Pencil and Paper Exercises Week 2 — Solutions

1. Prove that a binary relation R is transitive iff $R \circ R \subseteq R$.

   Answer:
   $\Rightarrow$: Assume R is a transitive binary relation on A.
   To be proved: $R \circ R \subseteq R$.
   Proof. Let $(x, y) \in R \circ R$. We must show that $(x, y) \in R$. From the definition of $\circ$: $\exists z \in A : (x, z) \in R, (z, y) \in R$. It follows from this and transitivity of R that $(x, y) \in R$.
   $\Leftarrow$: Assume $R \circ R \subseteq R$.
   To be proved: R is transitive.
   Proof. Let $(x, y) \in R, (y, z) \in R$. Then $(x, z) \in R \circ R$. Since $R \circ R \subseteq R$, it follows from this that $(x, z) \in R$. Thus, R is transitive.

2. Give an example of a transitive binary relation R with the property that $R \circ R \neq R$.

   Answer: The simplest example is a relation consisting of a single pair, say $R = \{(1, 2)\}$. This relation is transitive, but $R \circ R = \emptyset \neq R$.

3. Let R be a binary relation on a set A. Prove that $R \cup \{(x, x) \mid x \in A\}$ is the reflexive closure of R.

   Answer: Let $I = \{(x, x) \mid x \in A\}$. It is clear that $R \cup I$ is reflexive, and that $R \subseteq R \cup I$.
   To show that $R \cup I$ is the smallest relation with these two properties, suppose S is reflexive and $R \subseteq S$. Then by reflexivity of S, $I \subseteq S$. It follows that $R \cup I \subseteq S$.

4. Prove that $R \cup R\check{}$ is the symmetric closure of R.

   Answer: Clearly, $R \cup R\check{}$ is symmetric, and $R \subseteq R \cup R\check{}$.
   Let S be any symmetric relation that includes R. By symmetry of S and by the fact that $R \subseteq S$ it follows that $R\check{} \subseteq S$. Thus $R \cup R\check{} \subseteq S$.

5. A partition P of a set A is a set of subsets of A with the following properties:

   (a) every member of P is non-empty.

   (b) every element of A belongs to some member of P.

   (c) different members of P are disjoint.

   If R is an equivalence relation on A and $a \in A$ then $|a|_R$, the R-class of a, is the set $\{b \in A \mid bRa\}$. Show that the set of R-classes

   $$\{|a|_R \mid a \in A\}$$

   of an equivalence relation R on A forms a partition of A.

   Answer: We have to check the three properties of partitions.

Since R is a reflexive relation on A, we have that for each $a \in A$ it holds that $a \in |a|_R$. This takes care of (a) and (b).

Let $|a|_R$ and $|b|_R$ be two R-classes, and assume that they are not disjoint. Then we have $c \in |a|_R \cap |b|_R$.

This means that we have both aRc and bRc. Since R is symmetric, we have cRb, and by transitivity of R, aRb, and again by symmetry of R, bRa.

We can show now that $|a|_R = |b|_R$, as follows:

Assume $d \in |a|_R$. Then dRa. From this and aRb, by transitivity of R, dRb, i.e., $d \in |b|_R$. This shows: $|a|_R \subseteq |b|_R$.

Assume $d \in |b|_R$. Then dRb. From this and bRa, by transitivity of R, dRa, i.e., $d \in |a|_R$. This shows: $|b|_R \subseteq |a|_R$.

So if two R-classes $|a|_R$ and $|b|_R$ are not disjoint, then they are in fact equal. This takes care of (c).

6. Give the euclidean closure of the relation $\{(1,2),(2,3)\}$.

   Answer: the relation $\{(1,2),(2,2),(2,3),(3,2),(3,3)\}$.

7. Show that for any relation R, the relation $R^{\smallsmile} \circ (R \cup R^{\smallsmile})^* \circ R$ is symmetric, transitive and euclidean.

   Answer. Use E for $R^{\smallsmile} \circ (R \cup R^{\smallsmile})^* \circ R$. For symmetry of E, let $(x,y) \in E$. Then there is some $m \in \mathbb{N}$ with $(x,y) \in R^{\smallsmile} \circ (R \cup R^{\smallsmile})^m \circ R$. So there are z, u with $(x,z) \in R^{\smallsmile}$, $(z,u) \in (R \cup R^{\smallsmile})^m$, $(u,y) \in R$. But then $(y,u) \in R^{\smallsmile}$, $(u,z) \in (R \cup R^{\smallsmile})^m$, $(z,x) \in R$. It follows that $(y,x) \in R^{\smallsmile} \circ (R \cup R^{\smallsmile})^m \circ R$. and therefore $(y,x) \in E$.

   For transitivity, assume $(x,y) \in E$, $(y,z) \in E$. Then there are $m,k \in \mathbb{N}$ with $(x,y) \in R^{\smallsmile} \circ (R \cup R^{\smallsmile})^m \circ R$. $(y,z) \in R^{\smallsmile} \circ (R \cup R^{\smallsmile})^k \circ R$. It follows that $(x,z) \in R^{\smallsmile} \circ (R \cup R^{\smallsmile})^{m+k+2} \circ R$, and therefore $(x,z) \in E$.

   Euclideanness: any transitive and symmetric relation is euclidean.

8. Prove by induction that if R is an euclidean relation, then $R^{\smallsmile} \circ (R \cup R^{\smallsmile})^* \circ R \subseteq R$.

   Answer. Basis: $R^{\smallsmile} \circ R \subseteq R$. This follows from the euclideanness of R.

   Induction step. The induction hypothesis is that $R^{\smallsmile} \circ (R \cup R^{\smallsmile})^n \circ R \subseteq R$. We must prove that $R^{\smallsmile} \circ (R \cup R^{\smallsmile})^{n+1} \circ R \subseteq R$.

   We are done if we can prove the following two facts: (i) $R^{\smallsmile} \circ R \circ (R \cup R^{\smallsmile})^n \circ R \subseteq R$, and (ii) $R^{\smallsmile} \circ R^{\smallsmile} \circ (R \cup R^{\smallsmile})^n \circ R \subseteq R$.

   (i) Assume $(x,y) \in R^{\smallsmile} \circ R \circ (R \cup R^{\smallsmile})^n \circ R$. Then there is a z with $(x,z) \in R^{\smallsmile} \circ R$ and $(z,y) \in (R \cup R^{\smallsmile})^n \circ R$.

   By the fact that $R^{\smallsmile} \circ R$ is symmetric, $(z,x) \in R^{\smallsmile} \circ R$. From this and euclideanness of R, $(z,x) \in R$, and therefore $(x,z) \in R^{\smallsmile}$. Combining this with $(z,y) \in (R \cup R^{\smallsmile})^n \circ R$

we get that $(x, y) \in R^{\smile} \circ (R \cup R^{\smile})^n \circ R$, from which it follows by the induction hypothesis that $(x, y) \in R$.

(ii) Assume $(x, y) \in R^{\smile} \circ R^{\smile} \circ (R \cup R^{\smile})^n \circ R$. Then there is a z with $(x, z) \in R^{\smile}$ and $(z, y) \in R^{\smile} \circ (R \cup R^{\smile})^n \circ R$. From the first of these, $(z, x) \in R$. From the second of these, by induction hypothesis, $(z, y) \in R$. From $(z, x) \in R$ and $(z, y) \in R$ by euclideanness of R, $(x, y) \in R$.

9. Prove that

$$R \cup (R^{\smile} \circ (R \cup R^{\smile})^* \circ R)$$

is the euclidean closure of R.

Answer: Let E be the relation $R \cup (R^{\smile} \circ (R \cup R^{\smile})^* \circ R)$. Clearly, $R \subseteq E$.

We show that E is euclidean. Let $(x, y) \in E, (x, z) \in E$. Then there are three cases to consider.

Case 1. $(x, y) \in R, (x, z) \in R$. Then $(y, z) \in R^{\smile} \circ R$. Since $R^{\smile} \circ R \subseteq E$, it follows that $(y, z) \in E$.

Case 2. $(x, y) \in R$, $(x, z) \in R^{\smile} \circ (R \cup R^{\smile})^m \circ R$, for some $m \in \mathbb{N}$.

From the givens,

$$(y, z) \in R^{\smile} \circ R^{\smile} \circ (R \cup R^{\smile})^m \circ R.$$

It follows that $(y, z) \in R^{\smile} \circ (R \cup R^{\smile})^{m+1} \circ R$, and therefore $(y, z) \in E$.

Case 3. $(x, y) \in R^{\smile} \circ (R \cup R^{\smile})^* \circ R$, $(x, z) \in R^{\smile} \circ (R \cup R^{\smile})^* \circ R$. Now we get from the euclideanness of $R^{\smile} \circ (R \cup R^{\smile})^* \circ R$ (proved in a previous exercise) that $(y, z) \in R^{\smile} \circ (R \cup R^{\smile})^* \circ R$, and therefore $(y, z) \in E$.

Finally, we have to prove that E is the smallest relation with the required properties.

Let S be an arbitrary euclidean relation that includes R. We must show that $E \subseteq S$.

Since S is euclidean, we know from the previous exercise that $S^{\smile} \circ (S \cup S^{\smile})^* \circ S \subseteq S$.

Since $R \subseteq S$ we also have

$$R^{\smile} \circ (R \cup R^{\smile})^* \circ R \subseteq S^{\smile} \circ (S \cup S^{\smile})^* \circ S \subseteq S.$$

From this, together with $R \subseteq S$, it follows that $E \subseteq S$.

10. Prove that **lfp** $(\lambda S.S \cup (S \circ S))$ R is the transitive closure of R. (**lfp** f c is the least fixpoint of the operation f, starting from c.)

Answer. Let $f = \lambda S.S \cup (S \circ S)$, and let $T =$ **lfp** f R.

Then f is monotone, i.e., for all arguments X, $X \subseteq f(X)$. In particular, $R \subseteq f(R) \subseteq \ldots \subseteq T$. This shows that $R \subseteq T$.

We show that T is transitive. Since T is a fixpoint, $T = f(T)$, i.e., $T = T \cup (T \circ T)$. In other words, $T \circ T \subseteq T$. It follows from this that T is transitive by the result of the first exercise.

Finally, we show that T is the smallest transitive relation that includes R. Let S be an arbitrary transitive relation with $R \subseteq S$. From transitivity of S it follows that $S \circ S \subseteq S$. Therefore, $S = f(S)$, i.e., S is a fixpoint. Since T is the least fixpoint it follows that $T \subseteq S$.

11. Prove that **lfp** $(\lambda S.S \cup (S\check{} \circ S))$ R is the euclidean closure of R.

    Answer: similar to the reasoning in the previous exercise. Let $f = \lambda S.S \cup (S\check{} \circ S)$, and let $E = \textbf{lfp}\ f\ R$.

    Then f is monotone, i.e., for all arguments X, $X \subseteq f(X)$. In particular, $R \subseteq f(R) \subseteq \ldots \subseteq E$. This shows that $R \subseteq E$.

    We show that E is euclidean. Since E is a fixpoint, $E = f(E)$, i.e., $E = E \cup (E\check{} \circ E)$. In other words, $E\check{} \circ E \subseteq E$. It follows from this that E is euclidean by an argument given in the course slides.

    Finally, we show that E is the smallest euclidean relation that includes R. Let S be an arbitrary euclidean relation with $R \subseteq S$. From euclideanness of S it follows that $S\check{} \circ S \subseteq S$. Therefore, $S = f(S)$, i.e., S is a fixpoint. Since E is the least fixpoint it follows that $E \subseteq S$.

12. Give an example of a formula ϕ and a Kripke model M, with the following properties: (i) $M \models \phi$, and (ii) $M \mid \phi \not\models \phi$. In other words, public announcement of ϕ has the effect that ϕ becomes false.

    Answer. The simplest example is a model where p is true but agent a does not know that. Then publicly announcing 'p is true, but you don't know it' will result in a situation where what gets announced is falsified by the announcement itself. The formula for this is $p \wedge \neg K_a p$.