Selfishness Level of Strategic Games

Krzysztof R. Apt
CWI, Amsterdam, the Netherlands,
University of Amsterdam

based on joint work with

Guido Schäfer

CWI, Amsterdam, the Netherlands,
Vrije Universiteit Amsterdam

Strategic Games: Review

Strategic game for $|N| \geq 2$ players:

$$G := (N, \{S_i\}_{i \in N}, \{p_i\}_{i \in N}).$$

For each player i

- (possibly infinite) set S_i of strategies,
- payoff function $p_i : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$.

Main Concepts

- Notation: $s_i, s'_i \in S_i$,
  $s, s', (s_i, s_{-i}) \in S_1 \times \ldots \times S_n$.
- s is a Nash equilibrium if

$$\forall i \in \{1, \ldots, n\} \; \forall s'_i \in S_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

- Social welfare of s:

$$SW(s) := \sum_{j=1}^{n} p_j(s).$$

- s is a social optimum if $SW(s)$ is maximal.

Altruistic Games

- Given $G := (N, \{S_i\}_{i \in N}, \{p_i\}_{i \in N})$ and $\alpha \geq 0$.
- $G(\alpha) := (N, \{S_i\}_{i \in N}, \{r_i\}_{i \in N})$, where

$$r_i(s) := p_i(s) + \alpha SW(s).$$

- When $\alpha > 0$ the payoff of each player in $G(\alpha)$ depends on the social welfare of the players.
- $G(\alpha)$ is an altruistic version of G.

Selfishness Level (1)

- G is α-selfish if a Nash equilibrium of $G(\alpha)$ is a social optimum of $G(\alpha)$.
- Selfishness level of G:

$$\inf\{\alpha \in \mathbb{R}_+ \mid G \text{ is } \alpha\text{-selfish}\}.$$

  Recall $\inf(\emptyset) = \infty$.
- Selfishness level of G is α^+ iff the selfishness level of G is $\alpha \in \mathbb{R}_+$ but G is *not* α-selfish.

Selfishness Level (2)

Intuition

Selfishness level quantifies the minimal share of social welfare needed to induce the players to choose a social optimum.

Three Examples (1)

Prisoner's Dilemma

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

The Battle of the Sexes

	F	B
F	$2,1$	$0,0$
B	$0,0$	$1,2$

Matching Pennies

	H	T
H	$1,-1$	$-1,\ 1$
T	$-1,\ 1$	$1,-1$

Three Examples (2)

Prisoner's Dilemma: selfishness level is 1.

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

	C	D
C	$6,6$	$3,6$
D	$6,3$	$3,3$

The Battle of the Sexes: selfishness level is 0.

	F	B
F	$2,1$	$0,0$
B	$0,0$	$1,2$

Matching Pennies: selfishness level is ∞.

	H	T
H	$1,-1$	$-1,\ 1$
T	$-1,\ 1$	$1,-1$

Another Example

Game with a bad Nash equilibrium

	H	T	E
H	$1, -1$	$-1, \ \ 1$	$-1, -1$
T	$-1, \ \ 1$	$1, -1$	$-1, -1$
E	$-1, -1$	$-1, -1$	$-1, -1$

- The unique Nash equilibrium is (E, E).
- The selfishness level of this game is ∞.

Invariance of Selfishness Level

Lemma Consider a game G and $\alpha \geq 0$.

- For every a, G is α-selfish iff $G + a$ is α-selfish,
- For every $a > 0$, G is α-selfish iff aG is α-selfish.

Conclusion Selfishness level is invariant under positive linear transformations of the payoff functions.

Selfishness Level vs Price of Stability (1)

- Recall
  Price of stability = $SW(s)/SW(s')$,
  where s is a social optimum and s' a Nash equilibrium with the highest social welfare.
- Note
  Selfishness level of a finite game is 0 iff price of stability is 1.
- Theorem
  For every finite $\alpha > 0$ and $\beta > 1$ there is a finite game with selfishness level α and price of stability β.
- Theorem
  There exists a game that is 0^+-selfish
  (so α-selfish for every $\alpha > 0$, but is not 0-selfish).

Stable Social Optima

- Social optimum s stable if no player is better off by unilaterally deviating to another social optimum.
- That is, s is stable if for all $i \in N$ and $s'_i \in S_i$

  if (s'_i, s_{-i}) is a social optimum, then $p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i})$.

Characterization Result

Player i's appeal factor of s'_i given the social optimum s:

$$AF_i(s'_i, s) := \frac{p_i(s'_i, s_{-i}) - p_i(s_i, s_{-i})}{SW(s_i, s_{-i}) - SW(s'_i, s_{-i})}.$$

Theorem

- The selfishness level of G is finite iff a stable social optimum s exists for which
  $\alpha(s) := \max_{i \in N, s'_i \in U_i(s)} AF_i(s'_i, s)$
  is finite, where
  $U_i(s) := \{s'_i \in S_i \mid p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i})\}$.
- If the selfishness level of G is finite, then it equals $\min_{s \in SSO} \alpha(s)$, where SSO is the set of stable social optima.

Prisoner's Dilemma for n players

- Each $S_i = \{0, 1\}$,
- $p_i(s) := 1 - s_i + 2\sum_{j \neq i} s_j$.

Proposition Selfishness level is $\frac{1}{2n-3}$.

Public Goods Game

- n players,
- $b \in \mathbb{R}_+$: fixed budget,
- $c > 1$: a multiplier,
- $S_i = [0, b]$,
- $p_i(s) := b - s_i + \frac{c}{n} \sum_{j \in N} s_j$.

Proposition Selfishness level is $\max\left\{0, \frac{1-\frac{c}{n}}{c-1}\right\}$.

Notes

- Free riding: contributing 0 (it is a dominant strategy).
- For fixed c temptation to free ride increases with n.
- For fixed n temptation to free ride decreases as c increases.

Potential Games

$$G := (N, \{S_i\}_{i\in N}, \{p_i\}_{i\in N})$$

is an ordinal potential game if for some $P : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$ for all $i \in N$, $s_{-i} \in S_{-i}$ and $s_i, s'_i \in S_i$

$$p_i(s_i, s_{-i}) > p_i(s'_i, s_{-i}) \text{ iff } P(s_i, s_{-i}) > P(s'_i, s_{-i}).$$

Theorem Every finite ordinal potential game has a finite selfishness level.

Proof Each social optimum with the largest potential is a stable social optimum.

Fair Cost Sharing Games (1)

Fair cost sharing game: $G = (N, E, \{S_i\}_{i \in N}, \{c_e\}_{e \in E})$, where

- E is the set of facilities,
- $S_i \subseteq 2^E$ is the set of facility subsets available to player i, i.e., each $s_i \subseteq E$,
- $c_e \in \mathbb{Q}_+$ is the cost of facility $e \in E$.
- Let $x_e(s)$ be the number of players using facility e in s.
- The cost of facility $e \in E$ is evenly shared. So $c_i(s) := \sum_{e \in s_i} \frac{c_e}{x_e(s)}$.
- Social cost: $SC(s) = \sum_{i=1}^{n} c_i(s)$.

Fair Cost Sharing Games (2)

Singleton cost sharing game: for each s_i, $|s_i| = 1$.

- $c_{\max} := \max_{e \in E} c_e$,
- $c_{\min} := \min_{e \in E} c_e$,
- $L := \max_{i \in N,\ s_i \in S_i} |s_i|$
  (maximum number of facilities that a player can choose).

Proposition Selfishness level of

- a singleton cost sharing game is $\leq \frac{1}{2} c_{\max}/c_{\min} - 1$,
- a fair cost sharing game with non-negative integer costs is $\leq \frac{1}{2} L c_{\max} - 1$.

Note These bounds are tight.

Congestion Games

Congestion game: $G = (N, E, \{S_i\}_{i \in N}, \{d_e\}_{e \in E})$, where

- E is a finite set of facilities,
- $S_i \subseteq 2^E$ is the set of facility subsets available to player i,
- $d_e \in \mathbb{N}$ is the delay function for facility $e \in E$.
- Let $x_e(s)$ be the number of players using facility e in s.
- The goal of a player is to minimize his individual cost $c_i(s) := \sum_{e \in s_i} d_e(x_e(s))$.
- Social cost: $SC(s) = \sum_{i=1}^{n} c_i(s)$.

Symmetric congestion game: $S_i = S_j$ for all i, j.

Linear Congestion Games

Linear congestion game: each delay function is of the form $d_e(x) = a_e x + b_e$, where $a_e, b_e \in \mathbb{R}_+$.

- $\Delta_{\max} := \max_{e \in E}(a_e + b_e)$, $\Delta_{\min} := \min_{e \in E}(a_e + b_e)$,
- $L := \max_{i \in N,\ s_i \in S_i} |s_i|$,
- $\lambda_{\max}$: maximum discrepancy between two facilities,
- $a_{\min} := \min_{e \in E : a_e > 0} a_e$.

Proposition Selfishness level of

- a symmetric singleton linear congestion game is $\leq \frac{1}{2}(\Delta_{\max} - \Delta_{\min})/((1 - \lambda_{\max})a_{\min}) - \frac{1}{2}$,
- a linear congestion game with non-negative integer coefficients is $\leq \frac{1}{2}(L \cdot \Delta_{\max} - \Delta_{\min} - 1)$.

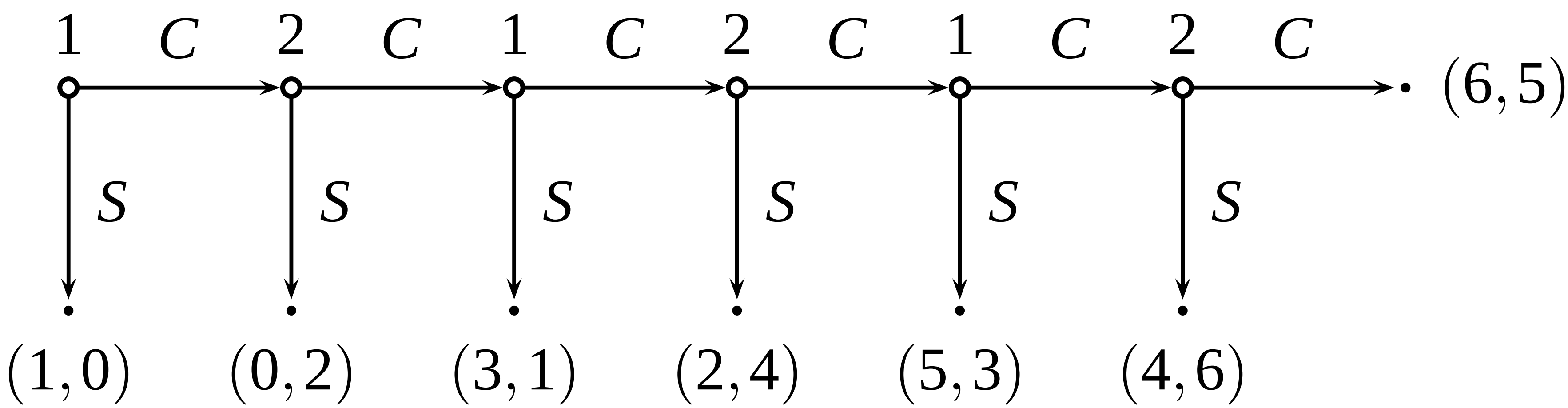
Games with Infinite Selfishness Level

- Cournot Competition
  - One infinitely divisible product (oil),
  - n companies decide simultaneously how much to produce,
  - price is decreasing in total output.
- Tragedy of the Commons
  - Contiguous common resource (bandwidth),
  - the payoff degrades when the resource is overused.
- Bertrand Competition
  - One product for sale.
  - 2 companies simultaneously select their prices.
  - The product is sold by the company that chose a lower price.

Concluding Remarks

Other games and equilibria notions can be studied.

Example Centipede game and subgame perfect equilibrium.



In its unique subgame perfect equilibrium the resulting payoffs are $(1,0)$.

We have $5+(6+5)\alpha \geq 6+(4+6)\alpha$ iff $\alpha \geq 1$.

So the (redefined) selfishness level is 1.

Some Quotations

- Dalai Lama:

  `The intelligent way to be selfish is to work for the welfare of others.`

  Microeconomics: Behavior, Institutions, and Evolution, S. Bowles '04.

- `An excellent way to promote cooperation in a society is to teach people to care about the welfare of others.`

  The Evolution of Cooperation, R. Axelrod, '84.

THANK YOU

Dziękuję za uwagę