The Many Faces of Rationalizability

Krzysztof R. Apt[1,2,3]

[1] CWI, Kruislaan 413, 1098 SJ Amsterdam, the Netherlands, apt@cwi.nl
[2] University of Amsterdam
[3] School of Computing, National University of Singapore

Abstract. The rationalizability concept was introduced in [4] and [12] to assess what can be inferred by rational players in a non-cooperative game in the presence of common knowledge. However, this notion can be defined in a number of ways that differ in seemingly unimportant minor details. We shed light on these differences, explain their impact, and clarify for which games these definitions coincide. Also we apply the same analysis to explain the differences and similarities between various ways the iterated elimination of strictly dominated strategies was defined in the literature. This allows us to clarify the results of [6] and [5] and improve upon them.
Our approach is based on a general study of the operators on complete lattices. We allow transfinite iterations of the considered operators and clarify the need for them. The advantage of such a general approach is that a number of results, including order independence for some of the notions of rationalizability and strict dominance, come for free.

JEL classification: C72

Keywords: rationalizability, iterated elimination of strategies, strict dominance, monotonicity

1 Introduction

Rationalizability was introduced in [4] and [12] to formalize the intuition that players in non-cooperatives games act by having common knowledge of each others' rational behaviour. Rationalizable strategies are then defined as a limit of an iterative process in which one repeatedly removes the strategies that are never best responses to the beliefs held about the other players.

However, this iterative process can be defined in a number of natural ways that differ in some minor details. In particular, two equivalent definitions were used in the original papers (if we abstract from the way the sets of beliefs were defined) and two others can be naturally envisaged. Each of these four definitions captures the original intuition in a meaningful way.

To study the differences and similarities between these definitions in a systematic way we consider first arbitary operators on complete lattices and study their transfinite iterations. This allows us to prove various results concerning the rationalizability by simply checking the properties of underlying operators. For example, order independence for specific definitions turns out to be a direct consequence of the fact that the underlying operator is monotonic.

Our analysis allows us to conclude that in general among four ways of defining rationalizability only the original two, due [4] and [12], are meaningful. The crucial feature of these two operators is that they refer to the best responses in the initial game and *not* the currently considered game. As a result they are monotonic. However, if we assume that to each belief in the initial game a best response exists, the iterations of all four operators coincide even though the other two are not monotonic. We also explain the need for transfinite iterations of the corresponding operators, even for the games that satisfy this assumption.

We also apply the same analysis to the notion of strict dominance. We explain that the iterated elimination of strictly dominated strategies can be defined in four natural ways. In contrast to the notion of rationalizability, the customary definition involves iterations of an operator that refers to the dominance in the currently considered game and *not* the initial game. As a result the underlying operator is not monotonic which clarifies why, as noted in [6], this definition of strict dominance is not order independent for arbitrary infinite games.

A monotonic version of the elimination of strictly dominated strategies was used, though not studied in [9], and a minor variation of it studied recently in [5]. The authors of the latter paper show that the resulting notion is stronger than the customary one, requires transfinite iterations, preserves the Nash equilibria, and is order independent. In our framework order independence of both definitions is a direct consequence of the monotonicity of the underlying operators.

We also consider a natural condition $\mathbf{D}(\alpha)$ of the initial game, parametrized by an ordinal α, that formalizes the statement that in each reduction reachable from the initial game in α iterations every strictly dominated strategy has an undominated dominator. Our formalization differs from the one used in [6] for which the authors showed that order independence in their sense does not need to hold. (Our definitions of order independence differ, since we allow transfinite iterations of the underlying operators.)

We show then that $\mathbf{D}(\omega)$ implies order independence in the sense of [6] of the usual definition of strict dominance and $\forall\alpha\mathbf{D}(\alpha)$ implies order independence of this definition in the sense used in our paper. Further, $\forall\alpha\mathbf{D}(\alpha)$ implies that the iterations of all four operators coincide.

The remainder of the paper is organized as follows. First we establish in Section 2 some general results about monotonic and contracting operators that provide a basis for our approach. Standard concepts on strategic games and examples of belief structures to which the results of this paper apply are introduced in Section 3. In Section 4 we discuss two operators that underly the original definitions of rationalizability. Both are monotonic, so the revelant properties of these operators and of their outcomes are direct consequences of the general results established in Section 2. Then in Section 5 we discuss two slightly different operators. We conclude that the first one cannot be meaningfully used to formalize the notion of rationalizability and the other one leads to too weak conclusions on the game modelling Bertrand competition for two firms. On the other hand, in Section 6 we show that the best response property in the initial game implies that the iterations of all four operators coincide.

Then in Sections 7 and 8 we discuss in detail four natural ways of defining iterated elimination of strictly dominated strategies. We focus our attention on the operator customarily used in the literature and establish for it some new order independence results. Finally, in Section 9 we assess the results of this paper stressing that monotonicity and transfinite iterations are most relevant for a study of reduction operators on games.

In the literature we found only one reference concerned with a similar comparative analysis of the notion of rationalizability. [1] studied the limited case of two-person games and beliefs consisting of the strategies of the opponent, and showed that the finite iterations of two operators concerned with the notion of rationalizablity coincide for compact games with continuous payoffs. We clarify his result in Section 6.

Parts of this research were reported in [3].

2 Mathematical preliminaries

We are interested in iterated reductions of strategic games entailed by various ways of removing strategies. To deal with them in a uniform way we define the relevant concepts for arbitrary operators on a fixed complete lattice $(D, \subseteq)$ with the largest element $\top$.

Anticipating the applications to games we denote the elements of D by G and H, with possible subscripts. In what follows we use ordinals and denote them by α, β, γ. Given a, possibly transfinite, sequence $(G_\alpha)_{\alpha<\gamma}$ of elements of D we denote their join and meet respectively by $\bigcup_{\alpha<\gamma} G_\alpha$ and $\bigcap_{\alpha<\gamma} G_\alpha$.

In the subsequent applications D will be the set of all restrictions of a fixed strategic game H and $\subseteq, \bigcup_{\alpha<\gamma}$ and $\bigcap_{\alpha<\gamma}$ natural set-theoretic relations or operations on them.

We now establish some general results first. The proofs of these results are straightforward and they readily apply to specific operators.

Definition 1. *Let T be an operator on $(D, \subseteq)$.*

- *We call T* **monotonic** *if for all G_1, G_2*

  $G_1 \subseteq G_2$ *implies* $T(G_1) \subseteq T(G_2)$.

- *We call T* **contracting** *if for all G*

  $T(G) \subseteq G$.

- *We say that an element G is a* **fixpoint** *of T if $T(G) = G$.*
- *We define by transfinite induction a sequence of elements T^α of D, where α is an ordinal, as follows:*

- $T^0 := \top$,
- $T^{\alpha+1} := T(T^\alpha)$,
- *for all limit ordinals* β, $T^\beta := \bigcap_{\alpha<\beta} T^\alpha$.

– *We call the least* α *such that* $T^{\alpha+1} = T(T^\alpha)$ *the* **closure ordinal** *of* T *and denote it by* α_T. *We call then* T^{α_T} *the* **outcome of** *(iterating)* T. □

In general the closure ordinal of T, and hence its outcome, do not need to exist. (Take for example an operator oscillating between two values.) However, they clearly exist when T is contracting since then for all α we have $T^{\alpha+1} \subseteq T^\alpha$.

Further, we have the following classic result due to [13], specialized to the case of restrictions.[1]

Theorem 1 (Fixpoint). *Every monotonic operator on* $(D, \subseteq)$ *has a largest fixpoint. This fixpoint is the outcome of* T, *i.e., it is of the form* T^{α_T} .

The weakest natural condition that implies existence of the outcome of an operator T is that that for all α we have $T^{\alpha+1} \subseteq T^\alpha$. This condition is clearly satisfied by both contracting and monotonic operators. We use it now in an observation that links an operator T with its contracting version $\overline{T}$ defined by:

$$\overline{T}(G) := T(G) \cap G.$$

Note 1. Consider an operator T on $(D, \subseteq)$ such that for all α we have $T^{\alpha+1} \subseteq T^\alpha$.

Then for all α it holds that $\overline{T}^\alpha = T^\alpha$. In particular, $\overline{T}^{\alpha_{\overline{T}}}$ exists and $\overline{T}^{\alpha_{\overline{T}}} = T^{\alpha_T}$. Further, if T is monotonic, then so is $\overline{T}$.

Proof. Straightforward by the transfinite induction. □

So if we are only interested in the operator iterations or its outcome, it does not matter whether we choose an operator T that is contracting or monotonic, or its contracting version $\overline{T}$.

Next, we introduce two concepts that are of importance when studying operators on the restrictions of a game.

Definition 2. *Consider an operator* T *on* $(D, \subseteq)$.

– *We say that the operator* R *is a* **relaxation** *of* T *if for all restrictions* G *of* H
  - $T(G) \subseteq R(G)$,
  - *if* $T(G) \subseteq G$, *then* $R(G) \subseteq G$,
  - *if* G *is a fixpoint of* R, *then it is a fixpoint of* T.

– *We say that* T *is* **order independent** *if the set*

$$\{G \mid G \text{ is an outcome of a relaxation of } T\}$$

has at most one element. □

[1] We use here its 'dual' version in which the iterations start at the largest and not at the least element of a complete lattice.

The intuition behind the conditions defining a relaxation is as follows. Suppose that G and $T(G)$ are games (more precisely restrictions, to be defined in the next section). Then the first condition states that R 'approximates' T from above. In turn, the second condition states that if T removes some strategies from G, then so does R. So if T is contracting, then R is contracting, as well, and their closure ordinals and the outcomes exist. Also, for a contracting operator T the first two conditions can be replaced by

– $T(G) \subseteq R(G) \subseteq G$.

Finally, the third condition states that if some strategy can be removed from a game G by the operator T (G is not a fixpoint of T), then the R operator will be able to do so, as well (G is not a fixpoint of R). Note that the reverse implication to the third condition is a consequence of the first two conditions.

Also, note that our definition of order independence of an operator T does not even require that T has an outcome. If it does, then each relaxation of it has the same outcome as T, if it has one.

We have then the following general result.

Theorem 2 (Order Independence). *Every monotonic operator on* $(D, \subseteq)$ *is order independent.*

Proof. Let T be a monotonic operator and R a relaxation of it. First we establish a simple claim concerning the fixpoints of T.

Claim. Suppose

– G is a fixpoint of T,
– $G \subseteq G_1$.

Then $G \subseteq R(G_1)$.

Proof. By the monotonicity of T we have $G \subseteq T(G_1)$. But by the definition of a relaxation $T(G_1) \subseteq R(G_1)$. □

Consider now a relaxation R that has an outcome and take the sequence of elements $(R^{\alpha})_{\alpha \leq \alpha_R}$ of D. Let G be a fixpoint of T. By Claim 2 using the transfinite induction we conclude that $G \subseteq R^{\alpha_R}$. In particular, $T^{\alpha_T} \subseteq R^{\alpha_R}$, since by the Fixpoint Theorem 1 T^{α_T}, the outcome of T exists and is the largest fixpoint of T. But by the definition of a relaxation R^{α_T} is a fixpoint of T, so also $R^{\alpha_R} \subseteq T^{\alpha_T}$. □

Intuitively, this result can be interpreted as follows. Suppose that an operator T removes *all* strategies from a restriction of a game H that meet some property and that we iterate this operator starting with H. Suppose now that at each stage we remove only *some* strategies that meet this property (instead of all). Then, when T is monotonic, we still end up with the same outcome (which is the largest fixpoint of the T operator).

In fact, this result is even more general, since we do not even assume that the operator T is contracting. Also, in the proof we did not use the second condition of the definition of a relaxation. In our approach this condition is needed only to deal with the contracting operators that are not monotonic.

Next, we show that any monotonic operator is 'fast' in the sense that it generates smaller elements than any of its relaxations and reaches their common outcome (guaranteed by the above theorem) earlier. While this is of course as expected, we shall see in Section 5 that this property does not hold for a natural operator that can be used to define rationalizability.

Theorem 3. *Let T be a monotonic operator on $(D, \subseteq)$ and R a relaxation of T. Then*

(i) *For all α we have $T^\alpha \subseteq R^\alpha$.*
(ii) $\alpha_T \leq \alpha_R$.

Proof.
(i) We prove the claim by the transfinite induction. Suppose it holds for some α, i.e., $T^\alpha \subseteq R^\alpha$. Then by the monotonicity of T

$$T(T^\alpha) \subseteq T(R^\alpha),$$

and since R is a relaxation

$$T(R^\alpha) \subseteq R(R^\alpha).$$

So $T^{\alpha+1} \subseteq R^{\alpha+1}$, i.e., the claim holds for $\alpha + 1$.

The induction step for a limit ordinal is immediate.

(ii) Take $\beta < \alpha_T$. Then $T^{\beta+1} \subset T^\beta$, so by (i) $T^{\beta+1} \subset R^\beta$. But by the Order Independence Theorem 2 the outcome of T is the outcome of R, so $R^{\beta+1} \subset R^\beta$, i.e., $\beta < \alpha_R$. This proves the claim. □

3 Game theoretic preliminaries

3.1 Strategic games and their restrictions

Let us move now to the subject of strategic games. Given n players ($n > 1$) we represent a strategic game (in short, a game) by a sequence

$$(S_1, \ldots, S_n, p_1, \ldots, p_n),$$

where for each $i \in [1..n]$

- S_i is the non-empty set of ***strategies*** available to player i,
- p_i is the ***payoff function*** for the player i, so $p_i : S_1 \times \ldots \times S_n \rightarrow \mathcal{R}$, where $\mathcal{R}$ is the set of real numbers.

Given a sequence of sets of strategies $S_1, \ldots, S_n$ and $s \in S_1 \times \ldots \times S_n$ we denote the ith element of s by s_i and use the following standard notation:

- $s_{-i} := (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$,
- $S_{-i} := S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$.

We denote the strategies of player i by s_i, possibly with some superscripts.

We say that $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ is a ***restriction*** of a game $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$ if each S_i is a (possibly empty) subset of T_i and each p_i is identified with its restriction to the smaller domain. When H is fixed we identify G with $(S_1, \ldots, S_n)$.

If some S_i is empty, G is not a game and the references to $p_j(s)$ (for any $j \in [1..n]$) are incorrect so we shall need to be careful about this. If all S_i are empty, we call G an ***empty restriction*** and denote it by $\emptyset_n$. The restrictions of H are naturally ordered by the componentwise set inclusion:

$$(S_1, \ldots, S_n) \subseteq (S'_1, \ldots, S'_n) \text{ iff } S_i \subseteq S'_i \text{ for all } i \in [1..n].$$

The notion of a ***union*** and of an ***intersection*** of a transfinite sequence $(G_\alpha)_{\alpha<\gamma}$ of restrictions of H (α and γ are ordinals) denoted respectively by $\bigcup_{\alpha<\gamma} G_\alpha$ and $\bigcap_{\alpha<\gamma} G_\alpha$ is defined in the expected way.

With the above operations the set of restrictions of H is a complete lattice with H being the largest element.

3.2 Belief structures

Throughout the paper $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$ is a *fixed* game.

Our intention is to explain various concepts of rationalizability abstracting from specific sets of beliefs that are assumed. Therefore we only assume that each player i in the game H has some further unspecified non-empty set of beliefs $\mathcal{B}_i$ about his opponents and that each payoff function p_i can be modified to an ***expected payoff*** function $p_i : S_i \times \mathcal{B}_i \to \mathcal{R}$.

In what follows we also assume that each set of beliefs $\mathcal{B}_i$ of player i in H can be *narrowed* to any restriction G of H. We denote the outcome of this ***narrowing*** of $\mathcal{B}_i$ to G by $\mathcal{B}_i \dot{\cap} G$. The set $\mathcal{B}_i \dot{\cap} G$ can be viewed as the set of beliefs of player i in the restriction G. We call then the pair $(\mathcal{B}, \dot{\cap})$, where $\mathcal{B} := (\mathcal{B}_1, \ldots, \mathcal{B}_n)$, a ***belief structure*** in the game H.

The following natural property of a belief structure $(\mathcal{B}, \dot{\cap})$ in H will be relevant.

A If $G_1 \subseteq G_2 \subseteq H$, then for all $i \in [1..n]$, $\mathcal{B}_i \dot{\cap} G_1 \subseteq \mathcal{B}_i \dot{\cap} G_2$.

This property simply states that for each player the set of his beliefs in a restriction G_1 of G_2 is a subset of the set of his beliefs in G_2.

The following four belief structures were considered in the literature. Given a finite non-empty set A we denote here by ΔA the set of probability distributions over A.

(i) $\mathcal{B}_i := T_{-i}$ for $i \in [1..n]$.
So beliefs are joint pure strategies of the opponents. For a restriction $G := (S_1, \ldots, S_n)$ of H we define

$$\mathcal{B}_i \mathbin{\dot{\cap}} G := S_{-i}.$$

We call then $(\mathcal{B}, \dot{\cap})$ the ***pure*** belief structure in H. This belief structure was considered in [4].

(ii) Assume H is finite. $\mathcal{B}_i := \Pi_{j \neq i} \Delta T_j$ for $i \in [1..n]$.
So beliefs are joint mixed strategies of the opponents. For a restriction $G := (S_1, \ldots, S_n)$ of H we define

$$\mathcal{B}_i \mathbin{\dot{\cap}} G := \Pi_{j \neq i} \Delta S_j.$$

This belief structure was considered in [4].

(iii) Assume H is finite. $\mathcal{B}_i := \Pi_{j \neq i} \Delta T_j$ for $i \in [1..n]$.
So beliefs are joint mixed strategies of the opponents. For a restriction $G := (S_1, \ldots, S_n)$ of H we define

$$\mathcal{B}_i \mathbin{\dot{\cap}} G := \Pi_{j \neq i} \overline{\Delta S_j},$$

where for a set M_j of mixed strategies of player j, $\overline{M_j}$ denotes its convex hull.
This belief structure was considered in [12].

(iv) Assume H is finite. $\mathcal{B}_i := \Delta T_{-i}$ for $i \in [1..n]$.
So beliefs are probability distributions over the set of joint pure strategies of the opponents. For a restriction $G := (S_1, \ldots, S_n)$ of H we define

$$\mathcal{B}_i \mathbin{\dot{\cap}} G := \Delta S_{-i}.$$

This belief structure was mentioned in [4] and considered in [11].

Given two finite sets A and B such that $A \subseteq B$, we identify each probability distribution on A with the probability distribution on B in which 0 is assigned to each element in $B \setminus A$. Then it is straightforward to see that property **A** is satisfied by the belief structures from examples (i)—(iv).

In principle, each $\mathcal{B}_i$ can be an arbitrary set of some higher-order beliefs (beliefs about beliefs etc.) In particular, it can be also a set of tuples of the form $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$, where each a_j is a pure strategy if j is even and a mixed strategy if j is odd.

Finally, given a restriction $G := (S_1, \ldots, S_n)$ of H and a belief $\mu_i \in \mathcal{B}_i \mathbin{\dot{\cap}} G$ we say that a strategy s_i of player i in game H is a ***best response to*** μ_i ***in*** G, and write $s_i \in BR_G(\mu_i)$, if

$$\forall s'_i \in S_i \; p_i(s_i, \mu_i) \geq p_i(s'_i, \mu_i).$$

Note that s_i does not need to be an element of S_i.

4 The $\boldsymbol{SR}$ and $\overline{\boldsymbol{SR}}$ operators

We now apply the above considerations to specific operators. Assume a belief structure $(\mathcal{B}, \dot{\cap})$ in the game H. We introduce the following operator SR (standing for 'stronger notion of rationalizability') on the set of restrictions of H:

$$SR(G) := (S'_1, \ldots, S'_n),$$

where for all $i \in [1..n]$

$$S'_i := \{s_i \in T_i \mid \exists \mu_i \in \mathcal{B}_i \,\dot{\cap}\, G \; s_i \in BR_H(\mu_i)\}.$$

So $SR(G)$ is obtained by removing from H all strategies that are never best responses *in* H (note this reference to H) to a belief held in G. That is, when removing the strategies we allow justifications (for their removal) from the initial game H.

Thanks to property **A** the operator SR is monotonic. So we can apply to SR and its contracting version $\overline{SR}$ Note 1 and as a result we can confine further analysis to the latter operator. The SR operator was considered in [4] (called there λ, see page 1015) and $\overline{SR}$ in [12] (implicitly, in Definition 1 on page 1032), each of them with different belief structures.

On the account of the results from the previous section the $\overline{SR}$ operator enjoys various properties. We list them as the following result.

Theorem 4 **($\overline{SR}$).**

(i) The largest fixpoint of $\overline{SR}$ exists and is its outcome.
(ii) $\overline{SR}$ is order independent.
(iii) For all relaxations R of $\overline{SR}$ and all α we have $\overline{SR}^{\alpha} \subseteq R^{\alpha}$. □

In general, infinite iterations of $\overline{SR}$ can be necessary. In fact, in some games ω iterations of $\overline{SR}$ are insufficient to reach the outcome, that is, the closure ordinal of $\overline{SR}$ can be larger than ω.

Example 1. Consider the following game H with two players. The set of strategies for each player is the set of natural numbers $\mathcal{N}$ augmented by -1. The payoff functions are defined as follows. For $k, \ell \geq 0$ we put:

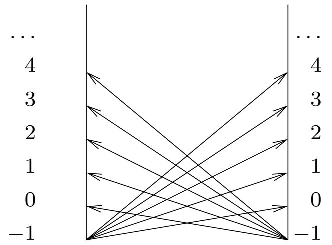
$$p_1(k,\ell) := \begin{cases} \ell + 1 \text{ if } k = \ell + 1 \\ 0 \quad\;\; \text{otherwise} \end{cases}$$

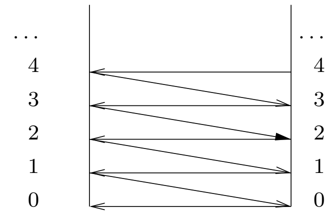
$$p_2(k,\ell) := \begin{cases} k \text{ if } k = \ell \\ 0 \text{ otherwise} \end{cases}$$

For the remaining pairs of strategies we put:

$$p_1(-1,\ell) := \ell + 1,$$
$$p_2(k,-1) := k.$$

Further we assume the pure belief structure, i.e., the beliefs are the strategies of the opponent. The following two diagrams explain the structure of this game. In each of them on the left we depict strategies of player 1 and on the right the strategies of player 2. An arrow from a to b stands for the statement 'strategy a is a best response to strategy b'.





Note that

- each -1 is a best response to any non-negative strategy,
- no best response to any -1 strategy exists,
- the best response to strategy $k \geq 0$ of player 1 is k,
- the best response to strategy $\ell \geq 0$ of player 2 is $\ell + 1$.

Denote $\mathcal{N} \cup \{-1\}$ by $\mathcal{N}'$. It is easy to see that

$$\overline{SR}^0 = (\mathcal{N}', \mathcal{N}'),$$
$$\overline{SR}^1 = (\mathcal{N}' \setminus \{0\}, \mathcal{N}'),$$
$$\overline{SR}^2 = (\mathcal{N}' \setminus \{0\}, \mathcal{N}' \setminus \{0\}),$$
$$\overline{SR}^3 = (\mathcal{N}' \setminus \{0, 1\}, \mathcal{N}' \setminus \{0\}),$$
$$\overline{SR}^4 = (\mathcal{N}' \setminus \{0, 1\}, \mathcal{N}' \setminus \{0, 1\}),$$
$$\ldots$$

So $\overline{SR}^{\omega} = (\{-1\}, \{-1\})$. But in the initial game no -1 is a best response to 'the other' -1. So $\overline{SR}((\{-1\}, \{-1\})) = (\emptyset, \emptyset)$, that is $\overline{SR}^{\omega+1} = (\emptyset, \emptyset)$. $\square$

Let us mention that a two-player game for which ω iterations of SR are insufficient to reach the outcome was already constructed in [8] . In his example each set of beliefs $\mathcal{B}_i$ consists of the mixed strategies of the opponent and the game is considerably more complex.

5 The $\boldsymbol{WR}$ and $\overline{\boldsymbol{WR}}$ operators

In this section we analyze the following operator WR (standing for 'weaker notion of rationalizability') on the set of restrictions of H:

$$WR(G) := (S'_1, \ldots, S'_n),$$

where for all $i \in [1..n]$

$$S'_i := \{s_i \in T_i \mid \exists \mu_i \in \mathcal{B}_i \dot{\cap} G \; s_i \in BR_G(\mu_i)\}.$$

So $WR(G)$ is obtained by removing from H all strategies that are never best responses *in* G (so *not* in H) to a belief held in G. That is, when removing strategies we use justifications (for their removal) from the currently considered game G. For each restriction G of H, $s_i \in BR_H(\mu_i)$ implies $s_i \in BR_G(\mu_i)$, so for all restrictions G we have $SR(G) \subseteq WR(G)$, which explains the above reference to a 'weaker' operator.

Unfortunately, the WR operator cannot be used as a meaningful basis for the rationalizability notion. Indeed, consider the following revealing example.

Example 2. Consider a two-players game H in which the set of strategies for each player is the left open unit interval $(0, 1]$ and in which the payoffs are defined as follows:

$$\begin{aligned}
&p_1(x, y) := x \text{ if } x \neq 1 \text{ or } y \neq 1\\
&p_1(1, 1) := 0\\
&p_2(x, y) := y \text{ if } x \neq 1 \text{ or } y \neq 1\\
&p_2(1, 1) := 0
\end{aligned}$$

Assume the pure belief structure. It is easy to check that

$$\begin{aligned}
&WR^0 = ((0, 1], (0, 1]),\\
&WR^1 = (\{1\}, \{1\}),\\
&WR^2 = ((0, 1), (0, 1)),\\
&WR^3 = (\{1\}, \{1\}),\\
&WR^4 = ((0, 1), (0, 1)),\\
&\ldots
\end{aligned}$$

So $WR^\omega = (\emptyset, \emptyset)$ and $(\emptyset, \emptyset)$ is the outcome of WR. But the finite iterations of WR oscillate between $(\{1\}, \{1\})$ and $((0, 1), (0, 1))$, so WR is not monotonic and not even contracting. So even though the outcome of this operator for the above game exists, it is not obtained by a natural, gradual, elimination process. □

This example also shows that we cannot apply to the WR operator Note 1 and in fact it is not clear that this operator always has an outcome. Consequently, we confine our attention to its contracting version, the $\overline{WR}$ operator, that always has an outcome. Unfortunately, $\overline{WR}$ fails to satisfy any of the properties of $\overline{SR}$ listed in the $\overline{SR}$ Theorem 4. First, it is not monotonic and the largest fixpoint of $\overline{WR}$ does not need to exist.

Example 3. Consider the game $H := (\{0, 1\}, \{0\}, p_1, p_2)$, where for $i \in \{0, 1\}$

$$p_1(i, 0) := i,$$

$$p_2(i, 0) := 0.$$

Assume the pure belief structure.

Note that $\overline{WR}(H) = (\{1\}, \{0\})$, whereas $(\{0\}, \{0\})$ is a fixpoint of $\overline{WR}$. So $(\{0\}, \{0\}) \subseteq H$, but $(\{0\}, \{0\})$ is not a restriction of $(\{1\}, \{0\})$.

Further, note that for each $i \in \{0, 1\}$ the restriction $(\{i\}, \{0\})$ is a fixpoint of $\overline{WR}$. However, their union, H, is not. □

Next, we exhibit a relaxation R of $\overline{WR}$ the outcome of which is different than the outcome of $\overline{WR}$.

Example 4. Consider a two-players game H in which the set of strategies for each player is the set of natural numbers. The payoff to each player is the number (strategy) he selected. Assume the pure belief structure. Clearly, no strategy is a best response to a strategy of the opponent. So $\overline{WR}(H) = \emptyset_2$ and $\emptyset_2$ is the outcome of $\overline{WR}$. However, for any $i \geq 0$ the restriction $(\{i\}, \{i\})$ is the outcome of a relaxation R of $\overline{WR}$ such that $R(H) = (\{i\}, \{i\})$. □

This example shows that the rationalizability notion entailed by the $\overline{WR}$ operator is not order independent. In this example the outcome of a relaxation of $\overline{WR}$ is strictly larger than the outcome of $\overline{WR}$. A more troublesome example is the following one in which the outcome of a relaxation of $\overline{WR}$ is strictly smaller than the outcome of $\overline{WR}$.

Example 5. Consider a version of Bertrand competition between two firms in which the marginal costs are 0 and in which the range of possible prices is the left-open real interval $(0, 100]$. So in this game H there are two players, each with the set $(0, 100]$ of strategies. We assume that the demand equals $100 - p$, where p is the lower price and that the profits are split in case of a tie. So the payoff functions are defined by:

$$p_1(s_1, s_2) := \begin{cases} s_1(100 - s_1) & \text{if } s_1 < s_2 \\ \dfrac{s_1(100 - s_1)}{2} & \text{if } s_1 = s_2 \\ 0 & \text{if } s_1 > s_2 \end{cases}$$

$$p_2(s_1, s_2) := \begin{cases} s_2(100 - s_2) & \text{if } s_2 < s_1 \\ \dfrac{s_2(100 - s_2)}{2} & \text{if } s_1 = s_2 \\ 0 & \text{if } s_2 > s_1 \end{cases}$$

Also, we assume the pure belief structure.

Since $s_1 = 50$ maximizes the value of $s_1(100 - s_1)$ in the interval $(0, 100]$, the strategy 50 is the unique best response to any strategy $s_2 > 50$ of the second player. Further, no strategy is a best response to a strategy $s_2 \leq 50$. By symmetry the same holds for the strategies of the second player. So $\overline{WR}(H) = (\{50\}, \{50\})$.

Further, $s_1 = 50$ is a best response in $(\{50\}, \{50\})$ to $s_2 = 50$ and symmetrically for the second player. So $\overline{WR}((\{50\}, \{50\})) = ((\{50\}, \{50\}))$.

Take now the relaxation R of $\overline{WR}$ defined by:

$$R(G) := \begin{cases} ((0,\ 50],\ (0,\ 50]) & \text{if } G = H \\ \overline{WR}(G) & \text{otherwise} \end{cases}$$

Then $R(H) = (0, 50], (0, 50])$ and $R(((0, 50], (0, 50])) = \emptyset_2$ since $\overline{WR}(((0, 50], (0, 50])) = \emptyset_2$. This shows that H cannot be reduced to an empty restriction by $\overline{WR}$ though it can by some relaxation of it.

Finally, note that in this example also the WR operator exhibits an erratic behaviour. Indeed, $WR(H) = (\{50\}, \{50\})$, i.e., $WR^1 = (\{50\}, \{50\})$. However, it is easy to see that $(49, 49) \in WR(\{50\}, \{50\})$, since $p_1(49, 50) > p_1(50, 50)$ and $p_2(50, 49) > p_2(50, 50)$. So WR^2 is not a restriction of WR^1. □

This example shows that $\overline{WR}$ is not fast (that is, a relaxation of it generates smaller restrictions) and suggests that it is unnecessarily weak. Also, it is counterintuitive that when $\overline{WR}$ is used to define rationalizability, in some natural games it is beneficial to eliminate at certain stages only *some* strategies that are never best responses.

The following observation assesses the relative strength of $\overline{SR}$ and $\overline{WR}$.

Lemma 1. *Let R be a relaxation of $\overline{WR}$. Then for all α we have $\overline{SR}^{\alpha} \subseteq R^{\alpha}$. In particular, for all α we have $\overline{SR}^{\alpha} \subseteq \overline{WR}^{\alpha}$ and consequently $\overline{SR}^{\alpha_{\overline{SR}}} \subseteq \overline{WR}^{\alpha_{\overline{WR}}}$.*

Proof. The proof is analogous to that of Theorem 3 with the following argument for the induction step for a successor ordinal. Suppose that for some α we have $\overline{SR}^{\alpha} \subseteq R^{\alpha}$. Then by the monotonicity of $\overline{SR}$

$$\overline{SR}(\overline{SR}^{\alpha}) \subseteq \overline{SR}(R^{\alpha}).$$

By the definition of the $\overline{SR}$ and $\overline{WR}$ operators

$$\overline{SR}(R^{\alpha}) \subseteq \overline{WR}(R^{\alpha}).$$

Further, since R is a relaxation of $\overline{WR}$,

$$\overline{WR}(R^{\alpha}) \subseteq R(R^{\alpha}).$$

So $\overline{SR}^{\alpha+1} \subseteq R^{\alpha+1}$. □

It is also easy to see that in the last example the outcome of the $\overline{SR}$ operator is an empty restriction. Indeed, $\overline{SR}(H) = (\{50\}, \{50\})$ and $\overline{SR}((\{50\}, \{50\})) = \emptyset_2$, since $s_1 = 49$ is a better response in H to $s_2 = 50$ than $s_1 = 50$ and symmetrically for the second player. So $\overline{SR}$ is strictly stronger than $\overline{WR}$ in the sense that its outcome can be strictly smaller than that of $\overline{WR}$.

In this example the outcome of $\overline{SR}$ is the same as the outcome of a relaxation of $\overline{WR}$. However, this does not need to hold in general. In other words, $\overline{SR}$ defines a genuinely more powerful rationalizability notion that cannot be derived from $\overline{WR}$.

Indeed, reconsider the game from Example 1. It is easy to see that the finite iterations of $\overline{WR}$ and $\overline{SR}$ coincide, so $\overline{WR}^{\omega} = (\{-1\}, \{-1\})$. But $(\{-1\}, \{-1\})$ is a fixpoint of $\overline{WR}$, so using $\overline{WR}$ we cannot reduce the initial game to an empty restriction. Moreover, the same holds for any relaxation R of $\overline{WR}$. Indeed, it is easy to see that for each relaxation R of $\overline{WR}$ its outcome always contains the joint strategy $(-1, -1)$.

6 When the outcomes of $\overline{SR}$ and $\overline{WR}$ coincide

To analyze the situations when the iterations of the $\overline{SR}$ and $\overline{WR}$ operators coincide we introduce the following property of the initial game H, given a belief structure $(\mathcal{B}, \dot{\cap})$ in H.

B For all beliefs $\mu_i \in \mathcal{B}_i$ a best response to μ_i in H exists.

For the finite games and pure belief structures property **B** obviously holds but it can clearly fail for the infinite games. For instance, it does not hold in the game considered in Example 4 since in this game no strategy is a best response to a strategy of the opponent. Also it does not hold in the game from Example 1 but we now offer an example of a game in which property **B** does hold and in which the iterations of length $> \omega$ of both $\overline{SR}$ and $\overline{WR}$ are unavoidable.

Example 6. We modify the game from Example 1 by adding the third player and by removing the strategy -1 from the set of strategies of each player. So the set of strategies for each player is the set of natural numbers $\mathcal{N}$. The payoff functions are now defined as follows:

$$p_1(k, \ell, m) := \begin{cases} \ell + 1 & \text{if } k = \ell + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$p_2(k, \ell, m) := \begin{cases} k \text{ if } k = \ell \\ 0 \text{ otherwise} \end{cases}$$

$$p_3(k, \ell, m) := 0.$$

Further we assume the pure belief structure. Note that

- the best response to $s_{-1} = (\ell, m)$ is $\ell + 1$,
- the best response to $s_{-2} = (k, m)$ is k,
- each $m \in \mathcal{N}$ is a best response to $s_{-3} = (k, \ell)$.

So to each joint strategy of the opponents a best response exists. That is, property **B** is satisfied. Analogously to Example 1 we have:

$$\begin{aligned}
&\overline{SR}^0 = (\mathcal{N}, \mathcal{N}, \mathcal{N}),\\
&\overline{SR}^1 = (\mathcal{N} \setminus \{0\}, \mathcal{N}, \mathcal{N}),\\
&\overline{SR}^2 = (\mathcal{N} \setminus \{0\}, \mathcal{N} \setminus \{0\}, \mathcal{N}),\\
&\overline{SR}^3 = (\mathcal{N} \setminus \{0, 1\}, \mathcal{N} \setminus \{0\}, \mathcal{N}),\\
&\overline{SR}^4 = (\mathcal{N} \setminus \{0, 1\}, \mathcal{N} \setminus \{0, 1\}, \mathcal{N}),\\
&\ldots
\end{aligned}$$

So $\overline{SR}^\omega = (\emptyset, \emptyset, \mathcal{N})$. Also $\overline{SR}((\emptyset, \emptyset, \mathcal{N})) = (\emptyset, \emptyset, \emptyset)$, so $\overline{SR}^{\omega+1} = (\emptyset, \emptyset, \emptyset)$.

Finally, it is easy to see that $\overline{WR}^\alpha = \overline{SR}^\alpha$ for all α (this is in fact, a consequence of a result proved below), so $\omega + 1$ is the closure ordinal of $\overline{WR}$, as well. $\Box$

For infinite games a natural situation when property **B** holds is the following. Recall that a game $H = (T_1, \ldots, T_n, p_1, \ldots, p_n)$ is called ***compact*** if the strategy sets are non-empty compact subsets of a complete metric space, and ***own-uppersemicontinuous*** if each payoff function p_i is uppersemicontinuous in the ith argument. In turn, p_i is called ***uppersemicontinuous in the*** i***th argument*** if the set $\{s'_i \in T_i \mid p_i(s'_i, s_{-i}) \geq r\}$ is closed for all $r \in \mathcal{R}$ and all $s_{-i} \in T_{-i}$.

As explained in [6] (see the proof of Lemma on page 2012) for such games and pure belief structures property **B** holds by virtue of a standard result from topology. If we impose a stronger condition on the payoff functions, namely that each of them is continuous —the game is called then ***continuous***— then we are within the framework considered in [4]. As shown there for compact and continuous games iterations of length $> \omega$ for the SR operator do not need to be considered, that is, its outcome (or equivalently the outcome of $\overline{SR}$) can be reached in ω iterations. To put it more succinctly: $\alpha_{SR} \leq \omega$.

In the presence of properties **A** and **B** still many differences between the operators $\overline{SR}$ and $\overline{WR}$ remain. In particular, $\overline{WR}$ does not need to be monotonic and its largest fixpoint does not need to exist. Indeed, it suffices to reconsider the game from Example 3.

To prove the positive results we identify first the crucial property of the initial game $H = (T_1, \ldots, T_n, p_1, \ldots, p_n)$ that holds in the presence of **B**.

Lemma 2. *Assume property **B**. Then for all relaxations R of $\overline{WR}$ and all $i \in [1..n]$*

$$\forall \alpha\, \forall \mu_i \in \mathcal{B}_i \mathbin{\dot\cap} R^\alpha\, \forall s_i \in T_i\, (s_i \in BR_{R^\alpha}(\mu_i) \rightarrow s_i \in BR_H(\mu_i)),$$

where $R^\alpha := (S_1, \ldots, S_n)$.

In words, for all α and all beliefs in $\mu_i \in \mathcal{B}_i \mathbin{\dot\cap} R^\alpha$ if s_i is a best response to μ_i in R^α, then it is in fact a best response to μ_i in H.

Proof. Suppose $s_i \in BR_{R^\alpha}(\mu_i)$, where $\mu_i \in \mathcal{B}_i \dot{\cap} R^\alpha$. R is contracting since $\overline{WR}$ is, so for all $\beta \leq \alpha$ we have $R^\alpha \subseteq R^\beta$. Therefore by property **A** for all $\beta \leq \alpha$ we have $\mu_i \in \mathcal{B}_i \dot{\cap} R^\beta$. In particular $\mu_i \in \mathcal{B}_i$. By property **B** a best response s_i^* to μ_i in H exists. Since R is a relaxation of $\overline{WR}$, s_i^* is a strategy of player i in all restrictions R^β for $\beta \leq \alpha$, in particular R^α. Hence

$$p_i(s_i^*, \mu_i) = p_i(s_i, \mu_i).$$

So s_i is also a best response to μ_i in H. $\square$

We prove now the following positive results under the assumption of property **B**. Part (i) shows that arbitrary iterations of the $\overline{SR}$ and $\overline{WR}$ operators coincide. Part (ii) establishes order independence of the operator $\overline{WR}$. This is not an immediate consequence of part (i) since in general the relaxations of $\overline{SR}$ and $\overline{WR}$ differ. Finally, in part (iii) we return to the WR operator that we considered and rejected in the previous section.

Theorem 5. *Assume properties* ***A*** *and* ***B****.*

(i) For all α we have $\overline{SR}^\alpha = \overline{WR}^\alpha$. In particular $\overline{SR}^{\alpha_{\overline{SR}}} = \overline{WR}^{\alpha_{\overline{WR}}}$.
(ii) The $\overline{WR}$ operator is order independent.
(iii) For all α we have $WR^\alpha = \overline{WR}^\alpha$. In particular $WR^{\alpha_{WR}}$ exists and $WR^{\alpha_{WR}} = \overline{WR}^{\alpha_{\overline{WR}}}$.

Proof.
(i) In view of Lemma 1 we only need to prove the inclusion $\overline{WR}^\alpha \subseteq \overline{SR}^\alpha$. We proceed by the transfinite induction. Suppose that $\overline{WR}^\alpha \subseteq \overline{SR}^\alpha$.

Let s_i be a strategy of player i in $\overline{WR}^{\alpha+1}$. By definition for some $\mu_i \in \mathcal{B}_i \dot{\cap} \overline{WR}^\alpha$ we have $s_i \in BR_{\overline{WR}^\alpha}(\mu_i)$, so by Lemma 2 $s_i \in BR_H(\mu_i)$. Further, $\overline{WR}^{\alpha+1} \subseteq \overline{WR}^\alpha$, so by the induction hypothesis s_i is a strategy of player i in $\overline{SR}^\alpha$. So s_i is a strategy of player i in $\overline{SR}^{\alpha+1}$.

The induction step for a limit ordinal is immediate.

(ii) Consider a relaxation R of $\overline{WR}$. Define a new relaxation R_1 by:

$$R_1(G) := \begin{cases} R(G) & \text{if } G = R^\alpha \text{ for some } \alpha \\ \overline{SR}(G) & \text{otherwise} \end{cases}$$

We claim that R_1 is a relaxation of $\overline{SR}$. To establish the first property of a relaxation take a restriction G of H. If $G = R^\alpha$ for some α, then $R_1(G) = R(G)$. Also $\overline{WR}(G) \subseteq R(G)$ and $\overline{SR}(G) \subseteq \overline{WR}(G)$, so $\overline{SR}(G) \subseteq R_1(G)$. If G is not of the form R^α, then $R_1(G) = \overline{SR}(G)$. The second property holds since by definition R_1 is contracting.

To establish the third property suppose G is a fixpoint of R_1. If $G = R^\alpha$ for some α, then $R_1(G) = R(G)$. So R^α is a fixpoint of R. But R is a relaxation of $\overline{WR}$, so R^α is a fixpoint of $\overline{WR}$. Let s_i be a strategy of player i in R^α. By the definition of $\overline{WR}$ for some $\mu_i \in \mathcal{B}_i \dot{\cap} R^\alpha$ we have $s_i \in BR_{R^\alpha}(\mu_i)$, so by Lemma 2 $s_i \in BR_H(\mu_i)$. Since s_i was arbitrary, this proves that R^α is a fixpoint of $\overline{SR}$.

If G is not R^α for any α, then $R_1(G) = \overline{SR}(G)$ and G is also a fixpoint of $\overline{SR}$. So we showed that R_1 is a relaxation of $\overline{SR}$.

Now note that the outcome of R is the same as the outcome of R_1. So by the $\overline{SR}$ Theorem 4(ii) the outcome of R is the same as the outcome of $\overline{SR}$. The claim now follows by part (i).

(iii) We proceed by the transfinite induction. Suppose the claim holds for all $\beta \leq \alpha$. To prove it for $\alpha + 1$ we first we prove that $WR^{\alpha+1} \subseteq WR^\alpha$. Let s_i be a strategy of player i in $WR^{\alpha+1}$. Then for some $\mu_i \in \mathcal{B}_i \dot{\cap} WR^\alpha$ we have $s_i \in BR_{WR^\alpha}(\mu_i)$. By the induction hypothesis $WR^\alpha = \overline{WR}^\alpha$, so by Lemma 2 $s_i \in BR_H(\mu_i)$.

Further, since $\overline{WR}$ is contracting, by property **A** for all $\beta \leq \alpha$ we have $\mu_i \in \mathcal{B}_i \dot{\cap} \overline{WR}^\beta$ and hence by the induction hypothesis for all $\beta \leq \alpha$ we have $\mu_i \in \mathcal{B}_i \dot{\cap} WR^\beta$. So s_i is a strategy of player i in all restrictions WR^β for $\beta \leq \alpha$, in particular WR^α. Since s_i was arbitrary, this proves that $WR^{\alpha+1} \subseteq WR^\alpha$.

Hence

$$WR^{\alpha+1} = WR^{\alpha+1} \cap WR^\alpha = \overline{WR}(WR^\alpha) = \overline{WR}^{\alpha+1},$$

where the last equality holds by the induction hypothesis.

The induction step for a limit ordinal is immediate. $\square$

We mentioned already that property **B** is satisfied by the compact and continuous games and pure belief structures. So part (i) is closely related to the result of [1] who showed that for such games the finite iterations of his versions of $\overline{SR}$ and $\overline{WR}$ operators coincide for the limited case of two-person games and pure belief structures. In his definition both operators are defined by considering the reduction for each player in succession and not in parallel.

In summary, by virtue of the results established in the last three sections, in the presence of property **B**, rationalizability can be equivalently defined using any of the introduced four operators $SR, \overline{SR}$, WR and $\overline{WR}$. However, when property **B** does not hold, only the first two operators are of interest.

7 Strict dominance: the SS and $\overline{SS}$ operators

To show the usefulness of our general approach to operators on games we now establish analogous results for the notion strict dominance. In what follows we analyze four operators that can be naturally used to define iterated elimination of strictly dominated strategies.

Given a restriction $G := (S_1, \ldots, S_n)$ of $H = (T_1, \ldots, T_n, p_1, \ldots, p_n)$ and two strategies s_i, s'_i from T_i we write $s'_i \succ_G s_i$ as an abbreviation for the statement $\forall s_{-i} \in S_{-i}\, p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i})$ and say then that s'_i ***strictly dominates*** s_i ***on*** G or, equivalently, that s_i ***is strictly dominated on*** G by s_i.

First, we introduce the following operator *SS* (standing for 'stronger strict dominance') on the set of restrictions of H:

$$SS(G) := (S'_1, \ldots, S'_n),$$

where for all $i \in [1..n]$

$$S'_i := \{s_i \in T_i \mid \neg\exists s'_i \in T_i \; s'_i \succ_G s_i\}.$$

So $SS(G)$ is obtained by removing from H all strategies that are strictly dominated on G by some strategy in H and *not* in G. The reasoning embodied in the definition of this operator is analogous to the one used for SR: we remove a strategy from H if a 'better' (here: strictly dominating) strategy exists in the initial game H, *even* if in the iteration process leading from H to G this better strategy might have been removed.

The *SS* operator was introduced in [9], where only its iterations up to ω were considered. As noted there this operator is clearly monotonic. So Note 1 applies and we can confine our considerations to the $\overline{SS}$ operator (defined by $\overline{SS}(G) := SS(G) \cap G$) that is also monotonic. On the account of the results of Section 2 this operator enjoys the same properties as the $\overline{SR}$ operator. We summarize them in the theorem below.

Theorem 6 ($\overline{SS}$).

(i) *The largest fixpoint of* $\overline{SS}$ *exists and is its outcome.*
(ii) $\overline{SS}$ *is order independent.*
(iii) *For all relaxations* R *of* $\overline{SS}$ *and all* α *we have* $\overline{SS}^{\alpha} \subseteq R^{\alpha}$. □

The $\overline{SS}$ operator was introduced and analyzed in [5], where the need for transfinite iterations was explained and where properties (i) and (ii) were proved.

8 Strict dominance: the WS and $\overline{WS}$ operators

The customary definition of the iterated elimination of strictly dominated strategies involves elimination of strategies that are strictly dominated by a strategy from the *currently considered* game and *not* the initial game H. A natural operator that formalizes this idea is the following one (*WS* stands for 'weaker strict dominance'):

$$WS(G) := (S'_1, \ldots, S'_n),$$

where $G := (S_1, \ldots, S_n)$ and for all $i \in [1..n]$

$$S'_i := \{s_i \in T_i \mid \neg\exists s'_i \in S_i \; s'_i \succ_G s_i\}.$$

So here, in contrast to SS, a strategy is removed if it is strictly dominated on G by some strategy *in* G itself. However, the WS operator is not acceptable for the same reason as the previously considered WR operator. In fact, the WS operator exhibits an even more erratic behaviour.

Example 7. Reconsider the two-players game H from Example 4 in which the set of strategies for each player is the set of natural numbers. The payoff to each player is the number he selected. In this game each strategy is strictly dominated. So $WS(H) = \emptyset_2$. But $WS(\emptyset_2) = H$, so the iterations of WS oscillate between H and $\emptyset_2$. Hence the outcome of WS does not exist.

A fortiori WS is not monotonic and not contracting. The most problematic is of course that this operator yields a non-empty restriction (in fact, the initial game) when applied to an empty restriction. □

So, as in the case of the WR operator, we confine our attention to the contracting version $\overline{WS}$ (defined by $\overline{WS}(G) := WS(G) \cap G$). In fact, in the literature the WS was not considered but rather $\overline{WS}$. However, $\overline{WS}$ fails to be monotonic, even for finite games. Indeed, the game from Example 3 again provides the evidence.

Below we shall use the following counterpart of Lemma 1, the proof of which is the same.

Lemma 3. *Let R be a relaxation of $\overline{WS}$. Then for all α we have $\overline{SS}^{\alpha} \subseteq R^{\alpha}$. In particular, for all α we have $\overline{SS}^{\alpha} \subseteq \overline{WS}^{\alpha}$ and consequently $\overline{SS}^{\alpha_{\overline{SS}}} \subseteq \overline{WS}^{\alpha_{\overline{WS}}}$.* □

The $\overline{WS}$ operator was shown to be order independent for finite games in [7]. It was studied for infinite games in [6] where it was noted that it fails to be order independent for arbitrary games. This is immediate to see using the game from Examples 4 and 7. Indeed, the outcome of $\overline{WS}$ is then $\emptyset_2$, whereas for any $i \geq 0$ the restriction $(\{i\}, \{i\})$ is the outcome of a relaxation R of $\overline{WS}$ such that $R(H) = (\{i\}, \{i\})$.

To explain the positive results of [6] we introduce the following notion. We say that G is an ***ω-outcome*** of an operator T on the set of restrictions of H if G is an outcome of T and $\alpha_T \leq \omega$. [2]

That is, G is an ω-outcome of T if $G = T^{\omega}$ and $T^{\omega+1} = T^{\omega}$. For a contracting operator, in contrast to the outcome, an ω-outcome does not need to exist. The study of [6] focuses on the set

$$\omega(\overline{WS}) := \{G \mid G \text{ is an } \omega\text{-outcome of a relaxation of } \overline{WS}\}.$$

If this set has at most one element, then they view $\overline{WS}$ as order independent. Recall that according to our definition an operator T is order independent if the set

$$\{G \mid G \text{ is an outcome of a relaxation of } T\}$$

[2] In [6] an ω-outcome is called a ***maximal*** $(\rightarrow^*)$-***reduction***.

has at most one element. We can then state the main results of [6] as follows. To clarify part (i) recall that by definition a restriction $(S_1, \ldots, S_n)$ is a game if each S_i is non-empty.

Theorem 7.

(i) If H is compact and own-uppersemicontinuous and the set $\omega(\overline{WS})$ has an element which is a game, then this is its only element.
(ii) If H is compact and continuous, then the set $\omega(\overline{WS})$ has precisely one element and this element is a compact and continuous game. □

[6] also considered the following property of the initial game H:

For all relaxations R of $\overline{WS}$ and all $k \geq 0$, every strictly dominated strategy in R^k has an undominated ('best') dominator.

This property limited to the initial game (so for $k = 0$) was considered in [10]. In [6] this property was formalized as $\forall k < \omega \mathbf{C}(k)$, where for an ordinal α we have:

$\mathbf{C}(\alpha)$ For all relaxations R of $\overline{WS}$ and all $i \in [1..n]$

$$\forall s_i \in S_i((\exists s'_i \in S_i s'_i \succ_{R^\alpha} s_i) \rightarrow \exists s^*_i \in S_i(s^*_i \succ_{R^\alpha} s_i \wedge \neg \exists s'_i \in S_i s'_i \succ_{R^\alpha} s^*_i)),$$

where $R^\alpha := (S_1, \ldots, S_n)$.

[6] showed that property $\forall k < \omega \mathbf{C}(k)$ is satisfied by the compact and own-uppersemicontinuous games but is not a sufficient condition for the order independence of $\overline{WS}$ in their sense. We now show that an alternative formalization of the original statement does yield an order independence result in the sense of [6]. Further, if this alternative formalization is adopted for all ordinals, then $\overline{WS}$ is also order independent in our sense.

Again, the subtlety lies here in minor details concerned with the reference to strict dominance in the initial game H. In fact, our formalization uses the following property $\mathbf{D}(\alpha)$, where for an ordinal α we have:

$\mathbf{D}(\alpha)$ For all relaxations R of $\overline{WS}$ and all $i \in [1..n]$

$$\forall s_i \in T_i((\exists s'_i \in T_i s'_i \succ_{R^\alpha} s_i) \rightarrow \exists s^*_i \in T_i(s^*_i \succ_{R^\alpha} s_i \wedge \neg \exists s'_i \in T_i s'_i \succ_{R^\alpha} s^*_i)),$$

where $R^\alpha := (S_1, \ldots, S_n)$.

Syntactically, the change is very simple: all the quantifiers now range over T_i and not S_i. So we refer now to strict dominance by a strategy in the initial game H and *not* in the currently considered reduction R^α.

First we establish a lemma that identifies the crucial property of the initial game $H = (T_1, \ldots, T_n, p_1, \ldots, p_n)$ under the assumption of $\mathbf{D}(\alpha)$.

Lemma 4. *Assume property $\boldsymbol{D}(\alpha)$. Then for all relaxations R of $\overline{WS}$ and all $i \in [1..n]$*

$$\forall s_i \in T_i((\exists s'_i \in T_i\ s'_i \succ_{R^\alpha} s_i) \rightarrow \exists s^*_i \in S_i\ s^*_i \succ_{R^\alpha} s_i),$$

where $R^\alpha := (S_1, \ldots, S_n)$.

That is, each s_i strictly dominated in R^α is in fact strictly dominated in R^α by some strategy in R^α.

Proof. Suppose $s_i \in T_i$ and $s'_i \in T_i$ are such that $s'_i \succ_{R^\alpha} s_i$. By property $\mathbf{D}(\alpha)$

$$\exists s^*_i \in T_i\ (s^*_i \succ_{R^\alpha} s_i \wedge \neg\exists s'_i \in T_i\ s'_i \succ_{R^\alpha} s^*_i).$$

R is contracting since $\overline{WS}$ is, so for $\beta \leq \alpha$ we have $R^\alpha \subseteq R^\beta$ and consequently $\forall \beta \leq \alpha\ \neg\exists s'_i \in T_i\ s'_i \succ_{R^\beta} s^*_i$. Hence s^*_i is a strategy of player i in all restrictions R^β for $\beta \leq \alpha$. In particular $s^*_i \in S_i$. □

We now establish a number of consequences of property $\mathbf{D}(\alpha)$ for various values of α. Part (i) is a counterpart of Theorem 5(i). Parts (ii) and (iii) are the promised order independence results for the $\overline{WS}$ operator. Finally, in part (iv) we compare the operators WS and $\overline{WS}$ and establish an analogue of Theorem 5(iii).

Theorem 8.

(i) Assume property $\forall \beta < \alpha \boldsymbol{D}(\beta)$ for an ordinal α. Then $\overline{SS}^\alpha = \overline{WS}^\alpha$. In particular, if $\forall \alpha \boldsymbol{D}(\alpha)$, then $\overline{SS}^{\alpha_{\overline{SS}}} = \overline{WS}^{\alpha_{\overline{WS}}}$.
(ii) Assume property $\boldsymbol{D}(\omega)$. Then $\omega(\overline{WS}) \subseteq \{\overline{WS}^\omega\}$, i.e., $\overline{WS}^\omega$ is the only possible ω-outcome of a relaxation of $\overline{WS}$.
(iii) Assume property $\forall \alpha \boldsymbol{D}(\alpha)$. Then the $\overline{WS}$ operator is order independent.
(iv) Assume property $\forall \beta < \alpha + 1\ \boldsymbol{D}(\beta)$. Then $WS^\alpha = \overline{WS}^\alpha$. In particular, if $\forall \alpha \boldsymbol{D}(\alpha)$, then $WS^{\alpha_{WS}}$ exists and $WS^{\alpha_{WS}} = \overline{WS}^{\alpha_{\overline{WS}}}$.

Proof.
(i) In view of Lemma 3 we only need to prove the inclusion $\overline{WS}^\alpha \subseteq \overline{SS}^\alpha$ and only the induction step for a successor ordinal requires a justification. So suppose that $\forall \beta < \alpha + 1\, \mathbf{D}(\beta)$. Let $\overline{WS}^\alpha := (S_1, \ldots, S_n)$.

Let s_i be a strategy of player i in $\overline{WS}^{\alpha+1}$. By definition $s_i \in S_i$ and $\neg\exists s'_i \in S_i\ s'_i \succ_{\overline{WS}^\alpha} s_i$, so by Lemma 4 $\neg\exists s'_i \in T_i\ s'_i \succ_{\overline{WS}^\alpha} s_i$. Hence $\neg\exists s'_i \in T_i s'_i \succ_{\overline{SS}^\alpha} s_i$, since by the induction hypothesis $\overline{WS}^\alpha \subseteq \overline{SS}^\alpha$. Also, because of the same inclusion, s_i is a strategy of player i in $\overline{SS}^\alpha$. So s_i is a strategy of player i in $\overline{SS}^{\alpha+1}$.

(ii) Let R be a relaxation of $\overline{WS}$ such that $\alpha_R \leq \omega$. By Lemma 3 we have $\overline{SS}^\omega \subseteq R^\omega$, so by part (i)

$$\overline{SS}^\omega \subseteq R^\omega. \tag{1}$$

Let $R^\omega := (S_1, \ldots, S_n)$. Take a strategy $s_i \in S_i$. Since R^ω is a fixpoint of $\overline{WS}$, $\neg\exists s'_i \in S_i \; s'_i \succ_{R^\omega} s_i$, so by Lemma 4 $\neg\exists s'_i \in T_i \; s'_i \succ_{R^\omega} s_i$. Since s_i was arbitrary, this shows that R^ω is a fixpoint of $\overline{SS}$. So by the $\overline{SS}$ Theorem 6(i) we have $R^\omega \subseteq \overline{SS}^{\alpha_{\overline{SS}}}$. Also, $\overline{SS}$ is contracting, so $\overline{SS}^{\alpha_{\overline{SS}}} \subseteq \overline{SS}^\omega$ and hence

$$R^\omega \subseteq \overline{SS}^\omega.$$

This inclusion combined with (1) yields $R^\omega = \overline{SS}^\omega$. By part ($i$) $R^\omega = \overline{WS}^\omega$.

So we showed that any ω-outcome of a relaxation of $\overline{WS}$ equals $\overline{WS}^\omega$. This concludes the proof.

(iii) The proof combines the arguments used in the proofs of Theorem 5(i) and of part (i). Consider a relaxation R of $\overline{WS}$. Define a new relaxation R_1 by:

$$R_1(G) := \begin{cases} R(G) & \text{if } G = R^\alpha \text{ for some } \alpha \\ \overline{SS}(G) & \text{otherwise} \end{cases}$$

We claim that R_1 is a relaxation of $\overline{SS}$. To establish the first property of a relaxation take a restriction G of H. If $G = R^\alpha$ for some α, then $R_1(G) = R(G)$. Also $\overline{WS}(G) \subseteq R(G)$ and $\overline{SS}(G) \subseteq \overline{WS}(G)$, so $\overline{SS}(G) \subseteq R_1(G)$. If G is not of the form R^α, then $R_1(G) = \overline{SS}(G)$. The second property holds since by definition R_1 is contracting.

To establish the third property of a relaxation suppose G is a fixpoint of R_1. If $G = R^\alpha$ for some α, then $R_1(G) = R(G)$. So R^α is a fixpoint of R. But R is a relaxation of $\overline{WS}$, so R^α is a fixpoint of $\overline{WS}$. Suppose now $R^\alpha := (S_1, \ldots, S_n)$.

Let s_i be a strategy of player i in R^α. By definition $s_i \in S_i$ and $\neg\exists s'_i \in S_i \; s'_i \succ_{R^\alpha} s_i$, so by Lemma 4 $\neg\exists s'_i \in T_i \; s'_i \succ_{R^\alpha} s_i$. Since s_i was arbitrary, this proves that R^α is a fixpoint of $\overline{SS}$.

If G is not R^α for any α, then $R_1(G) = \overline{SS}(G)$ and G is also a fixpoint of $\overline{SS}$. This shows that R_1 is a relaxation of $\overline{SS}$. But the outcome of R is the same as the outcome of R_1, so by the $\overline{SS}$ Theorem 6(ii) the outcome of R is the same as the outcome of $\overline{SS}$. The claim now follows by part (i).

(iv) We prove the claim by transfinite induction. Assume it holds for all $\beta \leq \alpha$. We prove it for $\alpha+1$. So suppose that $\forall \beta < \alpha+2 \mathbf{D}(\beta)$. Let $WS^\alpha := (S_1, \ldots, S_n)$. Let s_i be a strategy of player i in $WS^{\alpha+1}$. By definition

$$\neg\exists s'_i \in S_i \; s'_i \succ_{WS^\alpha} s_i. \tag{2}$$

By the induction hypothesis $WS^\alpha = \overline{WS}^\alpha$, so by Lemma 4 $\neg\exists s'_i \in T_i \, s'_i \succ_{\overline{WS}^\alpha} s_i$. So s_i is a strategy of player i in $SS(\overline{WS}^\alpha)$. But by part (i) $\overline{WS}^\alpha = \overline{SS}^\alpha$ and $\overline{WS}^{\alpha+1} = \overline{SS}^{\alpha+1}$. Also, since SS is monotonic, by Note 1 $\overline{SS}^\alpha = SS^\alpha$ and $\overline{SS}^{\alpha+1} = SS^{\alpha+1}$. So

$$SS(\overline{WS}^\alpha) = SS(\overline{SS}^\alpha) = SS^{\alpha+1} = \overline{WS}^{\alpha+1}.$$

Since s_i was arbitrary, this shows that $WS^{\alpha+1} \subseteq \overline{WS}^{\alpha+1}$. But also

$$\overline{WS}^{\alpha+1} = \overline{WS}(WS^{\alpha}) = WS^{\alpha+1} \cap WS^{\alpha} \subseteq WS^{\alpha+1},$$

where the first equality holds by the induction hypothesis. So $\overline{WS}^{\alpha+1} = WS^{\alpha+1}$.

The induction step for a limit ordinal is immediate. $\Box$

[6] showed that compact and own-uppersemicontinuous games satisfy for all $\alpha \leq \omega$ the property stated in Lemma 4. So, since in the proofs of parts (i)—(iv) property $\mathbf{D}(\alpha)$ is used only through a reference to this lemma, part (ii) is a minor improvement of the result of [6] listed above as Theorem 7(i). Also, for each α property $\mathbf{D}(\alpha)$ is obviously satisfied by the finite games, so part (iii) generalizes the already mentioned result of [7].

In summary, the iterations of two operators, SS and $\overline{SS}$, always coincide and each of them is order independent. Further, when $\forall k < \omega \mathbf{D}(k)$ holds, the first ω iterations of all four operators, SS, $\overline{SS}$, WS and $\overline{WS}$ coincide. In particular, this is the case for the compact and own-uppersemicontinuous games. Finally, when $\forall \alpha \mathbf{D}(\alpha)$ holds, all iterations of these operators coincide and any of them can be used to define the outcome of iterated elimination of strictly dominated strategy.

9 Conclusions

In this paper we analyzed two widely used ways of reducing strategic games concerned with the concepts of rationalizability and iterated elimination of strictly dominated strategies. We showed that both concepts can be defined in a number of ways that differ for arbitrary infinite games.

Our analysis was based on a general study of operators on complete lattices. This allowed us to clarify that for the notions of game reduction defined using monotonic operators order independence automatically holds. As an example of the usefulness of this observation, assume that the initial game $H = (T_1, \ldots, T_n, p_1, \ldots, p_n)$ is finite and consider the obvious modification of the SS operator to mixed strategies defined by:

$$MSS(G) := (S'_1, \ldots, S'_n),$$

where for all $i \in [1..n]$

$$S'_i := \{s_i \in T_i \mid \neg \exists m'_i \in \Delta T_i \; m'_i \succ_G s_i\},$$

and where the relation $\succ_G$ is now extended to mixed strategies in the obvious way.

The MSS operator is clearly monotonic, so by Note 1 and the Order Independence Theorem 2 both MSS and $\overline{MSS}$ are order independent. The corresponding operator $\overline{WSS}$ in which one eliminates the strategies that are strictly dominated by a mixed strategy taken from the currently considered game (and not H) is not monotonic but still is order independent. However, the proof is not anymore straightforward (see [11] and [2] for a simpler proof).

In some circumstances a reduction notion defined using a non-monotonic operator still can be analyzed using our approach, by relating the operator to a monotonic one. This is how we established the order independence of the $\overline{WS}$ operator under the assumption $\forall\alpha\mathbf{D}(\alpha)$ —by relating it to the $\overline{SS}$ operator which is monotonic.

An important aspect of our analysis is that we allow transfinite iterations of the corresponding operators. Their use in an analysis of reasoning used by rational agents can be baffling. These matters were discussed in [8], where a need for transfinite iterations in the definition of rationalizability was observed. More recently [5] argued for their usefulness in the context of iterated elimination of strictly dominated strategies. The distinction between finitary and infinitistic forms of reasoning is well understood in mathematical logic. Both forms have been widely used and we see no reason for limiting the study of games to finitary methods.

The final matter that merits attention is the striking difference between the way rationalizability and iterated elimination of strictly dominated strategies have been tradionally defined and used in the literature. Both concepts are supposed to capture reasoning used by rational players. Yet, in the former one the reference point for a deletion of a strategy is the *initial* game, while in the latter one the *currently considered* game. This difference is important, since the first approach yields a monotonic operator, while the second one not. From the mathematical point of view the first approach is clearly preferable. So, if we accept mathematical elegance and simplicity as a meaningful criterion for assessing formalization of concepts, this calls for a revision of the way iterated elimination of strictly dominated strategies should be defined and provides further support for the approach used in [9] and [5].

Acknowledgements

One of the referees of [3] drew our attention to the paper [1]. Also, we thank Xiao Luo for an interesting discussion on the subject of [3] and [5].

References

1. S. Ambroszkiewicz. Knowledge and best responses in games. *Annals of Operations Research*, 51:63–71, 1994.
2. K. R. Apt. Uniform proofs of order independence for various strategy elimination procedures. *Contributions to Theoretical Economics, 4(1)*, 2004. Article 5, 48 pages. Available via http://xxx.lanl.gov/abs/cs.GT/0403024.
3. K. R. Apt. Rationalizability and order independence. In *Proc. 10th Conference on Theoretical Aspects of Reasoning about Knowledge (TARK)*, pages 22–38. National University of Singapore, 2005. Available from `http://www.cwi.nl/~apt/ps/ratio05.ps`. Also to appear in the ACM Digital Library.
4. B. D. Bernheim. Rationalizable strategic behavior. *Econometrica*, 52(4):1007–1028, 1984.

5. Y.-C. Chen, N. Van Long, and X. Luo. Iterated strict dominance in general games. Available from http://www.sinica.edu.tw/~xluo/pa10.pdf, 2005.
6. M. Dufwenberg and M. Stegeman. Existence and uniqueness of maximal reductions under iterated strict dominance. *Econometrica*, 70(5):2007–2023, 2002.
7. I. Gilboa, E. Kalai, and E. Zemel. On the order of eliminating dominated strategies. *Operation Research Letters*, 9:85–89, 1990.
8. B. L. Lipman. A note on the implications of common knowledge of rationality. *Games and Economic Behavior*, 6:114–129, 1994.
9. P. Milgrom and J. Roberts. Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica*, 58(6):1255–1278, 1990.
10. P. Milgrom and J. Roberts. Coalition-proofness and correlation with arbitrary communication possibilities. *Games and Economic Behavior*, 17(4):113–128, 1996.
11. M. J. Osborne and A. Rubinstein. *A Course in Game Theory*. The MIT Press, Cambridge, Massachusetts, 1994.
12. D. G. Pearce. Rationalizable strategic behavior and the problem of perfection. *Econometrica*, 52(4):1029–1050, 1984.
13. A. Tarski. A lattice-theoretic fixpoint theorem and its applications. *Pacific J. Math*, 5:285–309, 1955.