A first order axiomatisation of least fixpoint on finite models

Jan van Eijck

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Let R be a relational variable of arity m, and let $\bar{x}$ be an m-tuple of variables. Let ϕ be a first order formula that is positive in R, i.e., all occurrences of R in ϕ are in the scope of an even number of negations. Then $\lambda R \lambda \bar{x} \phi$ is a function from m-ary relations to m-ary relations. Given a model M where R is interpreted as $[\![R]\!]$, it yields $[\![\lambda \bar{x}.\phi]\!]$, so we can say that it maps $[\![R]\!]$ to $[\![\lambda \bar{x}.\phi]\!]$. From the fact that ϕ is positive in R it follows that this function $\mathbf{f}$ is monotone, i.e., that $\mathbf{S} \subseteq \mathbf{S}'$ implies that $\mathbf{f}(\mathbf{S}) \subseteq \mathbf{f}(\mathbf{S}')$.

By the Knaster-Tarski fixpoint theorem ([2]; see, e.g., [1] for background), any monotone function has a fixpoint, and the least fixpoint of a monotone function $\mathbf{f}$ on $\mathcal{P}(D^m)$ can be reached by repeated application of the function $\mathbf{f}$, starting from $\emptyset$. To be more precise, the least fixpoint is given by $\bigcup_\kappa \mathbf{f}^{\uparrow\kappa}(\emptyset)$, where κ ranges over all ordinals of cardinality at most $|D^m|$, and where $\mathbf{f}^{\uparrow\kappa}$ is defined by:

$$
\begin{aligned}
\mathbf{f}^{\uparrow 0}(X) &= X, \\
\mathbf{f}^{\uparrow \kappa+1}(X) &= \mathbf{f}(\mathbf{f}^{\uparrow\kappa}(X)), \\
\mathbf{f}^{\uparrow \lambda}(X) &= \bigcup_{\kappa<\lambda} \mathbf{f}^{\uparrow\kappa}(X) \text{ for } \lambda \text{ a limit ordinal.}
\end{aligned}
$$

We use $[\mathrm{lfp}_{R,\bar{x}}\phi]$ for the least fixpoint of the function $\lambda R \lambda \bar{x}.\phi$. Thus, $[\mathrm{lfp}_{R,\bar{x}}\phi]$ denotes the smallest m-ary relation T with the property that $\lambda R \lambda \bar{x}.\phi(T)$ equals T.

Here is an example with a binary relation. Let ϕ be the following formula.

$$x_1 = x_2 \vee \exists y(Sx_1y \wedge Ryx_2).$$

Then ϕ is positive in R, and therefore $\lambda R\lambda x_1x_2.\phi$ denotes a monotone function from binary relations to binary relations. The least fixpoint $[\mathrm{lfp}_{R,x_1,x_2}\phi]$ of this function is the relation S^* (the reflexive and transitive closure of S).

Let a structure M, an m-ary relation symbol R and a formula $\phi\bar{x}$ that is positive in R be given. Define $\mathbf{R}^n$ as follows, by induction:

$$\begin{array}{rcl} \mathbf{R}^0 & := & \emptyset \\ \mathbf{R}^{n+1} & := & \{\bar{b} \mid M \models_{\{R\mapsto\mathbf{R}^n,\bar{x}\mapsto\bar{b}\}} \phi\} \end{array}$$

The following fact follows from the Knaster-Tarski theorem:

Fact 1 *On finite structures, $\bigcup_{n\in\mathbb{N}} \mathbf{R}^n$ equals the least fixpoint of $\lambda R\lambda\bar{x}.\phi$.*

Assume $\bar{a}$ is an arbitrary tuple in M. Define $\mathbf{R}^n_{\bar{a}}$ as follows:

$$\begin{array}{rcl} \mathbf{R}^0_{\bar{a}} & := & \{\bar{a}\} \\ \mathbf{R}^{n+1}_{\bar{a}} & := & \{\bar{b} \mid M \models_{\{R\mapsto\mathbf{R}^n_{\bar{a}},\bar{x}\mapsto\bar{b}\}} \phi\} \end{array}$$

Lemma 2 *On all finite structures it holds for all $n \in \mathbb{N}$ that*

$$\mathbf{R}^{n+1} = \bigcup_{\bar{a}\in\mathbf{R}^1} \mathbf{R}^n_{\bar{a}}.$$

Proof. Induction on n. Clearly, the assertion holds for $n = 0$, for $\mathbf{R}^1 = \bigcup_{\bar{a}\in\mathbf{R}^1}\{\bar{a}\} = \bigcup_{\bar{a}\in\mathbf{R}^1} \mathbf{R}^0_{\bar{a}}$.

Suppose $\mathbf{R}^{n+1} = \bigcup_{\bar{a}\in\mathbf{R}^1} \mathbf{R}^n_{\bar{a}}$. We show that $\mathbf{R}^{n+2} = \bigcup_{\bar{a}\in\mathbf{R}^1} \mathbf{R}^{n+1}_{\bar{a}}$.

$$\begin{array}{rcl} \mathbf{R}^{n+2} & = & \{\bar{b} \mid M \models_{\{R\mapsto\mathbf{R}^{n+1},\bar{x}\mapsto\bar{b}\}} \phi\} \\ & \stackrel{\text{ih}}{=} & \{\bar{b} \mid M \models_{\{R\mapsto\bigcup_{\bar{a}\in\mathbf{R}^1}\mathbf{R}^n_{\bar{a}},\bar{x}\mapsto\bar{b}\}} \phi\} \\ & \stackrel{*}{=} & \bigcup_{\bar{a}\in\mathbf{R}^1} \{\bar{b} \mid M \models_{\{R\mapsto\mathbf{R}^n_{\bar{a}},\bar{x}\mapsto\bar{b}\}} \phi\} \\ & = & \bigcup_{\bar{a}\in\mathbf{R}^1} \mathbf{R}^{n+1}_{\bar{a}} \end{array}$$

Equality $\overset{*}{=}$ expresses continuity of $\lambda R\lambda\bar{x}.\phi$, which holds for all monotone functions on finite domains. $\square$

Combining the fact and the lemma, we get:

Theorem 3 *On finite structures, $\bigcup_{n\in\mathbb{N},\bar{a}\in\mathbf{R}^1}\mathbf{R}^n_{\bar{a}}$ equals the least fixpoint of $\lambda R\lambda\bar{x}.\phi$.*

Write $\bar{x}\mathbf{R}^m\bar{y}$ for $\bar{y}\in\mathbf{R}^m_{\bar{x}}$. Introduce a $3m$-ary relation symbol L, with the following intended interpretation:

$$\lambda\bar{x}\lambda\bar{y}\lambda\bar{z}.\exists n,m\in\mathbb{N}(0<n\leq m\wedge\bar{x}\mathbf{R}^n\bar{y}\wedge\bar{x}\mathbf{R}^m\bar{z}\wedge\forall k\in\mathbb{N}(k<m\rightarrow\neg\bar{x}\mathbf{R}^k\bar{z})).$$

Notice that if L is interpreted like this, then $\lambda\bar{x}\lambda\bar{y}.L\bar{x}\bar{y}\bar{y}$ expresses the following:

$$\exists n\in\mathbb{N}.(n>0\wedge\bar{x}\mathbf{R}^n\bar{y}\wedge\bar{x}\neq\bar{y}).$$

Here $\bar{x}=\bar{y}$ abbreviates the conjunction $x_1=y_1\wedge\cdots\wedge x_m=y_m$. Let $\mathbf{I}$ denote the identity function for vectors of length m. Then $\lambda\bar{x}\lambda\bar{y}.L\bar{x}\bar{y}\bar{y}$ denotes $\mathbf{R}^+-\mathbf{I}$ (if we view $\mathbf{R}$ as a binary relation on vectors of length m).

Let ϕ_0 be the formula that results from ϕ by replacing each occurrence of R by $\lambda\bar{x}.\bar{x}\neq\bar{x}$ (the m-ary predicate that is always false). Then, given the intended interpretation of L, the following defines the least fixpoint of $\lambda R\lambda\bar{x}.\phi$ on finite structures. Here and henceforth, read the formulas as universally closed.

$$[\mathrm{lfp}_{R,\bar{x}}\phi](\bar{x})\leftrightarrow\phi_0\bar{x}\vee\exists\bar{y}(\phi_0\bar{y}\wedge L\bar{y}\bar{x}\bar{x}).\tag{DEF}$$

We will present six axioms L1–6 and we will show that these together enforce the intended interpretation of L.

Clearly, $\lambda\bar{x}\bar{y}.L\bar{x}\bar{y}\bar{u}$ is irreflexive, for any choice of $\bar{u}$. This is true in the intended interpretation of L because if $\bar{x}\mathbf{R}\bar{x}$ holds, and also $\bar{x}\mathbf{R}^m\bar{y}$ for $m>1$, then clearly $\bar{x}\mathbf{R}^k\bar{y}$ for some $k<m+1$. Irreflexivity of $\lambda\bar{x}\bar{y}.L\bar{x}\bar{y}\bar{u}$ is expressed by:

$$\neg L\bar{x}\bar{x}\bar{u}\tag{L1}$$

Also, $\lambda\bar{x}\bar{y}.L\bar{x}\bar{y}\bar{u}$ is transitive, for any $\bar{u}$. For if $\bar{x}\mathbf{R}^n\bar{y}$ and $\bar{x}\mathbf{R}^m\bar{z}$ with $0<n\leq m$ and for no $k<m$, $\bar{x}\mathbf{R}^k\bar{z}$, and $\bar{x}\mathbf{R}^p\bar{z}$ and $\bar{x}\mathbf{R}^q\bar{u}$ with $0<p\leq q$ and for

no $r < q$, $\bar{x}\mathbf{R}^r\bar{u}$, then $m = p$ and it follows that $\bar{x}\mathbf{R}^p\bar{y}$ and $\bar{x}\mathbf{R}^q\bar{u}$, and for no $r < q$, $\bar{x}\mathbf{R}^r\bar{u}$, i.e., $L\bar{x}\bar{y}\bar{u}$ holds. This transitivity requirement is expressed by:

$$(L\bar{x}\bar{y}\bar{z} \wedge L\bar{y}\bar{z}\bar{u}) \rightarrow L\bar{x}\bar{z}\bar{u} \tag{L2}$$

Axiom L3 expresses that $(\mathbf{R}^+ - \mathbf{I})$ is almost transitive:

$$(L\bar{x}\bar{y}\bar{y} \wedge L\bar{y}\bar{z}\bar{z} \wedge \bar{x} \neq \bar{z}) \rightarrow L\bar{x}\bar{z}\bar{z} \tag{L3}$$

Assume $\phi\bar{y}$ has no free occurrences of $\bar{x}$. Let $\phi_1\bar{x}\bar{y}$ be the result of replacing occurrences of R in $\phi\bar{y}$ by $\lambda\bar{z}.\bar{z} = \bar{x}$. Then $\phi_1\bar{x}\bar{y}$ expresses $\bar{x}\mathbf{R}\bar{y}$. We can use this to express that $(\mathbf{R} - \mathbf{I}) \subseteq (\mathbf{R}^+ - \mathbf{I})$, as follows:

$$(\phi_1\bar{x}\bar{y} \wedge \bar{x} \neq \bar{y}) \rightarrow L\bar{x}\bar{y}\bar{y} \tag{L4}$$

The next axiom expresses that if $\overline{xy} \in (\mathbf{R}^+ - \mathbf{I})$ then it is always possible to make a first $\mathbf{R}$-step on some shortest $\mathbf{R}$-path from $\bar{x}$ to $\bar{y}$.

$$L\bar{x}\bar{y}\bar{y} \rightarrow \exists\bar{z}(\phi_1\bar{x}\bar{z} \wedge L\bar{x}\bar{z}\bar{y}) \tag{L5}$$

Finally:

$$(L\bar{x}\bar{y}\bar{z} \wedge \bar{y} \neq \bar{z}) \rightarrow L\bar{y}\bar{z}\bar{z} \tag{L6}$$

This expresses that if $\bar{y}$ is somewhere along on a shortest $\mathbf{R}$-path from $\bar{x}$ to $\bar{z}$, and $\bar{y} \neq \bar{z}$, then $\overline{yz} \in (\mathbf{R}^+ - \mathbf{I})$.

This turns out to be a complete first order theory for least fixpoint on finite structures. The above discussion should have convinced the reader that the axioms are sound for the intended interpretation. We will now show that the theory consisting of L1–6 defines least fixpoint on finite models.

Theorem 4 *Let R be an m-ary relation symbol. Let $\phi\bar{x}$ be a first order formula that is positive in R. Let M be a finite model of L1–6, let $\mathbf{f}$ be the interpretation of $\lambda R\lambda\bar{x}.\phi$ in M, and let $\mathbf{S}$ be the interpretation of $[lfp_{R,\bar{x}}\phi]$ in M. Then $\mathbf{S}$ is the least fixpoint of $\mathbf{f}$.*

Proof. Let M, $\mathbf{f}$ and $\mathbf{S}$ be as stated in the theorem. To show that $\mathbf{S}$ is the least fixpoint of $\mathbf{f}$, by Theorem 3 it is enough to show that $\bigcup_{n\in\mathbb{N},\bar{a}\in\mathbf{R}^1} \mathbf{R}^n_{\bar{a}} = \mathbf{S}$.

$\Rightarrow$: Let $\mathbf{R}^{\geq n}_{\bar{a}}$ be given by

$$\mathbf{R}^{\geq n}_{\bar{a}} = \{\bar{b} \mid \bar{b} \in \mathbf{R}^n_{\bar{a}} \wedge \forall m < n \;\; \bar{b} \notin \mathbf{R}^m_{\bar{a}}\}.$$

We show by induction:

$$\text{For all } n \in \mathbb{N}: \ \bar{b} \in \mathbf{R}_{\bar{a}}^{\geq n+1} \text{ implies } \overline{ab b} \in [\![L]\!]. \qquad (*)$$

Base case: Let $\bar{b} \in \mathbf{R}_{\bar{a}}^{\geq 1}$. This is equivalent to $\bar{a} \neq \bar{b}$ and $\bar{b} \in \mathbf{R}_{\bar{a}}^{1}$. The second conjunct is equivalent to $\overline{ab} \in [\![\phi_1]\!]$. By an application of L4 we get that $\overline{abb} \in [\![L]\!]$.

Induction step: Assume $\bar{b} \in \mathbf{R}_{\bar{a}}^{\geq n+1}$ implies $\overline{abb} \in [\![L]\!]$. Suppose $\bar{b} \in \mathbf{R}_{\bar{a}}^{\geq n+2}$. Then $\bar{a} \neq \bar{b}$, and there is some $\bar{c} \in \mathbf{R}_{\bar{a}}^{1}$ with $\bar{a} \neq \bar{c}$, $\bar{c} \neq \bar{b}$, and $\bar{b} \in \mathbf{R}_{\bar{c}}^{\geq n+1}$. $\bar{c} \in \mathbf{R}_{\bar{a}}^{1}$ is equivalent to $\overline{ac} \in [\![\phi_1]\!]$. From $\overline{ac} \in [\![\phi_1]\!]$ and $\bar{a} \neq \bar{c}$ it follows by L4 that $\overline{acc} \in [\![L]\!]$. By induction hypothesis it follows from $\bar{b} \in \mathbf{R}_{\bar{c}}^{\geq n+1}$ that $\overline{cbb} \in [\![L]\!]$. Since $\bar{a} \neq \bar{b}$ we can apply L3. This yields $\overline{abb} \in [\![L]\!]$. This establishes (*).

Now assume $\bar{b} \in \bigcup_{n \in \mathbb{N}, \bar{a} \in \mathbf{R}^1} \mathbf{R}_{\bar{a}}^{n}$. Then either $\bar{b} \in \mathbf{R}^1 = [\![\phi_0]\!]$, or there is some $\bar{a} \in \mathbf{R}^1 = [\![\phi_0]\!]$ and some $n \in \mathbb{N}$ with $\bar{b} \in \mathbf{R}_{\bar{a}}^{\geq n+1}$, whence by (*), $\overline{abb} \in [\![L]\!]$. It follows from DEF that $\bar{b} \in \mathbf{S}$.

$\Leftarrow$: Let $\bar{b} \in \mathbf{S}$. We show that $\bar{b} \in \bigcup_{n \in \mathbb{N}, \bar{a} \in \mathbf{R}^1} \mathbf{R}_{\bar{a}}^{n}$.

If $\bar{b} \in [\![\phi_0]\!]$, then $\bar{b} \in \mathbf{R}_{\bar{b}}^{0}$, and done. Otherwise, by DEF, there is a $\bar{a}$ with $\bar{a} \in [\![\phi_0]\!]$ and $\overline{abb} \in [\![L]\!]$. By L5, there is a $\overline{a_1}$ with $\overline{aa_1} \in [\![\phi_1]\!]$ and $\overline{aa_1b} \in [\![L]\!]$. By L1, $\overline{a_1} \neq \bar{a}$. Suppose $\overline{a_1} = \bar{b}$. Then $\overline{ab} \in [\![\phi_1]\!]$ and therefore $\bar{b} \in \mathbf{R}_{\bar{a}}^{1}$, and done. Suppose $\overline{a_1} \neq \bar{b}$. Then by L6, $\overline{a_1bb} \in [\![L]\!]$, and from this it follows, by L5, that there is a $\overline{a_2}$ with $\overline{a_1a_2} \in [\![\phi_1]\!]$ and $\overline{a_1a_2b} \in [\![L]\!]$. By L1, $\overline{a_1} \neq \overline{a_2}$. If $\bar{a} = \overline{a_2}$ then from $\overline{a_1ab} \in [\![L]\!]$ and $\overline{aa_1b} \in [\![L]\!]$ it would follow by L2 that $\overline{a_1a_1b} \in [\![L]\!]$, and contradiction with L1. So $\bar{a} \neq \overline{a_2}$. If $\overline{a_2} = \bar{b}$ then $\bar{b} \in \mathbf{R}_{\bar{a}}^{2}$, and done. If $\overline{a_2} \neq \bar{b}$ then by L6, there is a $\overline{a_3}$, and so on. This creates a sequence of m-tuples

$$\bar{a} = \overline{a_0}, \ \overline{a_1}, \ \overline{a_2}, \ \ldots$$

with the $\overline{a_i}$ all different. By finiteness of the domain, this process has to stop with $\overline{a_n} = \bar{b}$ for some $n \in \mathbb{N}$. It follows that $\bar{b} \in \mathbf{R}_{\bar{a}}^{n}$. □

References

[1] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Order (Second Edition).* Cambridge University Press, Cambridge, 2002. First edition: 1990.

[2] Alfred Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5:285–309, 1955.