Principles of Constraint Programming

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Chapter 2
Constraint Satisfaction Problems: Examples

Objectives

- Define formally **Constraint Satisfaction Problems (CSP)**,
- **Modeling**: representation of a problem as a CSP.
- Clarify various aspects of modeling:
  - in general several natural representations exist,
  - some representations straightforward, some non-trivial,
  - some representations rely on a "background" theory.
- Show the generality of the notion of a CSP.

Constraint Satisfaction Problems (CSP)

Given:

- **Variables** $Y := y_1, \ldots, y_k$,
- **Domains** $D_1, \ldots, D_k$,

Constraint C on Y: subset of $D_1 \times \ldots \times D_k$.

Given:

- **Variables** $x_1, \ldots, x_n$,
- **Domains** $D_1, \ldots, D_n$,

Constraint Satisfaction Problem (**CSP**):

$$\{\mathcal{C} \; ; x_1 \in D_1, \ldots, x_n \in D_n\}$$

$\mathcal{C}$ – constraints, each on a subsequence of $x_1, \ldots, x_n$.

$(d_1, \ldots, d_n) \in D_1 \times \ldots \times D_n$ is a **solution** to

$$\{\mathcal{C} \; ; x_1 \in D_1, \ldots, x_n \in D_n\}$$

if for every constraint $C \in \mathcal{C}$ on $x_{i_1}, \ldots, x_{i_m}$

$$(d_{i_1}, \ldots, d_{i_m}) \in C.$$

Example: SEND + MORE = MONEY

Replace each letter by a different digit so that

$$\begin{array}{r} SEND \\ +\ \ MORE \\ \hline MONEY \end{array}$$

is a correct sum.

Unique solution:

$$\begin{array}{r} 9567 \\ +\ \ 1085 \\ \hline 10652 \end{array}$$

Variables: S, E, N, D, M, O, R, Y,

Domains:

$[1..9]$ for S, M,

$[0..9]$ for E, N, D, O, R, Y.

Alternatives for Equality Constraints

1. 1 equality constraint.

$$\begin{array}{rrrrrrrrrr} & & 1000 \cdot S & + & 100 \cdot E & + & 10 \cdot N & + & D \\ & + & 1000 \cdot M & + & 100 \cdot O & + & 10 \cdot R & + & E \\ = 10000 \cdot M & + & 1000 \cdot O & + & 100 \cdot N & + & 10 \cdot E & + & Y \end{array}$$

2. 5 equality constraints.

Use "carry" variables $C_1, \ldots, C_4 \in [0..1]$:

$$D + E = 10 \cdot C_1 + Y,$$
$$C_1 + N + R = 10 \cdot C_2 + E,$$
$$C_2 + E + O = 10 \cdot C_3 + N,$$
$$C_3 + S + M = 10 \cdot C_4 + O,$$
$$C_4 = M.$$

Alternatives for Disequality Constraints

1. 28 disequality constraints.

$x \neq y$ for $x, y \in \{S, E, N, D, M, O, R, Y\}$, $x \prec y$.

2. A single constraint for disequalities.

For variables $x_1, \ldots, x_n$ with domains $D_1, \ldots, D_n$:

$\texttt{all\_different}(x_1, \ldots, x_n)$
$\quad := \{(d_1, \ldots, d_n) \mid d_i \neq d_j \text{ for } i \neq j\}$.

Use

$\texttt{all\_different}(S, E, N, D, M, O, R, Y)$.

3. Modeling it as an IP problem.

For $x, y \in \{S, E, N, D, MO, R, Y\}$ transform $x \neq y$ to

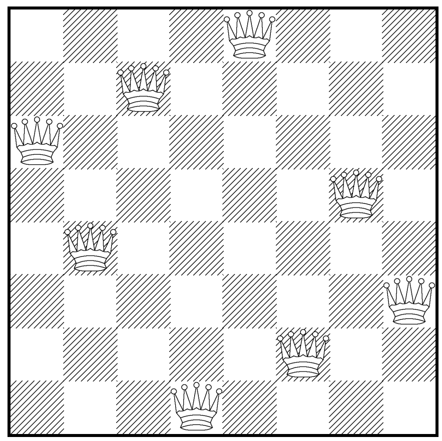
$x - y \leq 10 - 11 z_{x,y}$,
$y - x \leq 11 z_{x,y} - 1$,

where $z_{x,y} \in [0..1]$.

Disadvantage: 28 new variables.

N Queens

Problem Place n queens on the $n \cdot n$ chess board so that they do not attack each other.



Variables: $x_1, \ldots, x_n$,
Domains: [1..n],
Constraints:

For $i \in [1..n-1]$ and $j \in [i+1..n]$:

- $x_i \neq x_j$ (rows),
- $x_i - x_j \neq i - j$
  (South-West – North-East diagonals),
- $x_i - x_j \neq j - i$
  (North-West – South-East diagonals).

Zebra Puzzle

A small street has five differently **colored** houses on it.

Five men of different **nationalities** live in these five **houses**.

Each man has a different **profession**, each man likes a different **drink**, and each has a different **pet animal**.

Zebra puzzle, ctd

The Englishman lives in the red house.
The Spaniard has a dog.
The Japanese is a painter.
The Italian drinks tea.
The Norwegian lives in the first house on the left.
The owner of the green house drinks coffee.
The green house is on the right of the white house.
The sculptor breeds snails.
The diplomat lives in the yellow house.
They drink milk in the middle house.
The Norwegian lives next door to the blue house.
The violinist drinks fruit juice.
The fox is in the house next to the doctor's.
The horse is in the house next to the diplomat's.

Who has the zebra and who drinks water?

Zebra puzzle, ctd

25 **Variables**:

- nationality: **english, spaniard, japanese, italian, norwegian**,
- pet: **dog, snails, fox, horse, zebra**,
- profession: **painter, sculptor, diplomat, violinist, doctor**,
- drink: **tea, coffee, milk, juice, water**,
- colour: **red, green, white, yellow, blue**.

Domains: [1...5].

Constraints:

`all_different`(**red**, **green**, **white**, **yellow**, **blue**),

`all_different`(**english**, **spaniard**, **japanese**, **italian**, **norwegian**),

`all_different`(**dog**, **snails**, **fox**, **horse**, **zebra**),

`all_different`(**painter**, **sculptor**, **diplomat**, **violinist**, **doctor**),

`all_different`(**tea**, **coffee**, **milk**, **juice**, **water**).

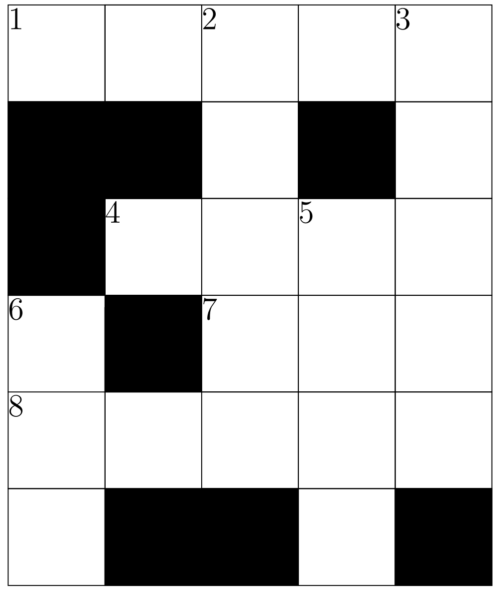
Constraints, ctd

- The Englishman lives in the red house:
  $\mathbf{english} = \mathbf{red}$,
- The Spaniard has a dog:
  $\mathbf{spaniard} = \mathbf{dog}$,
- The Japanese is a painter:
  $\mathbf{japanese} = \mathbf{painter}$,
- The Italian drinks tea:
  $\mathbf{italian} = \mathbf{tea}$,
- The Norwegian lives in the first house on the left:
  $\mathbf{norwegian} = 1$,
- The owner of the green house drinks coffee:
  $\mathbf{green} = \mathbf{coffee}$,
- The green house is on the right of the white house:
  $\mathbf{green} = \mathbf{white} + 1$,

Constraints, ctd

- The sculptor breeds snails:
  $\textbf{sculptor} = \textbf{snails}$,
- The diplomat lives in the yellow house:
  $\textbf{diplomat} = \textbf{yellow}$,
- They drink milk in the middle house:
  $\textbf{milk} = 3$,
- The Norwegian lives next door to the blue house:
  $|\textbf{norwegian} - \textbf{blue}| = 1$,
- The violinist drinks fruit juice:
  $\textbf{violinist} = \textbf{juice}$,
- The fox is in the house next to the doctor's:
  $|\textbf{fox} - \textbf{doctor}| = 1$,
- The horse is in the house next to the diplomat's:
  $|\textbf{horse} - \textbf{diplomat}| = 1$.

Crossword Puzzles



Fill the crossword grid with the words from:

- HOSES, LASER, SAILS, SHEET, STEER,
- HEEL, HIKE, KEEL, KNOT, LINE,
- AFT, ALE, EEL, LEE, TIE.

Variables: $x_1, \ldots, x_8$,

Domains: $x_7 \in \{$AFT, ALE, EEL, LEE, TIE$\}$, etc.

Constraints: one per crossing

$$C_{1,2} := \{(\text{HOSES, SAILS}), (\text{HOSES, SHEET}), (\text{HOSES, STEER}), (\text{LASER, SAILS}), (\text{LASER, SHEET}), (\text{LASER, STEER})\} \; .$$

etc.

Unique Solution

1 H	O	2 S	E	3 S
		A		T
	4 H	I	5 K	E
6 A		7 L	E	E
8 L	A	S	E	R
E			L	

Qualitative Temporal Reasoning

Consider the following problem.

> The **meeting** ran non-stop the whole day.
> Each person stayed at the meeting for a continuous period of time.
> The **meeting** began while **Mr Jones** was present and finished while **Ms White** was present.
> **Ms White** arrived after the meeting has began.
> In turn, **Director Smith** was also present but he arrived after **Jones** had left.
> **Mr Brown** talked to **Ms White** in presence of **Smith.**
> Could possibly **Jones** and **White** have talked during this meeting?

13 Temporal Relations (Allen '83)

`A` before `B`

`B` after `A`

`A` meets `B`

`B` met-by `A`

`A` overlaps `B`

`B` overlapped-by `A`

`A` starts `B`

`B` started-by `A`

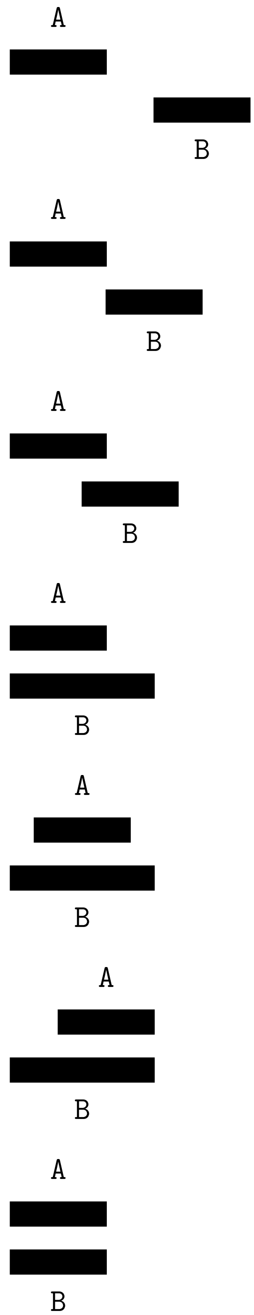
`A` during `B`

`B` contains `A`

`A` finishes `B`

`B` finished-by `A`

`A` equals `B`



The Composition Table

- Consider three events, `A, B` and `C`.

  Known: Temporal relations

  - `AB` between `A` and `B`,
  - `BC` between `B` and `C`.

  Question: What is the temporal relation `BC` between `A` and `C`?

- Allen '83 defined a 13×13 table.
- **Example**: if `A overlaps B` and `B` is `before C`, then `A` is `before C`.

  This yields entry

  `allen(overlaps, before, before).`

In total 409 entries.

Composition Table, part 1

	before	after	meets	met-by	overlaps	overl.-by
before	before	*TEMP*	before	before meets overlaps starts during	before	before meets overlaps starts during
after	*TEMP*	after	during finishes after met-by overl.-by	after	during finishes after met-by overl.-by	after
meets	before	after met-by overl.-by started-by contains	before	finishes finished-by equals	before	overlaps starts during
met-by	before overlaps meets contains finished-by	after	starts started-by equals	after	during finishes overl.-by	after
overlaps	before	after met-by overl.-by started-by contains	before	overl.-by started-by contains	before meets overlaps	*R-OVERLAP*
overl.-by	before meets overlaps contains finished-by	after	overlaps contains finished-by	after	*R-OVERLAP*	after met-by overl.-by
starts	before	after	before	met-by	before meets overlaps	during finishes overl.-by
started-by	before meets overlaps contains finished-by	after	overlaps contains finished-by	met-by	overlaps contains finished-by	overl.-by
during	before	after	before	after	before meets overlaps starts during	during finishes after met-by overl.-by
contains	before meets overlaps contains finished-by	after met-by overl.-by contains started-by	overlaps contains finished-by	overl.-by started-by contains	overlaps contains finished-by	overl.-by started-by contains
finishes	before	after	meets	after	overlaps starts during	after met-by overl.-by
finished-by	before	after met-by overl.-by started-by contains	meets	overl.-by started-by contains	overlaps	overl.-by started-by contains
equals	before	after	meets	met-by	overlaps	overl.-by

The Composition Table, part 2

	starts	started-by	during	contains	finishes	finished-by	equals
before	before	before	before meets overlaps starts during	before	before meets overlaps starts during	before	before
after	during finishes after met-by overl.-by	after	during finishes after met-by overl.-by	after	after	after	after
meets	meets	meets	overlaps starts during	before	overlaps starts during	before	meets
met-by	during finishes overl.-by	after	during finishes overl.-by	after	met-by	met-by	met-by
overlaps	overlaps	overlaps contains finished-by	overlaps starts during	before meets overlaps contains finished-by	overlaps starts during	before meets overlaps	overlaps
overl.-by	during finishes overl.-by	after met-by overl.-by	during finishes overl.-by	after meets overl.-by started-by contains	overl.-by	overl.-by started-by contains	overl.-by
starts	starts	starts started-by equals	during	before meets overlaps contains finished-by	during	before meets overlaps	starts
started-by	starts started-by equals	started-by	during finishes overl.-by	contains	overl.-by	contains	started-by
during	during	during finishes after met-by overl.-by	during	*TEMP*	during	before meets overlaps starts during	during
contains	overlaps contains finished-by	contains	*R-OVERLAP*	contains	overl.-by contains started-by	contains	contains
finishes	during	after met-by overl.-by	during	after met-by overl.-by started-by contains	finishes	finishes finished-by equals	finishes
finished-by	overlaps	contains	overlaps starts during	contains	finishes finished-by equals	finished-by	finished-by
equals	starts	started-by	during	contains	finishes	finished-by	equals

Representation as a CSP

- 5 events:
  - $\texttt{M}$ (meeting),
  - $\texttt{J}$ (Jones's presence),
  - $\texttt{B}$ (Brown's presence),
  - $\texttt{S}$ (Smith's presence),
  - $\texttt{W}$ (White's presence).
- This yields 10 variables, each associated with an ordered pair of events and each with a domain:

  $TEMP := \{$`before, after, meets, met-by, overlaps, overlapped-by, starts, started-by, during, contains,finishes, finished-by, equals`$\}$,

  $REAL\text{-}OVERLAP := TEMP - \{$`before, after, meets, met-by`$\}$

  - $x_{\texttt{J,M}} \in \{$`overlaps, contains, finished-by`$\}$,
  - $x_{\texttt{M,W}} \in \{$`overlaps`$\}$,
  - $x_{\texttt{M,S}} \in REAL\text{-}OVERLAP$,
  - $x_{\texttt{J,S}} \in \{$`before`$\}$,
  - $x_{\texttt{B,S}} \in REAL\text{-}OVERLAP$,
  - $x_{\texttt{B,W}} \in REAL\text{-}OVERLAP$,
  - $x_{\texttt{S,W}} \in REAL\text{-}OVERLAP$,
  - $x_{\texttt{J,B}}, x_{\texttt{J,W}}, x_{\texttt{M,B}} \in TEMP$.

Final question

Use rather

– $x_{\texttt{J,W}} \in REAL\text{-}OVERLAP$.

Is the above CSP consistent?

Constraints

- *allen*: the composition table as a ternary relation (409 entries).
- For each ordered triple `A,B,C` of the events a constraint $C_{\mathtt{A,B,C}}$ on the variables $x_{\mathtt{A,B}}$, $x_{\mathtt{B,C}}$, $x_{\mathtt{A,C}}$:

  $$C_{\mathtt{A,B,C}} := allen \cap (D_{\mathtt{A,B}} \times D_{\mathtt{B,C}} \times D_{\mathtt{A,C}}).$$

  where

  $x_{\mathtt{A,B}} \in D_{\mathtt{A,B}}$,

  $x_{\mathtt{B,C}} \in D_{\mathtt{B,C}}$,

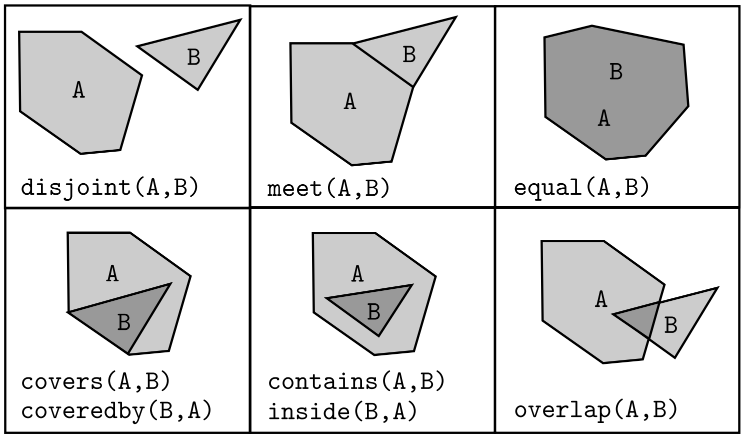
  $x_{\mathtt{A,C}} \in D_{\mathtt{A,C}}$.
- In total 10 constraints.

Qualitative Spatial Reasoning

Consider the following problem.

> Two houses are connected by a **road**. The **first house** is surrounded by its **garden** or is adjacent to its boundary while the **second house** is surrounded by its **garden.**
>
> What can we conclude about the relation between the second garden and the road?

8 Spatial Relations



$$\mathtt{RCC8} := \{\mathtt{disjoint,\ meet},\ \mathtt{equal,\ covers},\ \mathtt{coveredby,\ contains},\ \mathtt{inside,\ overlap}\}.$$

The composition table for RCC8

	disjoint	meet	equal	inside	coveredby	contains	covers	overlap
disjoint	RCC8	disjoint meet inside coveredby overlap	disjoint	disjoint meet inside coveredby overlap	disjoint meet inside coveredby overlap	disjoint	disjoint	disjoint meet inside coveredby overlap
meet	disjoint meet contains covers overlap	disjoint meet equal coveredby covers overlap	meet	inside coveredby overlap	meet inside	disjoint	disjoint meet	disjoint meet inside coveredby overlap
equal	disjoint	meet	equal	inside	coveredby	contains	covers	overlap
inside	disjoint	disjoint	inside	inside	inside	RCC8	disjoint meet inside coveredby overlap	disjoint meet inside coveredby overlap
coveredby	disjoint	disjoint meet	coveredby	inside	inside coveredby	disjoint meet contains covers overlap	disjoint meet equal coveredby covers overlap	disjoint meet overlap coveredby overlap
contains	disjoint meet contains covers overlap	contains covers overlap	contains	equal inside coveredby contains covers overlap	contains covers overlap	contains	contains	contains covers overlap
covers	disjoint meet contains covers overlap	meet contains covers overlap	covers	inside coveredby overlap	equal coveredby covers overlap	contains	contains covers	contains covers overlap
overlap	disjoint meet contains covers overlap	disjoint meet contains covers overlap	overlap	inside coveredby overlap	inside coveredby overlap	disjoint meet contains covers overlap	disjoint meet contains covers overlap	RCC8

Representation as a CSP

- 5 spatial objects:
  - `H1` (house 1),
  - `G1` (garden 1),
  - `H2` (house 2),
  - `G2` (garden 2),
  - `R` (road).
- 10 variables with domains, each associated with an ordered pair of spatial objects:
  - $x_{\mathtt{H1},\mathtt{G1}} \in \{\mathtt{inside},\mathtt{coveredby}\}$,
  - $x_{\mathtt{H2},\mathtt{G2}} \in \{\mathtt{inside}\}$,
  - $x_{\mathtt{H1},\mathtt{H2}} \in \{\mathtt{disjoint}\}$,
  - $x_{\mathtt{H1},\mathtt{R}} \in \{\mathtt{meet}\}$,
  - $x_{\mathtt{H2},\mathtt{R}} \in \{\mathtt{meet}\}$,
  - $x_{\mathtt{G1},\mathtt{G2}} \in \{\mathtt{disjoint},\mathtt{meet}\}$,
  - $x_{\mathtt{H1},\mathtt{G2}} \in \{\mathtt{disjoint},\mathtt{meet}\}$,
  - $x_{\mathtt{G1},\mathtt{H2}} \in \{\mathtt{disjoint},\mathtt{meet}\}$,
  - $x_{\mathtt{G1},\mathtt{R}} \in \mathrm{RCC8}$,
  - $x_{\mathtt{G2},\mathtt{R}} \in \mathrm{RCC8}$.

Constraints

- S_3: the composition table as a ternary relation (193 triples).
- For each ordered triple **A,B,C** of the objects a constraint $C_{\mathtt{A},\mathtt{B},\mathtt{C}}$ on the variables $x_{\mathtt{A},\mathtt{B}}, x_{\mathtt{B},\mathtt{C}}, x_{\mathtt{A},\mathtt{C}}$:

  $$C_{\mathtt{A},\mathtt{B},\mathtt{C}} := S_3 \cap (D_{\mathtt{A},\mathtt{B}} \times D_{\mathtt{B},\mathtt{C}} \times D_{\mathtt{A},\mathtt{C}}).$$

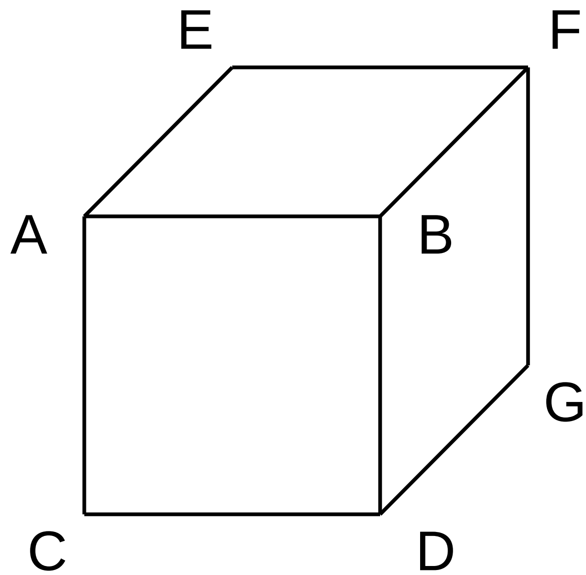
  where

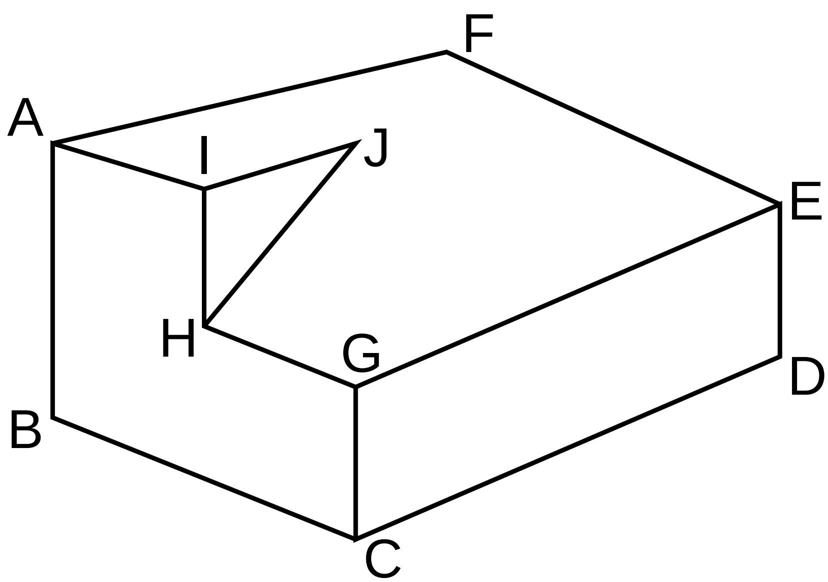
  $x_{\mathtt{A},\mathtt{B}} \in D_{\mathtt{A},\mathtt{B}}$,

  $x_{\mathtt{B},\mathtt{C}} \in D_{\mathtt{B},\mathtt{C}}$,

  $x_{\mathtt{A},\mathtt{C}} \in D_{\mathtt{A},\mathtt{C}}$.
- In total 10 constraints.

Analysis of Polyhedral Scenes

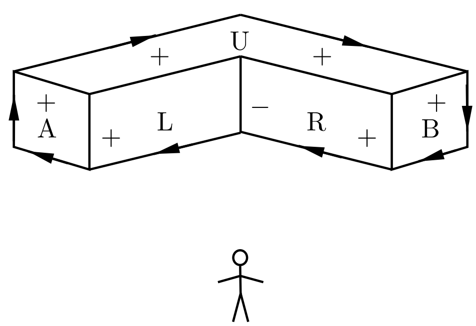


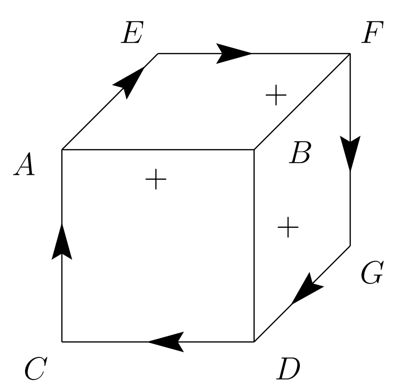


Four labels

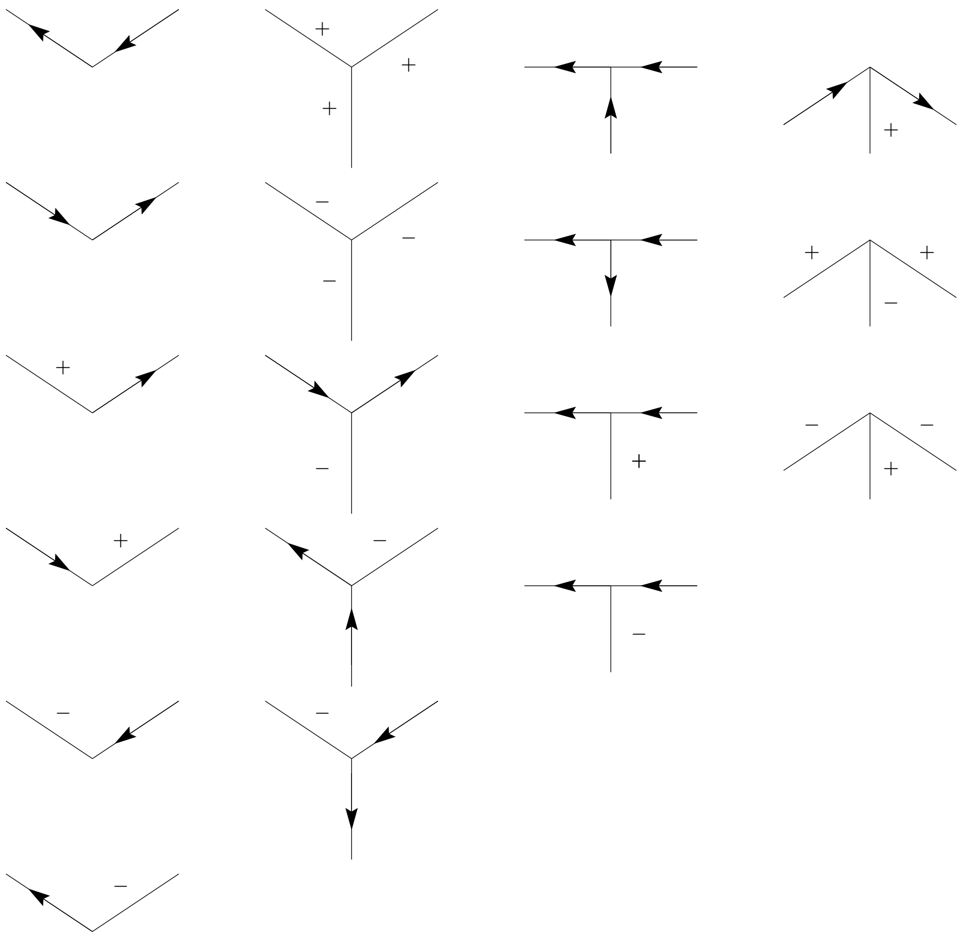
- $+$, mark **convex** edges,
  (takes 270 degrees to rotate)
- $-$, mark **concave** edges,
  (takes 90 degrees to rotate)
- $\rightarrow$ and $\leftarrow$ mark **boundary** edges
  (formed by two planes one of which is hidden).

Examples





Legal Junctions



Representation as a CSP

Variables: edges,

Domains: $\{+, -, \rightarrow, \leftarrow\}$,

Constraints: junctions.

Four type of constraints: L, *fork*, T, and $arrow$.

Example:

$$L := \{(\rightarrow, \leftarrow), (\leftarrow, \rightarrow), (+, \rightarrow), \\ (\leftarrow, +), (-, \leftarrow), (\rightarrow, -)\}.$$

Cube as a CSP:

$$arrow(AC, AE, AB),$$
$$fork(BA, BF, BD),$$
$$L(CA, CD),$$
$$arrow(DG, DC, DB),$$
$$L(EF, EA),$$
$$arrow(FE, FG, FB),$$
$$L(GD, GF).$$

Representation as a CSP, ctd

Also needed

$edge := \{(+,+), (-,-), (\rightarrow, \leftarrow), (\leftarrow, \rightarrow)\}.$

$edge$ captures the complementary character of $\rightarrow$ and $\leftarrow$.

Nine constraints:

$edge(AB, BA)$,
$edge(AC, CA)$,
$edge(CD, DC)$,
$edge(BD, DB)$,
$edge(AE, EA)$,
$edge(EF, FE)$,
$edge(BF, FB)$,
$edge(FG, GF)$,
$edge(DG, GD)$.

Constrained Optimization Problems

- **Given**:

  — a CSP

  $$\mathcal{P} := \langle \mathcal{C} \ ; \ x_1 \in D_1, \ldots, x_n \in D_n \rangle,$$

  — a function

  $$obj : Sol \rightarrow \mathcal{R}$$

- $(\mathcal{P}, obj)$ a **constrained optimization problem** (**COP**).

- **Task**: Find a solution d to $\mathcal{P}$ for which the value $obj(d)$ is optimal (below: maximal).

Example: Knapsack Problem

Given: a knapsack of a fixed **volume** and n objects, each with a **volume** and a **value**. Find a collection of these objects with **maximal total value** that fits in the knapsack.

Representation as a COP:

Given: knapsack **volume** v and n objects with **volumes** $a_1, ..., a_n$ and **values** $b_1, ..., b_n$.

Variables: $x_1, \ldots, x_n$,
Domains: $\{0, 1\}$,
Constraint:

$$\sum_{i=1}^{n} a_i \cdot x_i \leq v,$$

Objective function:

$$\sum_{i=1}^{n} b_i \cdot x_i.$$

Example: Golomb Ruler

- **Golomb ruler with m marks**: an ordered sequence of m natural numbers such that the distance between any two elements in this sequence is **unique**.
- The largest element of a Golomb ruler is its **length**.
- An **optimum Golomb ruler with m marks**: a Golomb ruler with m marks with a **minimal** length.

Optimum Golomb Ruler with 5 marks

$$0, 1, 4, 9, 11$$

is a Golomb ruler with 5 marks. Indeed, the distances are:

- for elements one apart: 1, 3, 5, 2,
- for elements two apart: 4, 8, 7,
- for elements three apart: 9, 10,
- for elements four apart: 11.

0,1,4,9,11 is an optimum Golomb ruler with 5 marks.

Largest known optimum Golomb ruler has 21 marks and is of length 333.

Representations as a COP

Fix m.

- **Pair**: two numbers i, j such that $1 \leq i < j \leq m$.
- Pairs i, j and k, l are
  – **different** if $i \neq k$ or $j \neq l$,
  – **disjoint** if $i \neq k$ and $j \neq l$.
- **Example**:
  1,3 and 1,4 are different but not disjoint.
  1,3 and 2,4 are disjoint (and so different).

Representation 1

Variables: $x_1, \ldots, x_m$,
Domains: $\mathcal{N}$,
Constraints:

- $x_i < x_{i+1}$ for $i \in [1..m-1]$,
- $x_j - x_i \neq x_l - x_k$ for all different pairs i, j and k, l.

Objective function: $-x_n$.

Representations as a COP, ctd

Representation 2

Constraints:

- $x_i < x_{i+1}$ for $i \in [1..m-1]$,
- $x_j - x_i \neq x_l - x_k$ for all disjoint pairs i, j and k, l.

Representation 3

Variables: $x_1, \ldots, x_m$, $z_{i,j}$ for each pair i, j,
Domains:
$\mathcal{N}$ for $x_1, \ldots, x_m$,
$\mathcal{N} \setminus \{0\}$ for $z_{i,j}$,
Constraints:

- $z_{i,j} = x_j - x_i$ for each pair i, j,
- $z_{i,j} \neq z_{k,l}$ for all different pairs i, j and k, l.

We can replace here "different" by "disjoint".

Representation 4

Replace the disequality constraints by a single `all_different` constraint on the variables $z_{i,j}$.

Different Representations as CSP

Less Contrived Examples

- *A Microcode Label Assignment Problem*
  - CSP representation: 187 finite integer domain variables,
  - IP representation: 2024 Boolean variables,
- *A Packing Problem*
  - CSP representation: 7 finite integer domain variables, 2 constraints,
  - IP representation: 42 Boolean variables, 18 constraints,
- *A Golf Scheduling Problem*
  - CP representation: 176 variables,
  - IP representation 1: 2574 variables,
  - IP representation 2: 592 variables.

Objectives

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- Show the generality of the notion of a CSP.