Principles of Constraint Programming

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Chapter 5
Local Consistency Notions

Objectives

- Introduce several local consistency notions:
  - node consistency,
  - arc consistency,
  - hyper-arc consistency,
  - directional arc consistency,
  - path consistency,
  - directional path consistency,
  - k-consistency,
  - strong k-consistency,
  - relational consistency.
- Use the proof theoretic framework to characterize these local consistency notions.

Node Consistency

- CSP is **node consistent** if for every variable x every unary constraint on x coincides with the domain of x.
- **Examples**

  Assume $\mathcal{C}$ contains no unary constraints.

  $\mathcal{N}$ – natural numbers,

  $\mathcal{Z}$ – integers.

  – $\langle \mathcal{C}, x_1 \geq 0, \ldots, x_n \geq 0 \; ; \; x_1 \in \mathcal{N}, \ldots, x_n \in \mathcal{N} \rangle$
  is node consistent.

  – $\langle \mathcal{C}, x_1 \geq 0, \ldots, x_n \geq 0 \; ;$
  $x_1 \in \mathcal{N}, \ldots, x_{n-1} \in \mathcal{N}, x_n \in \mathcal{Z} \rangle$
  is not node consistent.

Arc Consistency

- A constraint C on the variables x, y with the domains X and Y (so $C \subseteq X \times Y$) is **arc consistent** if
  - $\forall a \in X \exists b \in Y \ (a, b) \in C$,
  - $\forall b \in Y \exists a \in X \ (a, b) \in C$.
- A CSP is **arc consistent** if all its binary constraints are.
- **Examples**
  - $\langle x < y \ ; \ x \in [2..6], y \in [3..7] \rangle$
    is arc consistent.
  - $\langle x < y \ ; \ x \in [2..7], y \in [3..7] \rangle$
    is not arc consistent.

Status of Arc Consistency

- Arc consistency does not imply consistency.

  Example Take

  $$\langle x = y, x \neq y \ ; \ x \in \{a, b\}, y \in \{a, b\}\rangle.$$

- Consistency does not imply arc consistency.

  Example Take

  $$\langle x = y \ ; \ x \in \{a, b\}, y \in \{a\}\rangle.$$

- For some CSP's arc consistency does imply consistency.

  (A general result later.)

Proof Rules for Arc Consistency

ARC CONSISTENCY 1

$$\frac{\langle C \; ; \; x \in D_x, y \in D_y \rangle}{\langle C \; ; \; x \in D'_x, y \in D_y \rangle}$$

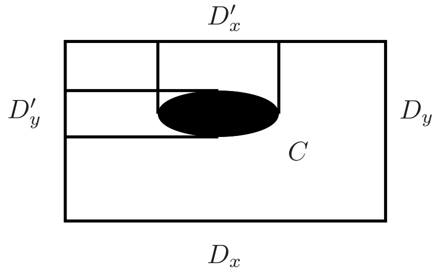
where $D'_x := \{a \in D_x \mid \exists\, b \in D_y \; (a, b) \in C\}$

ARC CONSISTENCY 2

$$\frac{\langle C \; ; \; x \in D_x, y \in D_y \rangle}{\langle C \; ; \; x \in D_x, y \in D'_y \rangle}$$

where $D'_y := \{b \in D_y \mid \exists\, a \in D_x \; (a, b) \in C\}$.

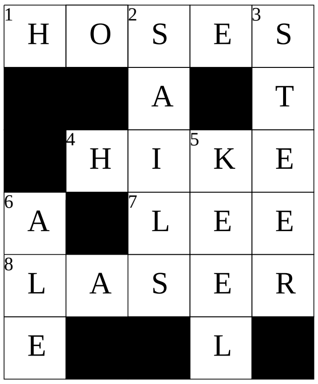
Intuition



Characterization of Arc Consistency

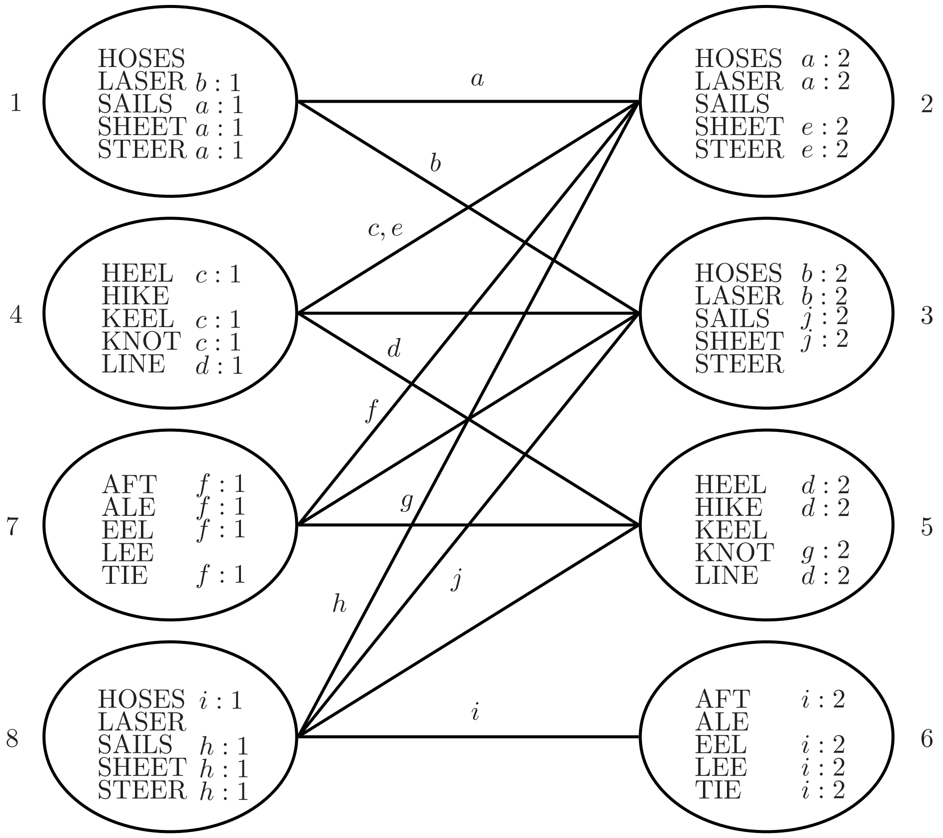
Note A CSP is arc consistent iff it is closed under the applications of the *ARC CONSISTENCY* rules 1 and 2.

Derivation: Example



$$a : C_{1,2},\ b : C_{1,3},\ c : C_{4,2},\ d : C_{4,5},\ e : C_{4,2},$$

$$f : C_{7,2},\ g : C_{7,5},\ h : C_{8,2},\ i : C_{8,6},\ j : C_{8,3}.$$



Hyper-arc Consistency

- A constraint C on the variables $x_1, \ldots, x_n$ with the domains $D_1, \ldots, D_n$ is **hyper-arc consistent** if

$$\forall i \in [1..n] \forall a \in D_i \; \exists d \in C \; a = d[x_i].$$

- CSP is **hyper-arc consistent** if all its constraints are.
- **Examples**
  - $\langle x \wedge y = z \; ; \; x = 1, y \in \{0,1\}, z \in \{0,1\} \rangle$ is hyper-arc consistent.
  - $\langle x \wedge y = z \; ; \; x \in \{0,1\}, y \in \{0,1\}, z = 1 \rangle$ is not hyper-arc consistent.

Characterization of Hyper-arc Consistency

$$\mathit{HYPER\text{-}ARC\ CONSISTENCY}$$

$$\frac{\langle C\ ;\ x_1 \in D_1, \ldots, x_n \in D_n\rangle}{\langle C\ ;\ \ldots, x_i \in D'_i, \ldots\rangle}$$

C a constraint on the variables $x_1, \ldots, x_n$,
$i \in [1..n]$,

$$D'_i := \{a \in D_i \mid \exists d \in C\ a = d[x_i]\}.$$

Note A CSP is hyper-arc consistent iff it is closed under the applications of the *HYPER-ARC CONSISTENCY* rule.

Directional Arc Consistency

Assume a linear ordering $\prec$ on the variables.

- A constraint C on x, y with the domains D_x and D_y is **directionally arc consistent w.r.t.** $\prec$ if
  - $\forall a \in D_x \exists b \in D_y \ (a, b) \in C$ provided $x \prec y$,
  - $\forall b \in D_y \exists a \in D_x \ (a, b) \in C$ provided $y \prec x$.
- A CSP is **directionally arc consistent w.r.t.** $\prec$ if all its binary constraints are.

Example

$$\langle x < y \ ; \ x \in [2..7], y \in [3..7] \rangle$$

is

- not arc consistent,
- directionally arc consistent w.r.t. $y \prec x$.
- not directionally arc consistent w.r.t. $x \prec y$.

Characterization of Directional Arc Consistency

Define $\mathcal{P}_{\prec}$:

$\mathcal{P}$ with the variables reordered w.r.t. $\prec$.

Example

Take $\mathcal{P}$:=

$\langle x < y, y \neq z \,;\, x \in [2..10], y \in [3..7], z \in [3..6]\rangle$

and

$$y \prec x \prec z.$$

Then $\mathcal{P}_{\prec}$:=

$\langle y > x, y \neq z \,;\, y \in [3..7], x \in [2..10], z \in [3..6]\rangle$.

Note A CSP $\mathcal{P}$ is directionally arc consistent w.r.t. $\prec$ iff the CSP $\mathcal{P}_{\prec}$ is closed under the applications of the *ARC CONSISTENCY* rule 1.

Limitations of Arc Consistency

Note

$$\langle x < y, y < z, z < x \; ; \; x, y, z \in \{1..100000\}\rangle.$$

is inconsistent.

Proof using arc consistency rules.

Applying *ARC CONSISTENCY* rule *1* we get

$$\langle x < y, y < z, z < x \, ; \, x \in \{1..99999\}, y, z \in \{1..100000\}\rangle,$$

etc.

Disadvantages:

- Large number of steps.
- Length depends on the size of the domains.

Direct proof: use transitivity of $<$.

Path consistency: a generalizes this form of reasoning to arbitrary binary relations.

Normalized CSP's

A CSP $\mathcal{P}$ is **normalized** if for each pair x, y of its variables at most one constraint on x, y exists.

Denote by $C_{x,y}$ the unique constraint on x, y if it exists and otherwise the "universal" relation on x, y.

R and S: two binary relations.

- **transposition** of R:

$$R^T := \{(b, a) \mid (a, b) \in R\},$$

- **composition** of R and S by

$$R \cdot S := \{(a, b) \mid \exists c\, ((a, c) \in R, (c, b) \in S)\}.$$

Path Consistency

A normalized CSP is **path consistent** if for each subset $\{x, y, z\}$ of its variables

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z}.$$

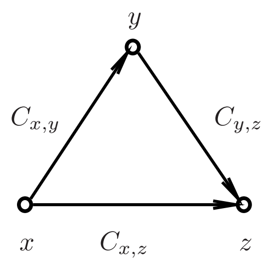
Note A normalized CSP is path consistent iff for each subsequence x, y, z of its variables

$$C_{x,y} \subseteq C_{x,z} \cdot C_{y,z}^T,$$

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z},$$

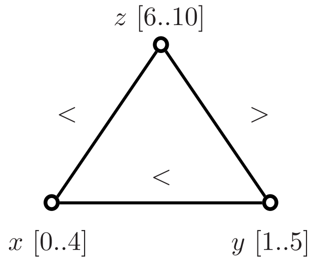
$$C_{y,z} \subseteq C_{x,y}^T \cdot C_{x,z}.$$

Intuition



Path Consistency: Example 1

$$\langle x < y, y < z, x < z;$$
$$x \in [0..4], y \in [1..5], z \in [6..10]\rangle$$



is path consistent. Indeed

$$C_{x,y} = \{(a, b) \mid a < b, a \in [0..4], b \in [1..5]\},$$

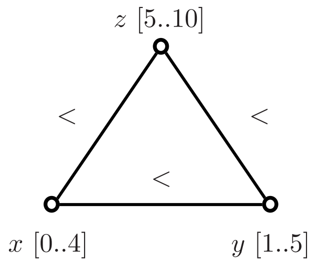
$$C_{x,z} = \{(a, c) \mid a < c, a \in [0..4], c \in [6..10]\},$$

$$C_{y,z} = \{(b, c) \mid b < c, b \in [1..5], c \in [6..10]\},$$

and all 3 conditions are satisfied.

Path Consistency: Example 2

$$\langle x < y, y < z, x < z;$$
$$x \in [0..4], y \in [1..5], z \in [5..10]\rangle$$



is not path consistent. Indeed, now

$$C_{x,z} = \{(a, c) \mid a < c, a \in [0..4], c \in [5..10]\}$$

and for $4 \in [0..4]$ and $5 \in [5..10]$ no $b \in [1..5]$ exists such that $4 < b$ and $b < 5$.

Characterization of Path Consistency

PATH CONSISTENCY 1

$$\frac{C_{x,y}, C_{x,z}, C_{y,z}}{C'_{x,y}, C_{x,z}, C_{y,z}}$$

where $C'_{x,y} := C_{x,y} \cap C_{x,z} \cdot C^T_{y,z}$,

PATH CONSISTENCY 2

$$\frac{C_{x,y}, C_{x,z}, C_{y,z}}{C_{x,y}, C'_{x,z}, C_{y,z}}$$

where $C'_{x,z} := C_{x,z} \cap C_{x,y} \cdot C_{y,z}$,

PATH CONSISTENCY 3

$$\frac{C_{x,y}, C_{x,z}, C_{y,z}}{C_{x,y}, C_{x,z}, C'_{y,z}}$$

where $C'_{y,z} := C_{y,z} \cap C^T_{x,y} \cdot C_{x,z}$.

Note A normalized CSP is path consistent iff it is closed under the applications of the *PATH CONSISTENCY* rules 1, 2 and 3.

m-Path Consistency

A normalized CSP is **m-path consistent** ($m \geq 2$) if for each subset $\{x_1, \ldots, x_{m+1}\}$ of its variables

$$C_{x_1,x_{m+1}} \subseteq C_{x_1,x_2} \cdot C_{x_2,x_3} \cdot \ldots \cdot C_{x_m,x_{m+1}}.$$

Note A normalized CSP is m-path consistent if for each subset $\{x_1, \ldots, x_{m+1}\}$ of its variables

> if $(a_1, a_{m+1}) \in C_{x_1,x_{m+1}}$, then for some $a_2, \ldots, a_m$ for all $i \in [1..m]$
> $(a_i, a_{i+1}) \in C_{x_i,x_{i+1}}$.

$a_2, \ldots, a_m$: **path** connecting a_1 and a_{m+1}.

Theorem Every normalized path consistent CSP is m-path consistent for each $m \geq 2$.

Proof. Induction on m.

Directional Path Consistency

Assume a linear ordering $\prec$ on the variables. A normalized CSP is **directionally path consistent w.r.t. $\prec$** if for each subset $\{x, y, z\}$ of its variables

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z} \text{ provided } x, z \prec y.$$

Note A normalized CSP is directionally path consistent w.r.t. $\prec$ iff for each subsequence x, y, z of its variables

$$C_{x,y} \subseteq C_{x,z} \cdot C^T_{y,z} \text{ provided } x, y \prec z,$$

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z} \text{ provided } x, z \prec y,$$

$$C_{y,z} \subseteq C^T_{x,y} \cdot C_{x,z} \text{ provided } y, z \prec x.$$

Examples

Reconsider
$\langle x < y, y < z, x < z;$
$x \in [0..4], y \in [1..5], z \in [5..10]\rangle$

Then

$$C_{x,y} = \{(a,b) \mid a < b, a \in [0..4], b \in [1..5]\},$$

$$C_{x,z} = \{(a,c) \mid a < c, a \in [0..4], c \in [5..10]\},$$

$$C_{y,z} = \{(b,c) \mid b < c, b \in [1..5], c \in [5..10]\}.$$

- It is directionally path consistent w.r.t. the ordering $\prec$ in which $x, y \prec z$.
  Indeed, for every pair $(a,b) \in C_{x,y}$ there exists $c \in [5..10]$ such that $a < c$ and $b < c$.
- It is directionally path consistent w.r.t. the ordering $\prec$ in which $y, z \prec x$.
  Indeed, for every pair $(b,c) \in C_{y,z}$ there exists $a \in [0..4]$ such that $a < b$ and $a < c$.

Characterization of Directional Path Consistency

Note A normalized CSP $\mathcal{P}$ is directionally path consistent w.r.t. $\prec$ iff $\mathcal{P}_{\prec}$ is closed under the applications of the *PATH CONSISTENCY* rule 1.

Instantiations

Fix a CSP $\mathcal{P}$.

- **Instantiation**: function on a subset of the variables of $\mathcal{P}$. It assigns to each variable a value from its domain.

  Notation:

  $$\{(x_1, d_1), \ldots, (x_k, d_k)\}.$$

- C: a constraint on $x_1, \ldots, x_k$.
  Instantiation $\{(x_1, d_1), \ldots, (x_k, d_k)\}$ **satisfies** C if $(d_1, \ldots, d_k) \in C$.

- I: instantiation with a domain X, $Y \subseteq X$.
  $I \mid Y$: **restriction of I to Y**.

- Instantiation I with domain X is **consistent** if for every constraint C of $\mathcal{P}$ on some Y with $Y \subseteq X$ $\ I \mid Y$ satisfies C.

- Consistent instantiation is **k-consistent** if its domain consists of k variables.

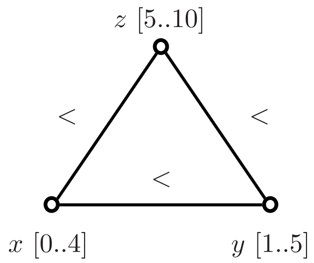
- An instantiation is a **solution** to $\mathcal{P}$ if it is consistent and defined on all variables of $\mathcal{P}$.

Example

Consider

$\langle x < y, y < z, x < z \,;\, x \in [0..4], y \in [1..5], z \in [5..10]\rangle$.

Let $I := \{(x, 0), (y, 5), (z, 6)\}$.



- $I \mid \{x, y\} = \{(x, 0), (y, 5)\}$.
  It satisfies $x < y$.
- $I \mid \{x, z\} = \{(x, 0), (z, 6)\}$.
  It satisfies $x < z$.
- $I \mid \{y, z\} = \{(y, 5), (z, 6)\}$.
  It satisfies $y < z$.
- So I is a 3-consistent instantiation. It is a solution to this CSP.

k-Consistency

- CSP is **1-consistent** if for every variable x with a domain D each unary constraint on x equals D.
- CSP is **k-consistent**, $k > 1$, if every $(k-1)$-consistent instantiation can be extended to a k-consistent instantiation **no matter** which new variable is chosen.

k-consistency aka **node consistency**

Note

- A node consistent CSP is arc consistent iff it is 2-consistent.
- A node consistent normalized binary CSP is path consistent iff it is 3-consistent.

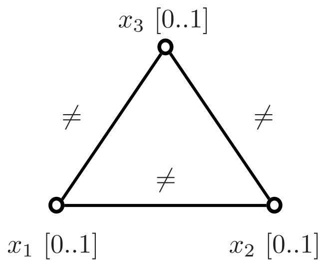
k-Consistency, ctd

Fix $k > 1$.

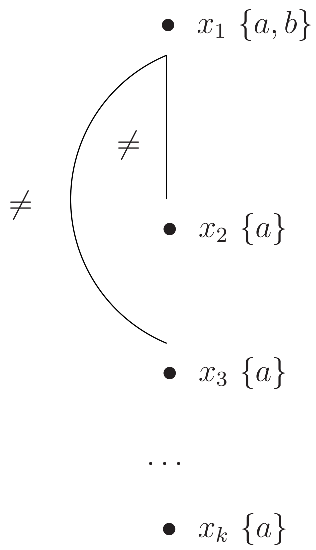
(i) There exists a CSP that is $(k-1)$-consistent but not k-consistent.

(ii) There exists a CSP that is not $(k-1)$-consistent but is k-consistent.

Proof of (i) for $k = 3$:



Proof of (ii):



Strong k-Consistency

CSP **strongly k-consistent**, $k \geq 1$, if it is i-consistent for every $i \in [1..k]$.

Theorem Take a CSP with k variables, $k \geq 1$, such that

- at least one domain is non-empty,
- it is strongly k-consistent.

Then it is consistent.

Proof. Construct a solution by induction.

Prove that

(i) there exists a 1-consistent instantiation,

(ii) for every $i \in [2..k]$ each $(i-1)$-consistent instantiation can be extended to an i-consistent instantiation.

Disadvantage Required level of strong consistency = # of variables.

Graphs and CSP's

Graph is **associated with a CSP $\mathcal{P}$**.

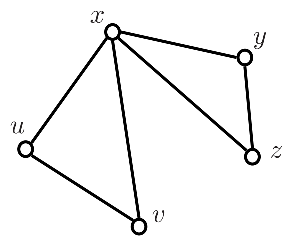
Nodes: variables of $\mathcal{P}$.

Arcs: connect two variables if they appear jointly in some constraint.

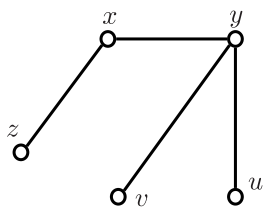
Examples

- $SEND + MORE = MONEY$ puzzle.
  The graph has 8 nodes, S, E, N, D, M, O, R, Y, and is complete.

- $\langle x + y = z, x + u = v \; ; \; \mathcal{DE} \rangle$



- $\langle x < z, x < y, y < u, y < v \; ; \; \mathcal{DE} \rangle$



Width of a Graph

G: a finite graph G.

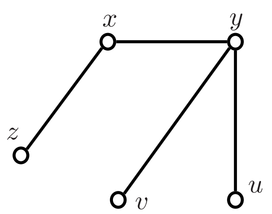
$\prec$: linear ordering on the nodes of G.

- **$\prec$-width** of a node of G: number of arcs in G that connect it to $\prec$-smaller nodes.
- **$\prec$-width** of G: maximum of the $\prec$-widths of its nodes.
- The **width** of G: minimum of $\prec$-widths for all linear orderings $\prec$.

Examples

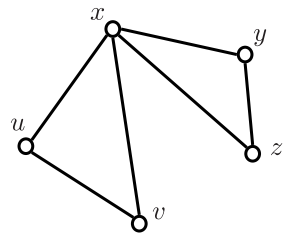
- $SEND + MORE = MONEY$ puzzle.

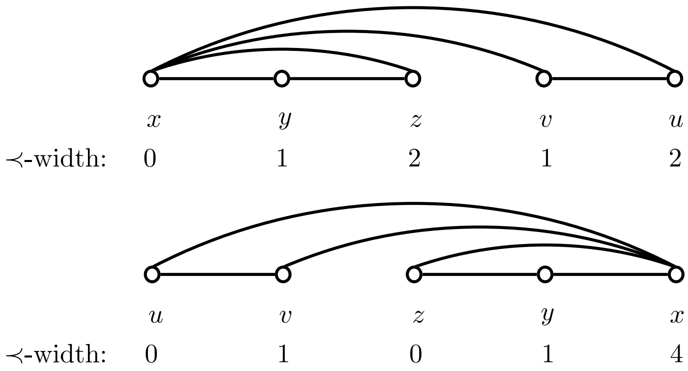
  Complete graph with 8 nodes, so its width $= 7$.



- It is a tree, so its width $= 1$.

Examples, ctd





Two examples of the $\prec$-widths of the nodes

Here width $= 2$.

Consistency via Strong k-Consistency

Theorem Given: a CSP such that

- all domains are non-empty,
- it is strongly k-consistent,
- the graph associated with it has width $k-1$.

Then this CSP is consistent.

Proof. (Sketch)

Assume n variables.

- Reorder the variables so that the resulting $\prec$-width is $k-1$.
- Prove by induction that
  - there exists consistent instantiation with domain $\{x_1\}$,
  - for every $j \in [1..n-1]$ each consistent instantiation with domain $\{x_1, \ldots, x_j\}$ can be extended to a consistent instantiation with domain $\{x_1, \ldots, x_{j+1}\}$.

Useful Corollaries

Corollary 1

Given: $\mathcal{P}$ and a linear ordering $\prec$ such that

- all domains are non-empty,
- $\mathcal{P}$ is
  - node consistent,
  - directionally arc consistent w.r.t. $\prec$,
- the $\prec$-width of the graph associated with $\mathcal{P}$ is 1.

Then $\mathcal{P}$ is consistent.

Corollary 2

Given: $\mathcal{P}$ and a linear ordering $\prec$ such that

- all domains are non-empty,
- $\mathcal{P}$ is
  - directionally arc consistent w.r.t. $\prec$,
  - directionally path consistent w.r.t. $\prec$,
- the $\prec$-width of the graph associated with it is 2.

Then $\mathcal{P}$ is consistent.

Relational Consistency

"Ultimate" notion of local consistency

- **Given**: $\mathcal{P}$ and a subsequence $\mathcal{C}$ of its constraints.

  $\mathcal{P} \mid \mathcal{C}$:

  – remove from $\mathcal{P}$ all constraints not in $\mathcal{C}$,

  – delete all domain expressions involving variables not present in any constraint in $\mathcal{C}$.

- $\mathcal{P}$ is **relationally (i, m)-consistent** if for every sequence $\mathcal{C}$ of m constraints and $X \subseteq Var(\mathcal{C})$ of size i:

  every consistent instantiation with the domain X can be extended to a solution to $\mathcal{P} \mid \mathcal{C}$.

Relational Consistency, ctd

Intuition:
For every sequence of m constraints and for every set X of i variables, each present in one of these m constraints:

> each consistent instantiation with the domain X can be extended to a solution to all these m constraints.

Some properties

- A node consistent binary CSP is arc consistent iff it is relationally $(1, 1)$-consistent.
- A node consistent CSP is hyper-arc consistent iff it is relationally $(1, 1)$-consistent.
- Every node consistent normalized relationally $(2, 3)$-consistent CSP is path consistent.
- Every strictly binary relationally $(k-1, k)$-consistent CSP is k-consistent.
- A CSP with m constraints is consistent iff it is relationally $(0, m)$-consistent.

Some Notation

- **Given**: constraint C on variables X, subsequence Y of X.

  $$\Pi_Y(C) := \{d[Y] \mid d \in C\}.$$

- X: sequence of variables,

  $X_1, \ldots, X_n$: subsequences of X.

  union of $X_1, \ldots, X_n$: shortest subsequence of X containing each X_i as a subsequence.

  Example: Take x_1, x_2, x_4, x_5.

  Union of $(x_2, x_4), (x_4, x_5), (x_2, x_5)$ is x_2, x_4, x_5.

- **Given**: a sequence of constraints $C_1, \ldots, C_m$ on variables $X_1, \ldots, X_m$.

  $$C_1 \bowtie \ldots \bowtie C_m := \{d \mid d[X_i] \in C_i \text{ for } i \in [1..m]\}.$$

  $C_1 \bowtie \ldots \bowtie C_m$ is a constraint on the "union" of $X_1, \ldots, X_m$.

- X: a sequence of variables

  $$\overline{C_X} := \bowtie \{C_Y \mid Y \text{ is a subsequence of } X\}.$$

Characterization of k-Consistency

Note d is a solution to $\langle C_1, \ldots, C_m \,;\, \mathcal{DE}\rangle$ iff $d \in C_1 \bowtie \ldots \bowtie C_m$.

A CSP $\mathcal{P}$ **regular** if for each sequence X of its variables a unique constraint on X exists. Denote it by C_X.

$$k\text{-}CONSISTENCY$$

$$\frac{C_X}{C_X \cap \Pi_X(\overline{C_{X,y}})}$$

Note If a regular CSP is closed under the applications of the k-*CONSISTENCY* rule for all subsequences X of $k-1$ variables and all variables y not in X, then it is k-consistent.

Characterization of Relational Consistency

RELATIONAL (i,m)-*CONSISTENCY*

$$\frac{C_X}{C_X \cap \Pi_X(C_1 \bowtie \ldots \bowtie C_m)}$$

Note If a regular CSP is closed under the applications of

RELATIONAL (i,m)-*CONSISTENCY*

rule for each subsequence of constraints $C_1, \ldots, C_m$ and each subsequence X of $Var(C_1, \ldots, C_m)$ of length i, then it is relationally (i,m)-consistent.

Objectives

- Introduce several local consistency notions:
  - node consistency,
  - arc consistency,
  - hyper-arc consistency,
  - directional arc consistency,
  - path consistency,
  - directional path consistency,
  - k-consistency,
  - strong k-consistency,
  - relational consistency.
- Use the proof theoretic framework to characterize these local consistency notions.