\tau)\}$

proof of tail equivalence ...

- $S(\tau) \uparrow$ **in** τ**: sample-path argument**

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proof of tail equivalence . . .

- $S(\tau) \uparrow$ in τ: sample-path argument
- $\mathbf{E}\left[S(\tau)\right] = \dfrac{\tau}{1-\rho}$
- Variance: distinguish two cases
  - ◇ $\mathbf{E}\left[B^2\right] < \infty$
  - ◇ $\mathbf{E}\left[B^\alpha\right] < \infty$ and $\mathbf{E}\left[B^\zeta\right] = \infty$, $1 < \alpha < \zeta < 2$

proof of tail equivalence ...

- $\mathbf{E}\left[B^2\right] < \infty$: **For** $k = 2, 3, \ldots$,

$$\mathbf{E}\left[S(\tau)^k\right] \;=\; \left(\frac{\tau}{c-\rho}\right)^k + O(\tau^{k-1}), \qquad \tau \to \infty$$

Take $\kappa > \zeta$ **and** $\delta \in (0,1)$

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- $\mathbf{E}\left[B^\alpha\right] < \infty$ and $\mathbf{E}\left[B^\zeta\right] = \infty$, $1 < \alpha < \zeta < 2$

$$\mathbf{Var}\left[S(\tau)\right] \;\sim\; \int_{u=0}^{\tau} (\tau - u)\mathbf{P}\left\{W_{\lambda,B} > u\right\} \mathrm{d}u$$

Take $\kappa = 2$ and $\delta < \alpha - 1$

“Tail equivalence”

Theorem

If $\overline{B}(x) \in \mathcal{IR}$ *and* $S(\tau)$ *satisfies the three properties then*

$$\lim_{x\to\infty} \frac{\mathbf{P}\left\{S > \frac{x}{c-\rho}\right\}}{\mathbf{P}\left\{B > x\right\}} \;=\; 1$$

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- **When B is “small”, S can not be “large”**
- **When B is “large”, S must be “large” as well**

proof of tail equivalence . . .

When B is "small", S can not be "large":

$$\lim_{x\to\infty} \frac{\mathbf{P}\left\{S > \frac{x}{c-\rho}; B \leq x(1-\varepsilon)\right\}}{\mathbf{P}\left\{B > x(1-\varepsilon)\right\}} = 0$$

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Proof Markov's inequality:

$$\mathbf{P}\left\{S(\tau) - \mathbf{E}\left[S(\tau)\right] > t\right\} \le \frac{\mathbf{E}\left[\left|S(\tau) - \mathbf{E}\left[S(\tau)\right]\right|^{\kappa}\right]}{t^{\kappa}}, \qquad \tau \ge 0,\, t > 0$$

$$
\begin{aligned}
\mathbf{P}\left\{S > \frac{x}{c-\rho}; B \le x(1-\varepsilon)\right\} &= \int_{\tau=0}^{x(1-\varepsilon)} \mathbf{P}\left\{S(\tau) > \frac{x}{c-\rho}\right\} \mathrm{d}B(\tau) \\
&\le \int_{\tau=0}^{x(1-\varepsilon)} \frac{\mathbf{E}\left[\left| S(\tau) - \mathbf{E}\left[S(\tau)\right] \right|^{\kappa}\right]}{\left(\frac{x}{c-\rho} - \mathbf{E}\left[S(x(1-\varepsilon))\right]\right)^{\kappa}} \mathrm{d}B(\tau) \\
&\le (1+o(1)) \frac{\overline{B}(x(1-\varepsilon))\,(x(1-\varepsilon))^{\kappa-\delta}}{\left(\frac{x\varepsilon}{c-\rho}\right)^{\kappa}}
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Dividing by $\overline{B}(x(1-\varepsilon))$, and letting $x \to \infty$, proves

$$
\lim_{x\to\infty} \frac{\mathbf{P}\left\{S > \frac{x}{c-\rho}; B \leq x(1-\varepsilon)\right\}}{\mathbf{P}\left\{B > x(1-\varepsilon)\right\}} = 0
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$\square$

proof of tail equivalence . . .

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Proof It suffices to show that the lim inf is ≥ 1.

$$\begin{aligned} &\mathbf{P}\left\{S > \frac{x}{c-\rho}; B > x(1+\varepsilon)\right\} \\ &= \int_{\tau=x(1+\varepsilon)}^{\infty} \mathbf{P}\left\{S(\tau) > \frac{x}{c-\rho}\right\} \mathrm{d}B(\tau) \\ &\geq \underbrace{\mathbf{P}\left\{S(x(1+\varepsilon)) > \frac{x}{c-\rho}\right\}}_{\longrightarrow 1} \mathbf{P}\left\{B > x(1+\varepsilon)\right\}. \end{aligned}$$

$\square$

Proof of tail equivalence . . .

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If $\overline{B}(x) \in \mathcal{IR}$ *and* $S(\tau)$ *satisfies the three properties then*

$$\lim_{x\to\infty} \frac{\mathbf{P}\left\{S > \frac{x}{c-\rho}\right\}}{\mathbf{P}\left\{B > x\right\}} \;=\; 1$$

Proof Part (i). 'Small B'

$$\mathbf{P}\left\{S > \frac{x}{c-\rho}\right\} \;\leq\; \underbrace{\mathbf{P}\left\{S > \frac{x}{c-\rho}; B \leq x(1-\varepsilon)\right\}}_{\text{negligible}} + \mathbf{P}\left\{B > x(1-\varepsilon)\right\}$$

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In both parts use $\overline{B}(x) \in \mathcal{IR}$ $\square$

FBPS and SRPT

Lemma *If* $\overline{B}(x) \in \mathcal{IR}$, $\mathbf{E}[B^\alpha] < \infty$ *and* $\mathbf{E}\left[B^\zeta\right] = \infty$, *for some* $1 < \alpha < \zeta < 2$, *then* $S(\tau)$ *satisfies the three properties with* $\kappa = 2$ *and* $\delta < \alpha - 1$.

Proof [FBPS] Yashkov (1987)

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- $\mathbf{E}[S(\tau)] = \dfrac{\tau}{1-\lambda h_1(\tau)} + \dfrac{\lambda h_2(\tau)}{2(1-\lambda h_1(\tau))^2}$

- $\mathbf{Var}[S(\tau)] = \dfrac{\lambda h_3(\tau)}{3(1-\lambda h_1(\tau))^3} + \dfrac{\lambda\tau h_2(\tau)}{(1-\lambda h_1(\tau))^3} + \dfrac{3(\lambda h_2(\tau))^2}{4(1-\lambda h_1(\tau))^4}$

- **with** $h_j(\tau) = j\int_{x=0}^{\tau} x^{j-1}\overline{B}(x)\mathrm{d}x$

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- **with** $h_j(\tau) = j\int_{x=0}^{\tau} x^{j-1}\overline{B}(x)\mathrm{d}x$
- $h_1(\tau) \to \beta_1$ **and** $h_j(\tau) = \mathrm{o}(\tau^{j-\alpha+\varepsilon})$, $j=2,3$ $\square$

Proof [SRPT] Schrage and Miller (1966)

The waiting time is negligible compared to the residence time $R(\tau)$

$$\begin{aligned} \mathbf{E}\left[R(\tau)\right] &= \int_{t=0}^{\tau} \frac{1}{1-\rho(t)} \mathrm{d}t, \\ \mathbf{Var}\left[R(\tau)\right] &= \lambda \int_{t=0}^{\tau} \frac{\int_{u=0}^{t} u^2 \mathrm{d}B(u)}{(1-\rho(t))^3} \mathrm{d}t, \end{aligned}$$

with

$$\rho(\tau) := \lambda \int_{t=0}^{\tau} t \mathrm{d}B(t).$$

□

Last-Come First-Served Non-Preemptive

Focus on

$S_{LNP}(\tau) =$ **sojourn time of a customer that arrives when the remaining service requirement of the customer in service equals** τ

$$S_{LNP}(\tau) = \tau + \sum_{n=1}^{N(\tau)} P_n + B$$

$N(\tau) =$ **number of arrived customers during the remaining service requirement** τ

$P_n =$ **i.i.d. sequence of** $M/G/1$ **busy periods**

$B =$ **customer's own service requirement**

last-come first-served non-preemptive (2)

$$\mathbf{P}\{S_{LNP} > t\} = (1-\rho)\,\mathbf{P}\{B > t\} + \rho\,\mathbf{P}\{S_{LNP}(B^r) > t\}, \qquad t \geq 0$$

B^r = **unconditional residual service requirement of the customer in service**

$$\mathbf{P}\{S_{LNP}(B^r) > t\} = \int_{\tau=0}^{\infty} \mathbf{P}\{S_{LNP}(\tau) > t\} \mathrm{d}B^r(\tau), \qquad t \geq 0$$

Show for non-integer $\nu > 2$, **if** $B(\cdot) \in \mathcal{R}_{-\nu}$ **and, hence,** $B^r(\cdot) \in \mathcal{R}_{-\alpha}$ **with** $\alpha := \nu - 1$, **then**

$$\mathbf{P}\{S_{LNP}(B^r) > \frac{x}{1-\rho}\} \sim \mathbf{P}\{B^r > x\}.$$

and

$$\mathbf{P}\{S_{LNP} > \frac{x}{1-\rho}\} \sim \rho\,\mathbf{P}\{B^r > x\},$$

Differential Equations

Let $m < \nu < m+1$**, for some integer** $m \geq 2$

$\mathbf{E}\{B^m\} < \infty$ **and** $\mathbf{E}\{P^m\} < \infty$

As $\Delta \to 0$

$$\begin{aligned}\mathbf{E}\{\left[S_{LNP}(\tau+\Delta)^k\right]\} &= (1-\lambda\Delta)\mathbf{E}\{\left[(\Delta+S_{LNP}(\tau))^k\right]\} \\ &\quad +\lambda\Delta\mathbf{E}\{\left[(\Delta+S_{LNP}(\tau)+P)^k\right]\}+o(\Delta)\end{aligned}$$

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$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{E}\{S_{LNP}(\tau)^k\} &= \frac{k}{1-\rho}\mathbf{E}\{S_{LNP}(\tau)^{k-1}\} \\ &\quad +\lambda\sum_{j=0}^{k-2}\binom{k}{j}\mathbf{E}\{S_{LNP}(\tau)^j\}\mathbf{E}\{P^{k-j}\}\end{aligned}$$

with $\mathbf{E}\{S(0)^k\} = \mathbf{E}\{B^k\}$

differential equations (2)

$$\mathbf{E}\{S_{LNP}(\tau)\} = \beta + \frac{\tau}{1-\rho}$$

$$\mathbf{E}\{S_{LNP}(\tau)^k\} = \left(\frac{\tau}{1-\rho}\right)^k + p_{k-1}(\tau)$$

$p_{k-1}(\tau) =$ **polynomial in** τ **of degree** $k-1$

Take $\kappa = m$ **and use**

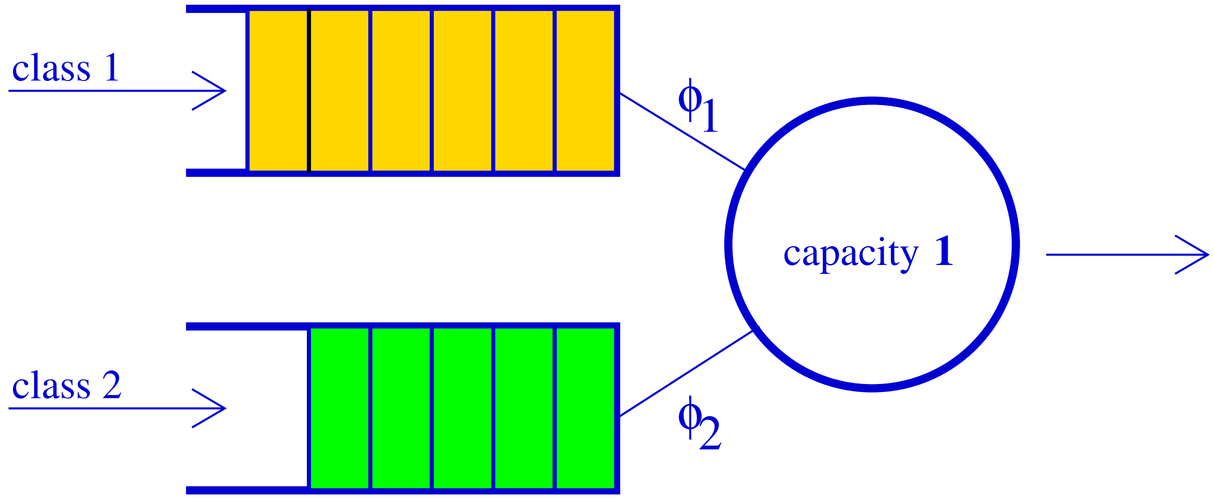
$$\mathbf{P}\{S(\tau) > t\} \leq \frac{\mathbf{E}\{S(\tau)^\kappa\} - (\mathbf{E}\{S(\tau)\})^\kappa}{(t - \mathbf{E}\{S(\tau)\})^\kappa}, \qquad \tau \geq 0,\ t > \mathbf{E}\{S(\tau)\}$$

where $\kappa \geq 2$ □

Part IV.a. Class-based Scheduling:

GPS with two classes

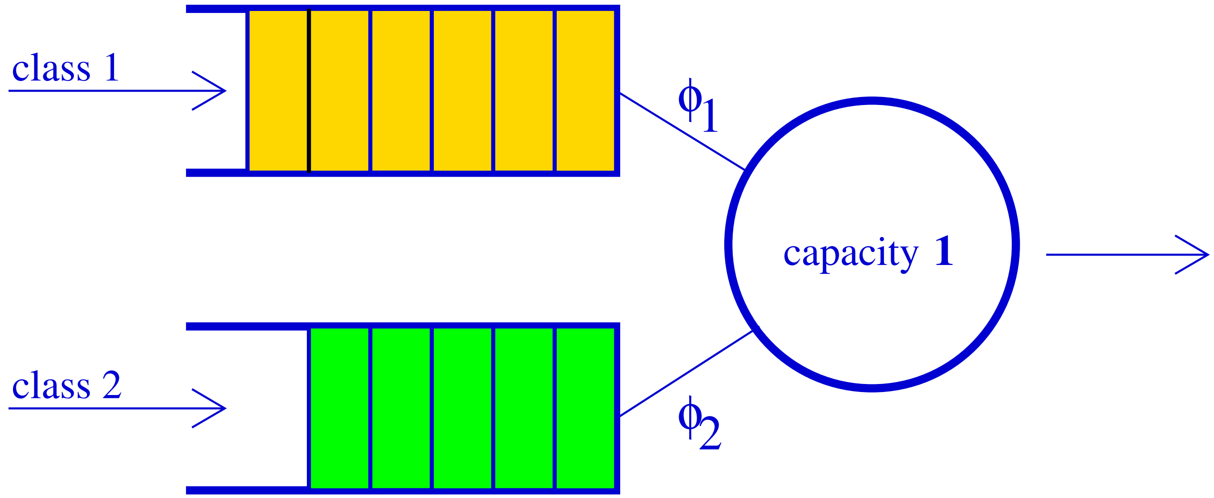
Two classes



- **Between classes: GPS (Generalized Processor Sharing)**

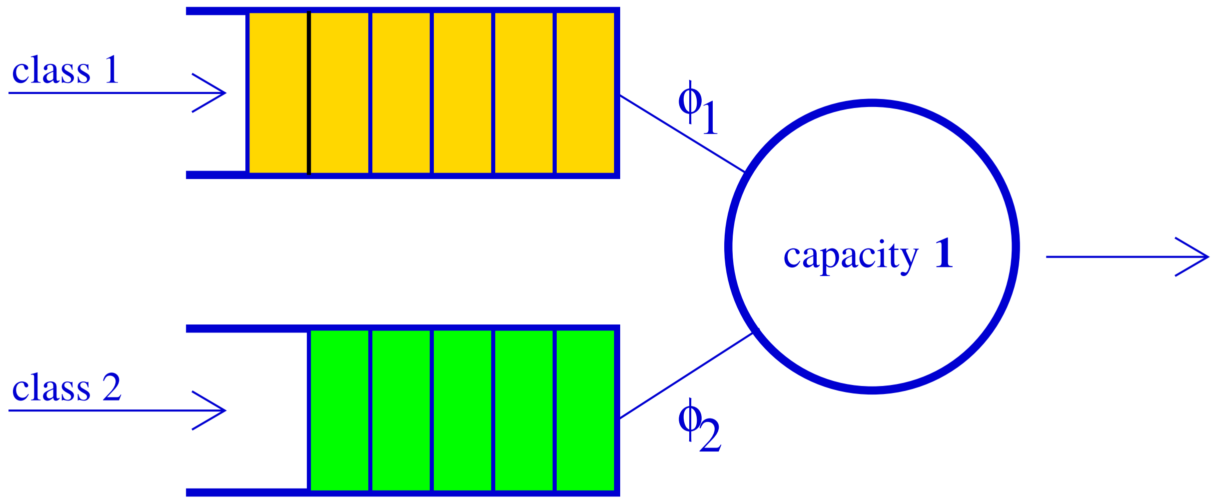
  ◇ $\phi_1 + \phi_2 = 1$

Two classes



- **Between classes: GPS (Generalized Processor Sharing)**
  - $\phi_1 + \phi_2 = 1$
- **Within class 1: PS (Processor Sharing)**
  - **rate per flow between ϕ_1/n_1 and $1/n_1$**

Two classes



- **Between classes: GPS (Generalized Processor Sharing)**
  - $\phi_1 + \phi_2 = 1$
- **Within class 1: PS (Processor Sharing)**
  - **rate per flow between** ϕ_1/n_1 **and** $1/n_1$
- **Within class 2: work-conserving, for instance PS**

Strict priorities ($\phi_1 = 0$)

M/G/1 PS with random service interruptions

- **Poisson arrivals with rate** λ
- **Service requirement distribution** $B(x) \in \mathcal{IR}$

Strict priorities ($\phi_1 = 0$)

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  - ◇ "on-periods": exponential with mean $1/\nu$
  - ◇ "off-periods": general with finite moments m_1, m_2, m_3
  - ◇ mean service rate: $c = \frac{1}{1+\nu m_1}$
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M/G/1 PS with random service interruptions

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- **Service requirement distribution $B(x) \in \mathcal{IR}$**
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  - ◇ **"off-periods": general with finite moments m_1, m_2, m_3**
  - ◇ **mean service rate: $c = \frac{1}{1+\nu m_1}$**
- **Using method based on conditional moments**
- $\mathbf{P}\{V > x\} \sim \mathbf{P}\{B > x(c-\rho)\}$

Uniform stability ($\rho_1 < \phi_1$)

- **Class-1 traffic:**
  - **Poisson arrival process of rate** λ_1
  - **Service requirement distribution** $B_1(\cdot) \in \mathcal{I}R$
  - $\rho_1 < \phi_1$

Uniform stability ($\rho_1 < \phi_1$)

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- $A_i(s,t)$ **= class-i traffic generated during** $(s,t]$

$$\lim_{t \to \infty} \frac{1}{t-s} A_i(s,t) = \lim_{u \to -\infty} \frac{1}{s-u} A_i(u,s) = \rho_i, \ \forall s, \ \textbf{w.p. 1}$$

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Theorem:

$$\mathbf{P}\{\mathbf{S}_0 > t\} \sim \mathbf{P}\{\mathbf{B}_0 > (1 - \psi_2 - \rho_1)t\} \sim \mathbf{P}\{\mathbf{S}_0^{1-\psi_2} > t\}$$

$\psi_2 := \min\{\rho_2, \phi_2\}$

Heuristics

- $S_0 = B_0 + B_1(0, S_0) + B_2(0, S_0)$

- $\rho_2 < \phi_2$
  - $\Rightarrow S_0 \approx B_0 + (\rho_1 + \rho_2) S_0$

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  - $\Rightarrow S_0 \approx B_0/(1 - \rho_1 - \rho_2)$

Heuristics

- $S_0 = B_0 + B_1(0, S_0) + B_2(0, S_0)$

- $\rho_2 < \phi_2$
  - $\Rightarrow S_0 \approx B_0 + (\rho_1 + \rho_2) S_0$
  - $\Rightarrow S_0 \approx B_0 / (1 - \rho_1 - \rho_2)$

- $\rho_2 \geq \phi_2$
  - $\Rightarrow S_0 \approx B_0 + (\rho_1 + \phi_2) S_0$
  - $\Rightarrow S_0 \approx B_0 / (1 - \rho_1 - \phi_2)$

- **Alternative scenarios ...**

Heuristics

$B_1(0, S_0) \approx \rho_1 S_0$ **and** $B_2(0, S_0) \approx \min\{\rho_2, \phi_2\} S_0$ **because**

- $B_i(0, t) = W_i(0) + A_i(0, t) - W_i(t)$

- **Large deviations**

  ◇ $A_i(0, S_0) \approx \rho_i S_0$

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  - $A_i(0, S_0) \approx \rho_i S_0$
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Heuristics

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- **Large deviations**
  - ◇ $A_i(0, S_0) \approx \rho_i S_0$
  - ◇ $W_i(0) = o(B_0)$
  - ◇ $W_i(t) \leq W_i^{\phi_i}(t)$, **also for class 1!**

Heuristics

$B_1(0,S_0) \approx \rho_1 S_0$ **and** $B_2(0,S_0) \approx \min\{\rho_2,\phi_2\} S_0$ **because**

- $B_i(0,t) = W_i(0) + A_i(0,t) - W_i(t)$

- **Large deviations**

  ◇ $A_i(0,S_0) \approx \rho_i S_0$

  ◇ $W_i(0) = o(B_0)$

  ◇ $W_i(t) \le W_i^{\phi_i}(t)$, **also for class 1!**

  ⋆ $\rho_i < \phi_i \Rightarrow W_i(S_0) = o(B_0)$

Heuristics

$B_1(0, S_0) \approx \rho_1 S_0$ **and** $B_2(0, S_0) \approx \min\{\rho_2, \phi_2\} S_0$ **because**

- $B_i(0,t) = W_i(0) + A_i(0,t) - W_i(t)$
- **Large deviations**
  - $\diamond$ $A_i(0, S_0) \approx \rho_i S_0$
  - $\diamond$ $W_i(0) = o(B_0)$
  - $\diamond$ $W_i(t) \leq W_i^{\phi_i}(t)$, **also for class 1!**
    - $\star$ $\rho_i < \phi_i \Rightarrow W_i(S_0) = o(B_0)$
    - $\star$ $\rho_2 > \phi_2 \Rightarrow B_2(0, S_0) \approx \phi_2 S_0 \Rightarrow W_2(S_0) \approx (\rho_2 - \phi_2) S_0$

Discussion

- $\rho_1 < \phi_1$
  - ◇ **Only condition on class 2: existence of** ρ_2
  - ◇ **Reduced-load equivalence**

- $\rho_1 > \phi_1$
  - ◇ **"Induced burstiness"**
  - ◇ **Unless** $\mathrm{P}\{\sup_{t\geq 0}\{A_2(0,t) - \phi_2 t\} > x\} = o(\bar{B}_1(x))$ **as** $x \to \infty$
  - ◇ **Strict priorities**

Proof using lower & upper bounds

- **"Permanent" class-1 customer arriving at time 0**

- $B_0(0,t)$ **= amount of service received by one customer**

$$\mathbf{P}\{\mathbf{S}_0 > t\} = \mathbf{P}\{\mathbf{B}_0 > B_0(0,t)\}$$

- **Identity** $B_0(0,t) + B_1(0,t) + B_2(0,t) = t$

- $W(t) := W_1(t) + W_2(t)$ **= total backlog in the system not including permanent customer**

Reference system

- M/G/1 PS with capacity ϕ_1
  - Arrivals
  - Service requirements
  - Permanent customer at time 0

Lemma At any time t each customer present in the GPS system has received at most the same amount of service in the reference system

- $W_1(t) \leq W_1^{\phi_1}(t)$
- $B_0(0,t) \geq B_0^{\phi_1}(0,t)$
- $\mathbf{S}_0 \leq \mathbf{S}_0^{\phi_1}$

First bound

- **M/G/1 PS queue with permanent customer(s) is stable**

$$\mathbf{P}\{\mathbf{W}_1^{\phi_1} \leq x\} := \lim_{t\to\infty} \mathbf{P}\{W_1^{\phi_1}(t) \leq x\}$$

- $c < \rho_2$: $Z_2^c(s) := \sup_{u \geq s} \{c(u-s) - A_2(s,u)\}$ **has a proper non-defective distribution**

Lemma

$$B_0(0,t) \leq (1-\rho_1-\psi_2+2\epsilon)t + (\rho_1-\epsilon)t - A_1(0,t) + W_1^{\phi_1}(t) + Z_2^{\psi_2-\epsilon}(0)$$

- **Sample path upper bound for** $B_0(0,t)$
- **Lower bound for** S_0
- $1-\rho_1-\psi_2$ **= average service rate for permanent customer**

Proof

$$\begin{aligned} B_0(0,t) &\le t - A_1(0,t) + W_1(t) - B_2(0,t) \\ &\le t - (\rho_1 - \epsilon)t + (\rho_1 - \epsilon)t - A_1(0,t) + W_1^{\phi_1}(t) - B_2(0,t) \end{aligned}$$

$s \in [0,t]$: $W_2(s) = 0$ **and** $W_2(u) > 0$ **for** $u \in (s,t)$

$$\begin{aligned} B_2(0,t) &\ge A_2(0,s) + \phi_2(t-s) \ge A_2(0,s) + \psi_2(t-s) \\ &\ge (\psi_2 - \epsilon)t + A_2(0,s) - (\psi_2 - \epsilon)s \ge (\psi_2 - \epsilon)t - Z_2^{\psi_2 - \epsilon}(0) \end{aligned}$$

$\square$

Lower bound sojourn time

Theorem: If $B_1(\cdot) \in \mathcal{I}R$ **and** $\rho_1 < \phi_1$**, then**

$$\liminf_{t\to\infty} \frac{\mathbf{P}\{\mathbf{S}_0 > t\}}{\mathbf{P}\{\mathbf{B}_0 > (1-\rho_1-\psi_2)t\}} \geq 1$$

Proof

$$\mathbf{P}\{\mathbf{S}_0 > t\} = \mathbf{P}\{\mathbf{B}_0 > B_0(0,t)\}$$

$$\geq \mathbf{P}\{\mathbf{B}_0 > (1-\rho_1-\psi_2+2\epsilon)t + (\rho_1-\epsilon)t - A_1(0,t) + W_1^{\phi_1}(t) + Z_2^{\psi_2-\epsilon}($$

$$\geq \mathbf{P}\{\mathbf{B}_0 > (1-\rho_1-\psi_2+4\epsilon)t\}$$
$$\times \underbrace{\mathbf{P}\{(\rho_1-\epsilon)t - A_1(0,t) + W_1^{\phi_1}(t) \leq \epsilon t\}}_{\longrightarrow 1} \underbrace{\mathbf{P}\{Z_2^{\psi_2-\epsilon}(0) \leq \epsilon t\}}_{\longrightarrow 1}$$

- $A_1(0,t)$ **and** $W_1^{\phi_1}(t)$ **not independent**

$$
\begin{aligned}
& \mathbf{P}\{(\rho_1-\epsilon)t - A_1(0,t) + W_1^{\phi_1}(t) \le \epsilon t\} \\
\ge\ & \mathbf{P}\{A_1(0,t) \ge (\rho_1-\epsilon)t,\ W_1^{\phi_1}(t) \le \epsilon t\} \\
\ge\ & \mathbf{P}\{A_1(0,t) \ge (\rho_1-\epsilon)t\} - \mathbf{P}\{W_1^{\phi_1}(t) > \epsilon t\}
\end{aligned}
$$

- $\mathbf{P}\{A_1(0,t) \ge (\rho_1-\epsilon)t\} \to 1$

- $\mathbf{P}\{W_1^{\phi_1}(t) > \epsilon t\} \to 0$

Finally, use $B_1(\cdot) \in \mathcal{I}R$

Second bound

- **Sample path lower bound for** $B_0(0,t)$
- **Upper bound for** S_0
- **Needed if** $\rho_2 < \phi_2$

Lemma

$$B_0(0,t) \geq t - W(0) - A(0,t)$$

Proof

$$B_0(0,t) = t - B_1(0,t) - B_2(0,t) = t - W(0) - A(0,t) + W(t)$$

□

Upper bound sojourn time

Theorem: If $B_1(\cdot) \in \mathcal{I}R$ **and** $\rho_1 < \phi_1$ **then**

$$\limsup_{t\to\infty} \frac{\mathbf{P}\{\mathbf{S}_0 > t\}}{\mathbf{P}\{\mathbf{B}_0 > (1-\rho_1-\psi_2)t\}} \leq 1$$

$\psi_2 = \min\{\rho_2, \phi_2\}$

Proof

- $\rho_2 \geq \phi_2 = \psi_2$:

$$\mathbf{S}_0 \leq \mathbf{S}_0^{\phi_1} \approx \frac{1}{\phi_1 - \rho_1} B_0 = \frac{1}{1-\phi_2-\rho_1} B_0$$

upper bound sojourn time ...

- $\rho_2 = \psi_2 < \phi_2$:

$$\mathbf{P}\{\mathbf{S}_0 > t\}$$

$$\leq \mathbf{P}\{\mathbf{S}_0^{\phi_1} > t, \, \mathbf{B}_0 > (1-\rho-\epsilon)t - W(0) + (\rho+\epsilon)t - A(0,t)\}$$

$$\leq \mathbf{P}\{\mathbf{B}_0 > (1-\rho-2\epsilon)t\} + \overbrace{\mathbf{P}\{\mathbf{S}_0^{\phi_1} > t, \, \mathbf{B}_0 \leq (\phi_1-\rho_1-\epsilon)t\}}^{=o(\mathbf{P}\{\mathbf{B}_0>(\phi_1-\rho_1-\epsilon)t\})}$$

$$+\mathbf{P}\{\mathbf{B}_0 > (\phi_1-\rho_1-\epsilon)t\} \underbrace{\mathbf{P}\{W(0) + A(0,t) - (\rho+\epsilon)t > \epsilon t\}}_{\longrightarrow 0}$$

$$\leq \mathbf{P}\{\mathbf{B}_0 > (1-\rho-2\epsilon)t\} + o(\mathbf{P}\{\mathbf{B}_0 > (\phi_1-\rho_1-\epsilon)t\})$$

$$= \mathbf{P}\{\mathbf{B}_0 > (1-\rho-2\epsilon)t\} + o(\mathbf{P}\{\mathbf{B}_0 > (1-\rho)t\})$$

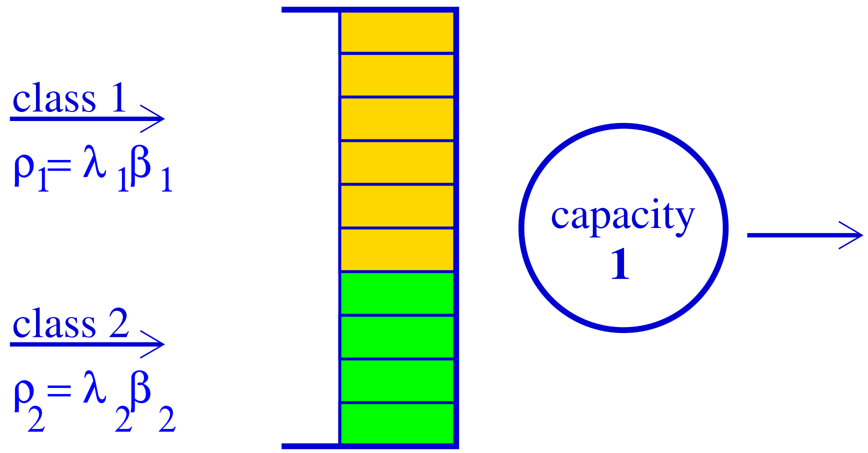
Part IV.b. Flow-based Scheduling:

Multi-class processor sharing

Background

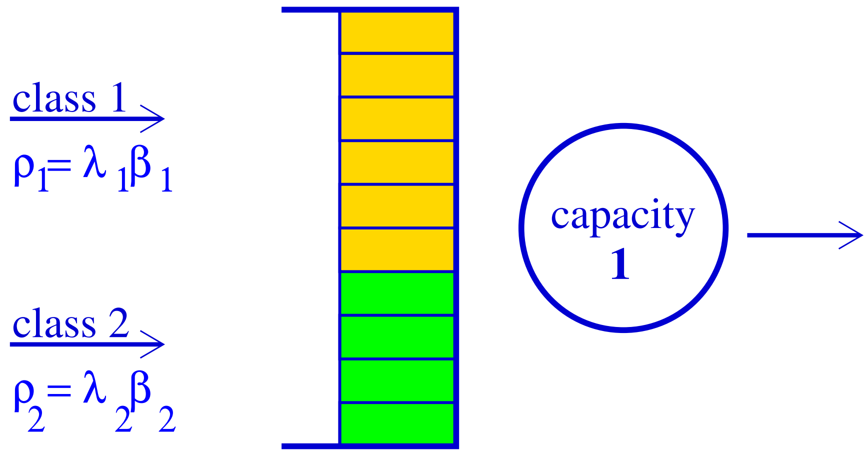
- Extreme variability in flow sizes in the Internet
- TCP-like controlled data transfer
- Processor sharing
  - ◇ Admission control
  - ◇ No admission control

Model



- **Two (or more) heterogeneous traffic classes**

Model



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- **Class 1: light-tailed** $\Pr\{B_1 > x\} = q_1(x)e^{-x^{\eta_1}}$

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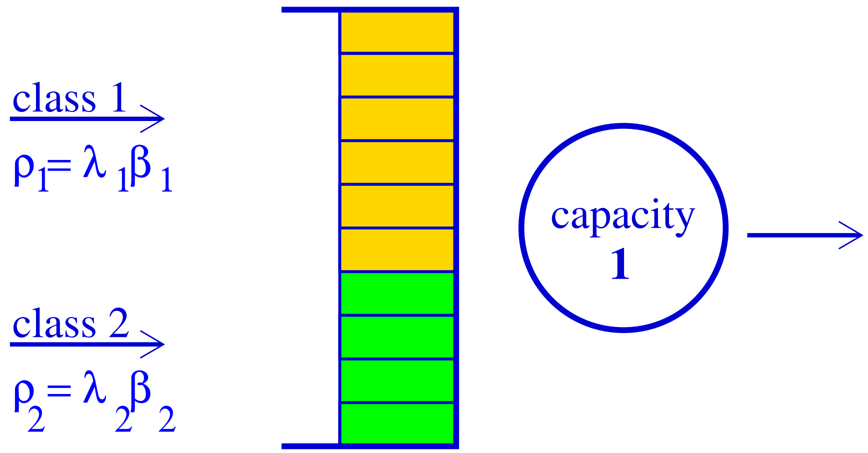

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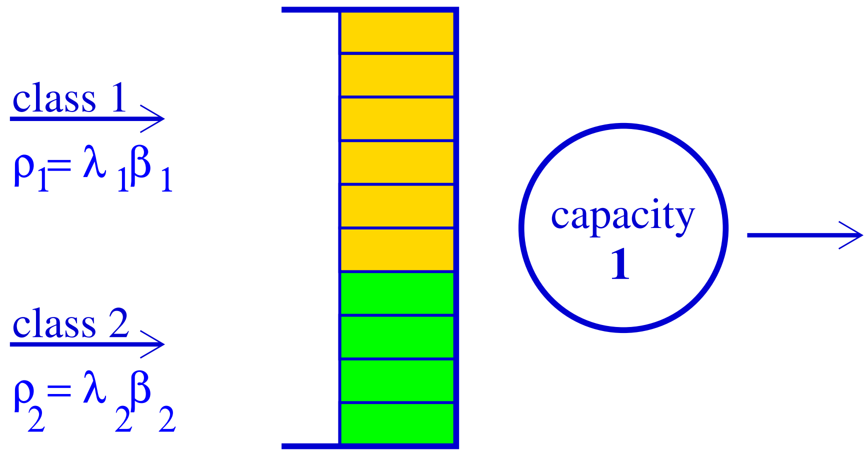
- **Class 2: heavy-tailed** $\Pr\{B_2 > x\} \sim x^{-\nu_2} l_2(x)$

Model



- **Two (or more) heterogeneous traffic classes**
- **Class 1: light-tailed** $\Pr\{B_1 > x\} = q_1(x)e^{-x^{\eta_1}}$
- **Class 2: heavy-tailed** $\Pr\{B_2 > x\} \sim x^{-\nu_2} l_2(x)$
- **PS: Each of n users present receives service at rate $1/n$**
- K: **threshold on # flows**

Perspective

- M/G/1 with heavy tails
  [Zwart & Boxma, Zwart, NQ, Jelenković & Momčilović]

$$\Pr\{V > x\} \sim \Pr\{B > x(1-\rho)\}$$

- M/M/1 [Borst, Boxma, Morrison, NQ]

$$\Pr\{V > x\} \sim c_1\, x^{5/6}\, e^{-c_2\, x \,-\, c_3\, x^{1/3}},$$

- Mixture of both (with limited system occupancy K)

Cohen '79, Kelly '79

- N = number of users
- $X_1, \ldots, X_N$ = remaining service requirements

Cohen '79, Kelly '79

- $\mathbf{N}$ = number of users
- $\mathbf{X_1, \ldots, X_N}$ = remaining service requirements

$$\mathsf{Pr}\{\mathbf{N = n; (X_1, \ldots, X_n) > (x_1, \ldots, x_n)}\} =$$

$$\frac{(1-\rho)\rho^{\mathbf{n}}}{1-\rho^{\mathbf{K+1}}} \prod_{\mathbf{k=1}}^{\mathbf{n}} \left(\frac{\rho_1}{\rho} \mathsf{Pr}\{\mathbf{B_1^r > x_k}\} + \frac{\rho_2}{\rho} \mathsf{Pr}\{\mathbf{B_2^r > x_k}\} \right)$$

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Delay asymptotics with admission control

Main result

$$\mathsf{Pr}\{\mathbf{V_1} > \mathbf{x}\} \sim \mathsf{Pr}\{\mathbf{B_1} > \frac{\mathbf{x}}{\mathbf{K}}\}\frac{(\mathbf{1}-\rho)\rho_\mathbf{2}^{\mathbf{K}-\mathbf{1}}}{\mathbf{1}-\rho^{\mathbf{K}+\mathbf{1}}}\left(\mathsf{Pr}\{\mathbf{B_2^r} > \frac{\mathbf{x}}{\mathbf{K}}\}\right)^{\mathbf{K}-\mathbf{1}}$$

Right-hand side:

Delay asymptotics with admission control

Main result

$$\Pr\{V_1 > x\} \sim \Pr\{B_1 > \frac{x}{K}\} \frac{(1-\rho)\rho_2^{K-1}}{1-\rho^{K+1}} \left(\Pr\{B_2^r > \frac{x}{K}\}\right)^{K-1}$$

Right-hand side:

- **Service requirement of new class-1 user** $\geq x/K$

Delay asymptotics with admission control

Main result

$$\Pr\{V_1 > x\} \sim \Pr\{B_1 > \frac{x}{K}\} \frac{(1-\rho)\rho_2^{K-1}}{1-\rho^{K+1}} \left(\Pr\{B_2^r > \frac{x}{K}\}\right)^{K-1}$$

Right-hand side:

- Service requirement of new class-1 user $\geq x/K$
- Exactly $K-1$ class-2 users present, each with remaining service requirement $\geq x/K$

Intuition for equivalence

- **Service rate** $\geq 1/K \Rightarrow B_1 > x/K$

- **Two possibilities,** $\gamma < 1$

  I $S(0, x) \leq x/K + x^\gamma$

  II $B_1 \geq x/K + x^\gamma$

Intuition for equivalence

- Service rate $\geq 1/K \Rightarrow B_1 > x/K$
- Two possibilities, $\gamma < 1$

  I $S(0, x) \leq x/K + x^{\gamma}$

  II $B_1 \geq x/K + x^{\gamma}$

Lemma:
(I) requires exactly $K-1$ users present when the new user arrives, each with remaining service requirement $\geq x/K$

Intuition for equivalence

- Service rate $\geq 1/K \Rightarrow B_1 > x/K$

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  II $B_1 \geq x/K + x^{\gamma}$

Lemma:
(I) requires exactly $K-1$ users present when the new user arrives, each with remaining service requirement $\geq x/K$

For $\gamma > 0$

$$\Pr\{B_1 > \frac{x}{K} + x^{\gamma}\} = o(1)\Pr\{B_1 > \frac{x}{K}\}\Pr\{B_2^r > \frac{x}{K}\}^{K-1}$$

as $x \to \infty$, so that (II) is highly unlikely

Lemma

$$\Pr\{S(0,x) \le \frac{x}{K} + x^{\gamma}\} \sim \frac{(1-\rho)\rho_2^{K-1}}{1-\rho^{K+1}} \left(\Pr\{B_2^r > \frac{x}{K}\}\right)^{K-1}$$

Proof

Lower bound: Immediate

Lemma

$$\Pr\{S(0,x) \le \frac{x}{K} + x^{\gamma}\} \sim \frac{(1-\rho)\rho_2^{K-1}}{1-\rho^{K+1}} \left(\Pr\{B_2^r > \frac{x}{K}\}\right)^{K-1}$$

Proof

Lower bound: Immediate

Upper bound:

- $T(s,t) =$ amount of time during $[s,t]$ with $\le K{-}2$ OTHER users
- Service at rate $\ge 1/(K-1)$

$$\begin{aligned} S(0,x) &\ge (x - T(0,x))\frac{1}{K} + T(0,x)\frac{1}{K-1} \\ &= \frac{x}{K} + T(0,x)\left(\frac{1}{K-1} - \frac{1}{K}\right) \end{aligned}$$

Lemma (proof continued)

$$\mathbf{S(0,x)} \geq \frac{\mathbf{x}}{\mathbf{K}} + \mathbf{T(0,x)}\left(\frac{\mathbf{1}}{\mathbf{K-1}} - \frac{\mathbf{1}}{\mathbf{K}}\right)$$

$\Rightarrow$

$$\Pr\{\mathbf{S(0,x)} \leq \frac{\mathbf{x}}{\mathbf{K}} + \mathbf{x}^{\gamma}\} \leq \Pr\{\mathbf{T(0,x)} \leq \mathbf{x}^{\gamma} / \left(\frac{\mathbf{1}}{\mathbf{K-1}} - \frac{\mathbf{1}}{\mathbf{K}}\right)\}$$

$$\cdots$$

Lemma (proof continued)

$$S(0,x) \geq \frac{x}{K} + T(0,x)\left(\frac{1}{K-1} - \frac{1}{K}\right)$$

$\Rightarrow$

$$\Pr\{S(0,x) \leq \frac{x}{K} + x^{\gamma}\} \leq \Pr\{T(0,x) \leq x^{\gamma}/\left(\frac{1}{K-1} - \frac{1}{K}\right)\}$$

$$\cdots$$

$$\leq \frac{1-\rho}{1-\rho^{K+1}}\left(\rho_1 \Pr\{B_1^r > (1-\delta)\frac{x}{K}\} + \rho_2 \Pr\{B_2^r > (1-\delta)\frac{x}{K}\}\right)^{K-1}$$

$$+ \frac{1-\rho}{1-\rho^{K+1}} \sum_{n=1}^{K} \frac{\left(\rho_2 \Pr\{B_2^r > (1-\delta)\frac{x}{K}\}\right)^{K-1-n}\left(\delta x \lambda_2 \Pr\{B_2 > (1-2\delta)\frac{x}{K}\}\right)^{n}}{n!}$$

$$+ o((\Pr\{B_2^r > (1-\delta)\frac{x}{K}\})^{K-1})$$

Next $\delta \downarrow 0$

No admission control

- $\Pr\{B_1 > x\} = e^{-\mu_1 x^{\alpha_1}}$
- $K = \infty$

No admission control

- $\Pr\{B_1 > x\} = e^{-\mu_1 x^{\alpha_1}}$

- $K = \infty$

Then

$$\Pr\{V_1 > x\} \geq (1 + o(1))(1 - \rho)c_2\sqrt{2\pi/c_3}\frac{x^{\frac{1}{2}r_1}}{(\ln x)^{1-\frac{1}{2}r_1}}e^{-c_1(x \ln x)^{r_1}}$$

where

$$r_1 = \frac{\alpha_1}{1 + \alpha_1}$$

Tail of V_1 is much heavier than that of B_1!

Proof

$$\Pr\{\mathbf{V_1 > x}\} \geq \sum_{\mathbf{k=0}}^{\infty} (\mathbf{1} - \rho)\rho_{\mathbf{2}}^{\mathbf{k}} \Pr\{\mathbf{B_1} > \frac{\mathbf{x}}{\mathbf{k+1}}\} \left(\Pr\{\mathbf{B_2^r} > \frac{\mathbf{x}}{\mathbf{k+1}}\}\right)^{\mathbf{k}}$$

Proof

$$\Pr\{V_1 > x\} \geq \sum_{k=0}^{\infty} (1-\rho)\rho_2^k \Pr\{B_1 > \frac{x}{k+1}\} \left(\Pr\{B_2^r > \frac{x}{k+1}\}\right)^k$$

$$\geq \sum_{k=0}^{\infty} \int_{u=k}^{k+1} (1-\rho)\rho_2^u \Pr\{B_1 > \frac{x}{u}\} \left(\Pr\{B_2^r > \frac{x}{u}\}\right)^u du$$

Proof

$$\mathsf{Pr}\{\mathbf{V_1} > \mathbf{x}\} \geq \sum_{\mathbf{k=0}}^{\infty} (\mathbf{1}-\rho)\rho_{\mathbf{2}}^{\mathbf{k}} \mathsf{Pr}\{\mathbf{B_1} > \frac{\mathbf{x}}{\mathbf{k+1}}\} \left(\mathsf{Pr}\{\mathbf{B_2^r} > \frac{\mathbf{x}}{\mathbf{k+1}}\}\right)^{\mathbf{k}}$$

$$\geq \sum_{\mathbf{k=0}}^{\infty} \int_{\mathbf{u=k}}^{\mathbf{k+1}} (\mathbf{1}-\rho)\rho_{\mathbf{2}}^{\mathbf{u}} \mathsf{Pr}\{\mathbf{B_1} > \frac{\mathbf{x}}{\mathbf{u}}\} \left(\mathsf{Pr}\{\mathbf{B_2^r} > \frac{\mathbf{x}}{\mathbf{u}}\}\right)^{\mathbf{u}} \mathsf{d}\mathbf{u}$$

$$\sim (\mathbf{1}-\rho) \int_{\mathbf{u=0}}^{\infty} \mathsf{d}\mathbf{u}\, \mathsf{e}^{\mathbf{u} \ln \rho_{\mathbf{2}} - \mu_{\mathbf{1}} (\frac{\mathbf{x}}{\mathbf{u}})^{\alpha_{\mathbf{1}}} + \mathbf{u}(\mathbf{1}-\nu_{\mathbf{2}}) \ln \frac{\mathbf{x}}{\mathbf{u}}}$$

Summary

- Part III. Delay asymptotics
  - ◇ proof of tail equivalence based on conditional moments
  - ◇ processor sharing (with service interruptions)
  - ◇ foreground-background processor sharing; shortest remaining processing time
- Part IV.a. Class-based scheduling
  - ◇ Delay of elastic traffic in a two-class GPS system
  - ◇ $\rho_1 < \phi_1$: reduced-load equivalence
  - ◇ Additional assumptions needed when $\rho_1 \geq \phi_1$
  - ◇ $\phi_1 = 0$: M/G/1 PS with strict priorities

Summary

- **Part IV.b. Flow-based scheduling**
  - ◇ **Flow-based scheduling**
  - ◇ **Heavy-tailed class affects light-tailed class (despite the PS discipline)**
  - ◇ $K < \infty$: **tail of** $\Pr\{V_1 > x\}$ **is not heavier than that of** $\Pr\{B_1 > \frac{x}{K}\}$
  - ◇ $K = \infty$: $\Pr\{V_1 > x\}$ **is heavier**
  - ◇ **Trade-off between blocking and long transfer delays**
  - ◇ **Limitation on # active flows: admission control, TCP time-outs and user impatience**

Heavy Tails: Performance Models and Scheduling Disciplines

Parts III & IV – Delay asymptotics, class-based scheduling and flow-based scheduling

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