Potential Games

Krzysztof R. Apt

CWI, Amsterdam, the Netherlands,

University of Amsterdam

Overview

- Best response dynamics.
- Potential games.
- Congestion games.
- Examples.
- Price of Stability.

Best Response Dynamics

- Consider a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$.
- An algorithm to find a Nash equilibrium:

choose $s \in S_1 \times \ldots \times S_n$;
while s is not a NE **do**
  choose $i \in \{1, \ldots, n\}$ such that
    s_i is not a best response to s_{-i};
  $s_i :=$ a best response to s_{-i}
od

- Trivial Example: the Battle of the Sexes game.
  Start anywhere.

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Best Response Dynamics, ctd

Best response dynamics may miss a Nash equilibrium.

Example (Shoham and Leyton-Brown '09)

	H	T	E
H	$1, -1$	$-1, \ \ 1$	$-1, -1$
T	$-1, \ \ 1$	$1, -1$	$-1, -1$
E	$-1, -1$	$-1, -1$	$-1, -1$

Here (E, E) is a unique Nash equilibrium.

Potential Games

(Monderer and Shapley '96)

- Consider a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$.
- A function $P : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$ is a potential function for G if

$$\forall i \in \{1, \ldots, n\} \; \forall s_{-i} \in S_{-i} \; \forall s_i, s_i' \in S_i$$
$$p_i(s_i, s_{-i}) - p_i(s_i', s_{-i}) = P(s_i, s_{-i}) - P(s_i', s_{-i}).$$

- Intuition: P tracks the changes in the payoff when some player deviates.
- Potential game: a game that has a potential function.

Example

- Prisoner's dilemma for n players.

$$p_i(s) := \begin{cases} 2\sum_{j\neq i} s_j + 1 & \text{if } s_i = 0 \\ 2\sum_{j\neq i} s_j & \text{if } s_i = 1 \end{cases}$$

- For $i = 1, 2$

$$p_i(0, s_{-i}) - p_i(1, s_{-i}) = 1.$$

- So $P(s) := -\sum_{j=1}^{n} s_j$ is a potential function.

Another Example

The Battle of the Sexes

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Each potential function P has to satisfy

- $P(F, F) - P(B, F) = 2$,
- $P(F, F) - P(F, B) = 1$,
- $P(B, B) - P(F, B) = 1$,
- $P(B, B) - P(B, F) = 2$.

Just use: $P(F, F) = P(B, B) = 2$, $P(F, B) = 1$, $P(B, F) = 0$.

Non-example:

Matching Pennies See the next slide.

Potential Games, ctd

Theorem (Monderer and Shapley '96)
Every finite potential game has a Nash equilibrium.

Proof 1.

- The games $(S_1, \ldots, S_n, p_1, \ldots, p_n)$ and $(S_1, \ldots, S_n, P, \ldots, P)$ have the same set of Nash equilibria.
- Take s for which P reaches maximum. Then s is a Nash equilibrium of $(S_1, \ldots, S_n, P, \ldots, P)$.

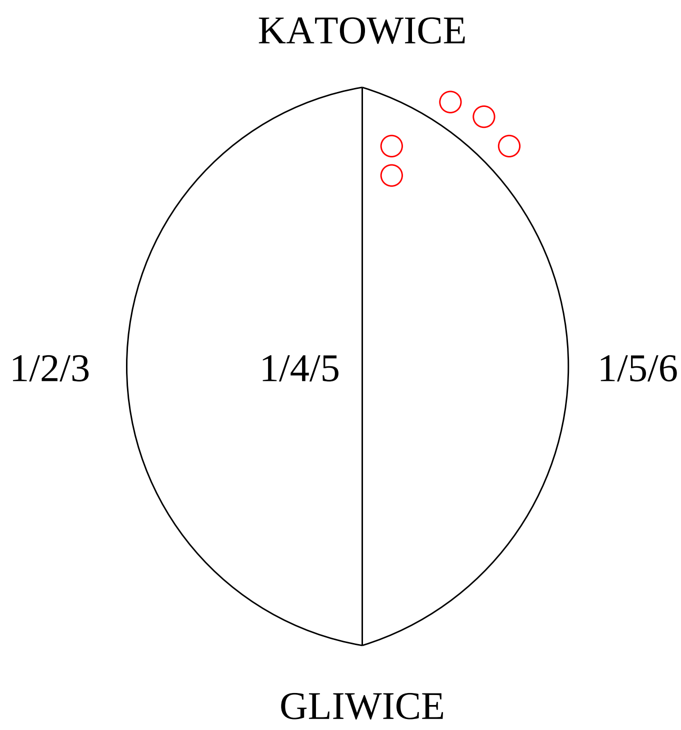
Proof 2.
For finite potential games the best response dynamics terminates.

Congestion Games

- $n > 1$ players,
- set M of facilities (road segments, primary production factors, $\ldots$),
- each strategy is a non-empty subset of M,
- each player has a possibly different set of strategies,
- *cost*$_j : \{1, \ldots, n\} \to \mathbb{R}$ is the cost function for using $j \in M$,
- *cost*$_j(k)$ is the cost to each user of facility j when there are k users of j,
- *users*$(r, s) = |\{i \in \{1, \ldots, n\} \mid r \in s_i\}|$ is the number of users of facility r in s,
- $c_i(s) := \sum_{r \in s_i}$ *cost*$_r($*users*$(r, s))$,
- We use here cost functions c_i instead of payoff functions p_i. To convert to payoffs use $p_i(s) := -c_i(s)$.

Example

- 5 drivers.
- Each driver chooses a road from Katowice to Gliwice.
- More drivers choose the same road: bigger delays.

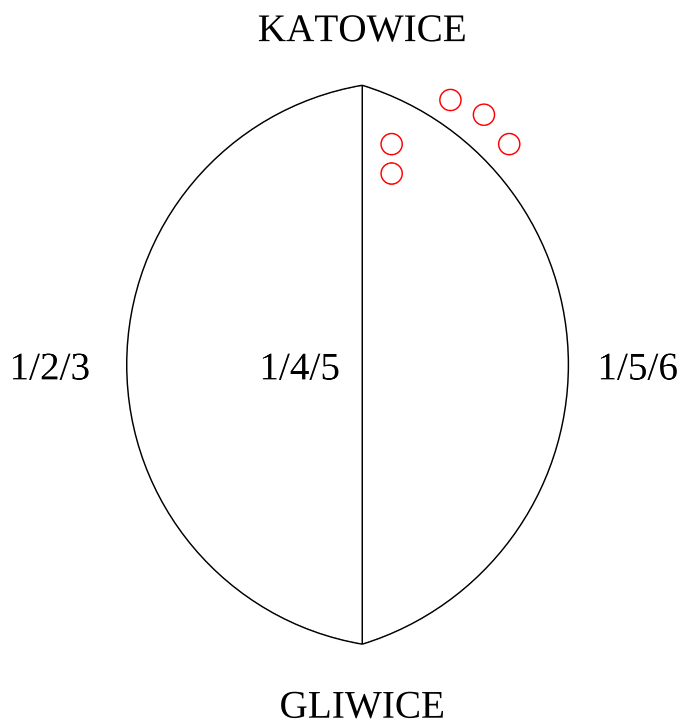


Example as a Congestion Game

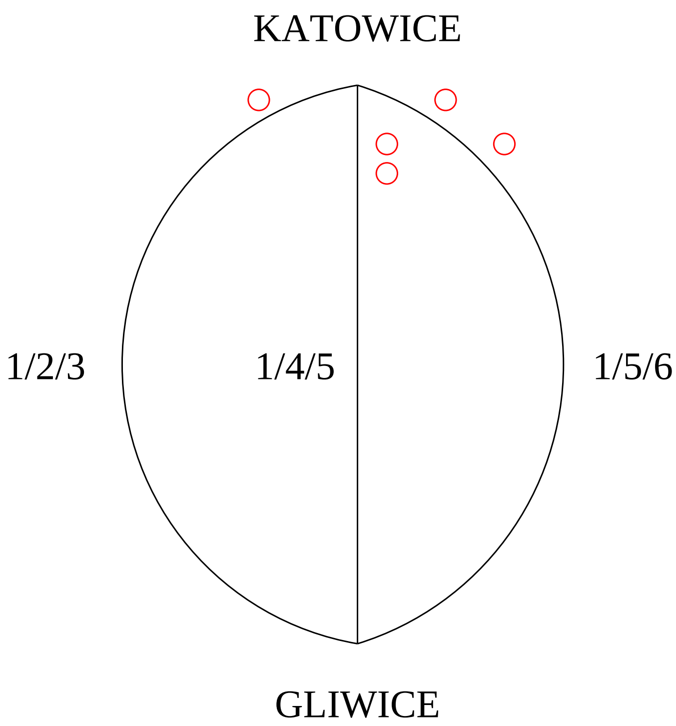
- 5 players,
- 3 facilities (roads),
- each strategy: (a singleton set consisting of) a road,
- cost function:

$$c_i(s) := \begin{cases} 1 & \text{if } s_i = 1 \text{ and } |\{j \mid s_j = 1\}| = 1 \\ 2 & \text{if } s_i = 1 \text{ and } |\{j \mid s_j = 1\}| = 2 \\ 3 & \text{if } s_i = 1 \text{ and } |\{j \mid s_j = 1\}| \geq 3 \\ 1 & \text{if } s_i = 2 \text{ and } |\{j \mid s_j = 2\}| = 1 \\ \ldots & \\ 6 & \text{if } s_i = 3 \text{ and } |\{j \mid s_j = 3\}| \geq 3 \end{cases}$$

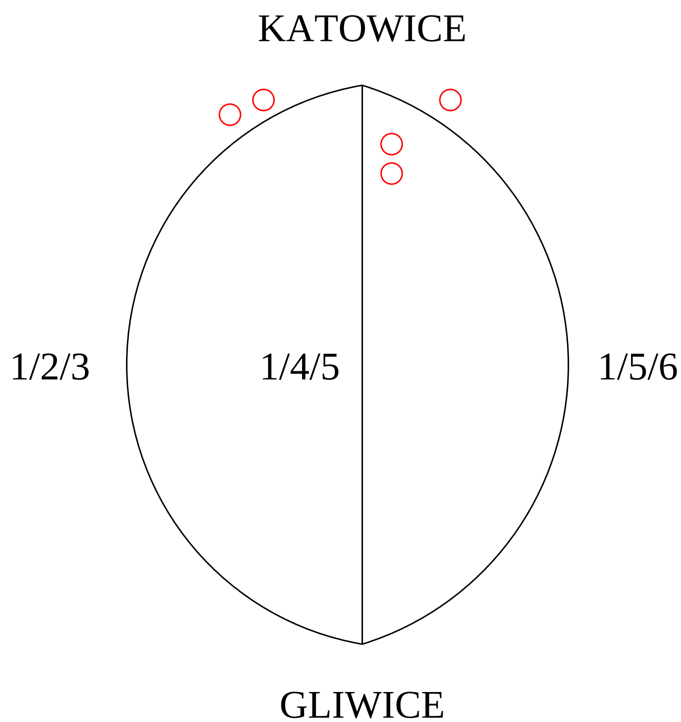
Possible evolution (1)



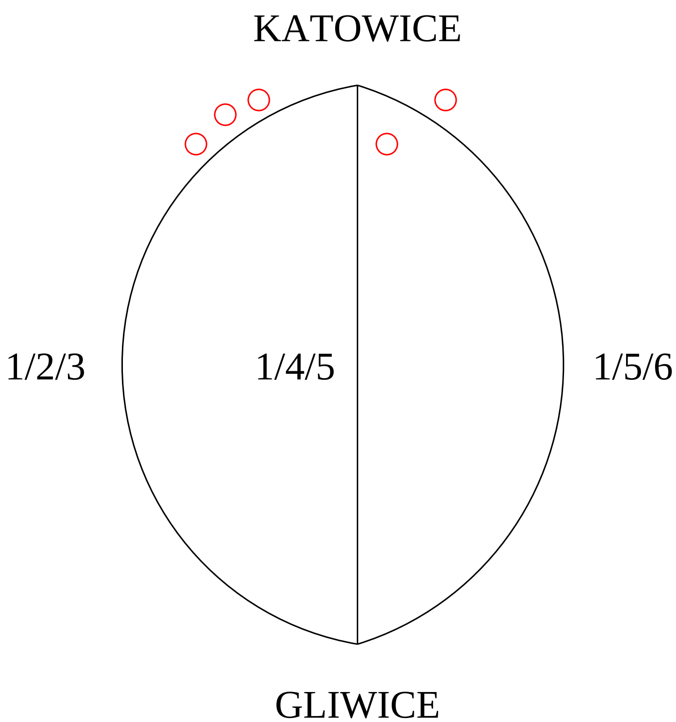
Possible evolution (2)



Possible evolution (3)



Possible evolution (4)



So we reached a Nash equilibrium using the best response dynamics.

Congestion Games, ctd

Theorem (Rosenthal, '73)
Every congestion game is a potential game.

Proof.
Given a joint strategy s we define $\cup s := \cup_{i=1}^{n} s_i$.

$$P(s) := \sum_{r \in \cup s} \sum_{k=1}^{\textit{users}(r,s)} \textit{cost}_r(k),$$

where (recall)

$$\textit{users}(r, s) = |\{i \in \{1, \ldots, n\} \mid r \in s_i\}|$$

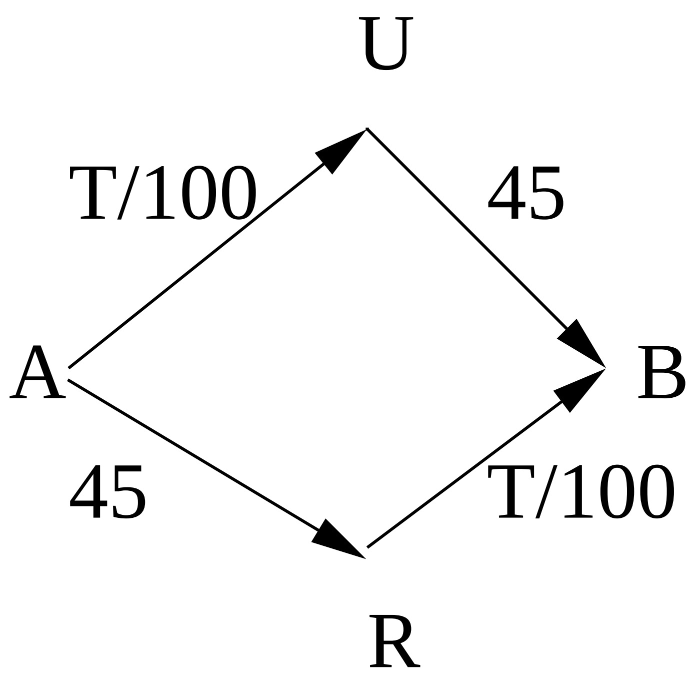
is a potential function.

Conclusion Every congestion game has a Nash equilibrium.

Another Example

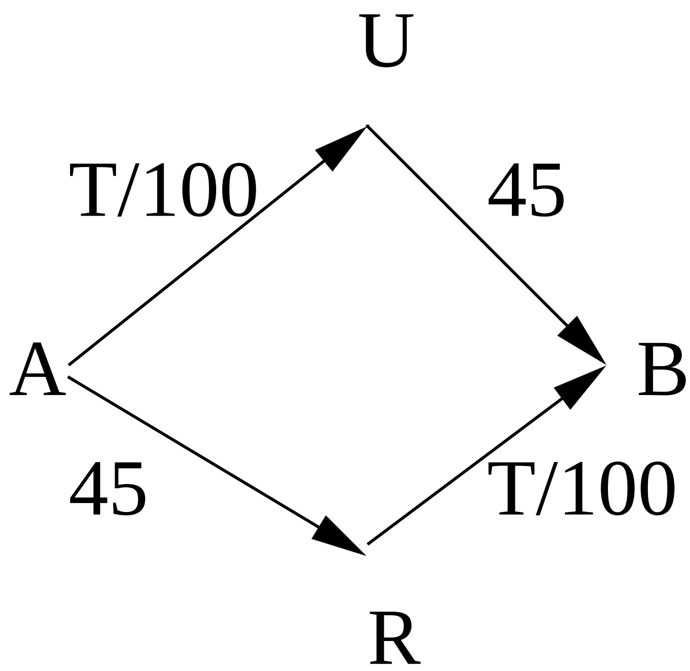
Assumptions:

- 4000 drivers drive from A to B.
- Each driver has 2 possibilities (strategies).



Problem: Find a Nash equilibrium (T = number of drivers).

Nash Equilibrium

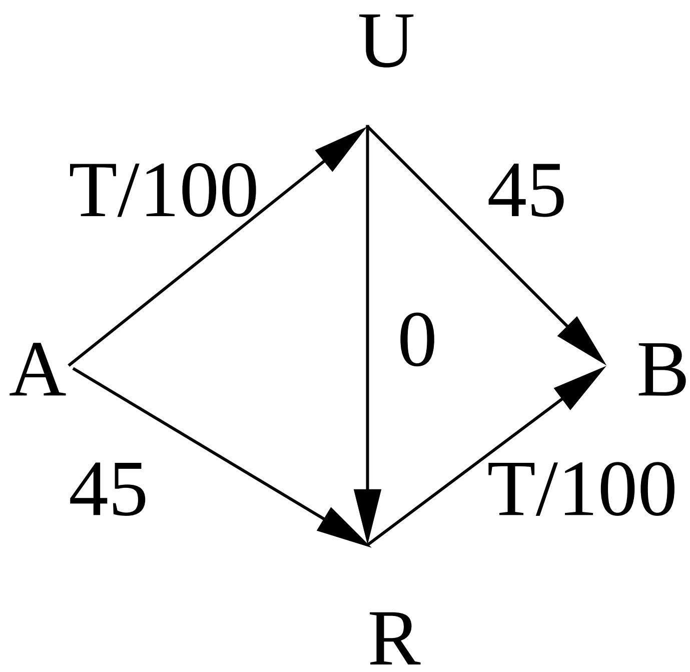


Answer: 2000/2000.

Travel time: 2000/100 + 45 = 45 + 2000/100 = 65.

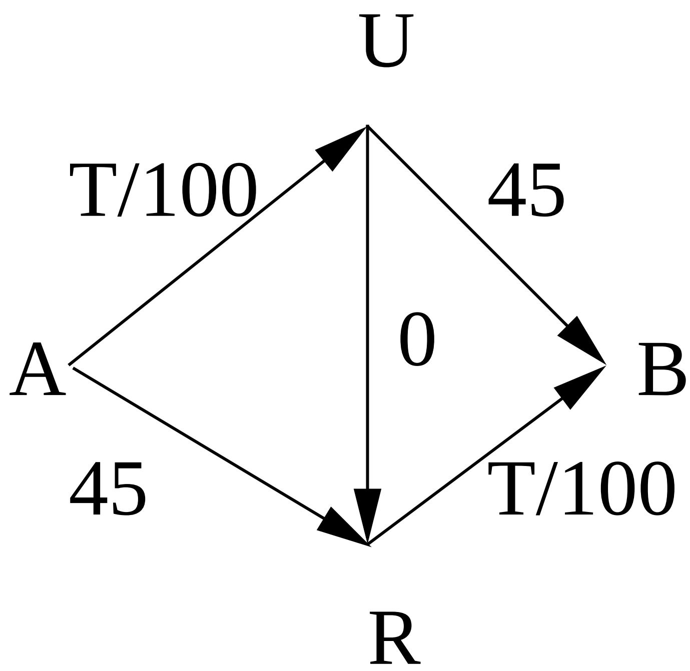
Braess Paradox

- Add a fast road from U to R.
- Each drives has now 3 possibilities (strategies):
  A - U - B,
  A - R - B,
  A - U - R - B.



Problem: Find a Nash equilibrium.

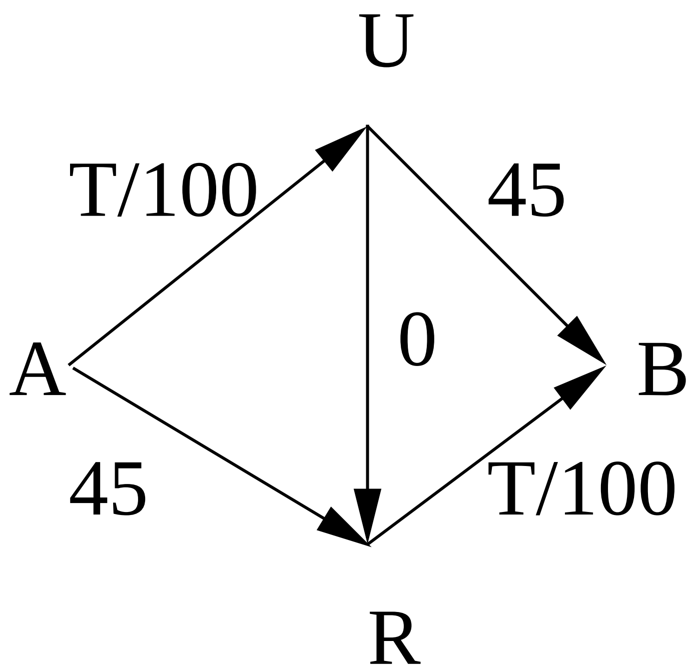
Nash Equilibrium



Answer: Each driver will choose the road A - U - R - B.

Why?: The road A - U - R - B is always a best response.

Small Complication



Travel time: 4000/100 + 4000/100 = 80!

Does it Happen?

From Wikipedia ('Braess Paradox'):

- In Seoul, South Korea, a speeding-up in traffic around the city was seen when a motorway was removed as part of the Cheonggyecheon restoration project.
- In Stuttgart, Germany after investments into the road network in 1969, the traffic situation did not improve until a section of newly-built road was closed for traffic again.
- In 1990 the closing of 42nd street in New York City reduced the amount of congestion in the area.
- In 2008 Youn, Gastner and Jeong demonstrated specific routes in Boston, New York City and London where this might actually occur and pointed out roads that could be closed to reduce predicted travel times.

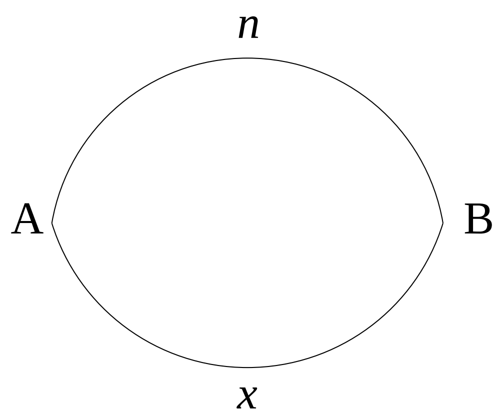
Price of Stability

Definition

PoS: $\dfrac{\text{social welfare of the best Nash equilibrium}}{\text{social welfare of the social optimum}}$

Question: What is PoS for congestion games?

Example

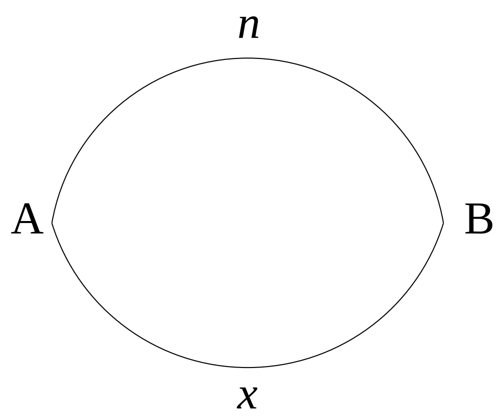


n - even number of players.

x - number of drivers on the lower road.

- Two Nash equilibria
  $1/(n-1)$, with social welfare $n+(n-1)^2$.
  $0/\,n$, with social welfare n^2.
- Social optimum
  Take $f(x) = x \cdot x + (n-x) \cdot n = x^2 - n \cdot x + n^2$.
  We want to find the minima of f.
  $f'(x) = 2x - n$, so $f'(x) = 0$ if $x = \frac{n}{2}$.

Example



- Best Nash equilibrium
  $1/(n-1)$, with the social welfare $n + (n-1)^2$.
- Social optimum
  $f(x) = x^2 - n \cdot x + n^2$.
  Social optimum = $f(\frac{n}{2}) = \frac{3}{4}n^2$.
- PoS = $\frac{(n+(n-1)^2)}{\frac{3}{4}n^2} = \frac{4}{3}\frac{n+(n-1)^2}{n^2}$.
- $\lim_{n\to\infty}$ PoS = $\frac{4}{3}$.

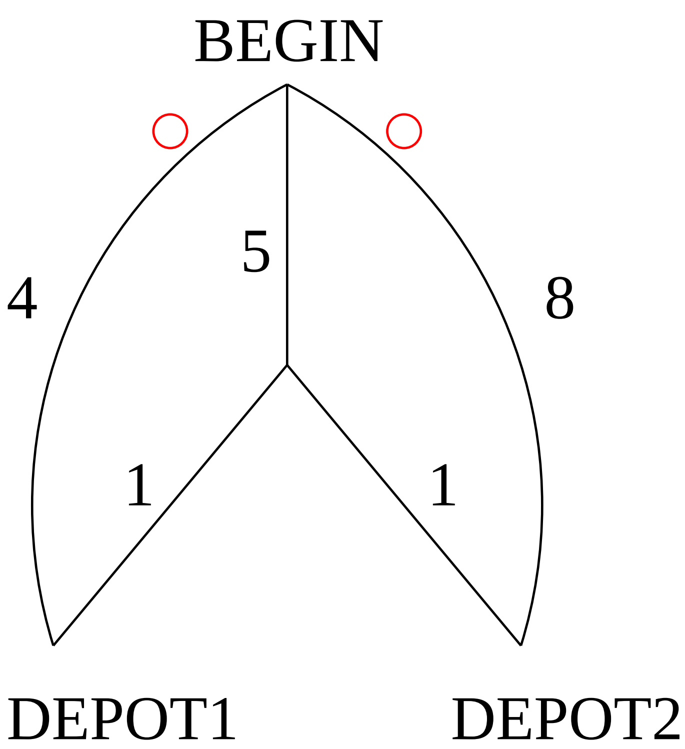
Price of Stability

- Theorem (Roughgarden and Tárdos, 2002)
  Assume the delay functions (for example $T/100$) are linear.
  Then PoS for the congestion games is $\leq \frac{4}{3}$.
- A good Nash equilibrium can be reached using the best response dynamics.
- Unfortunately: it can take exponentially long before the equilibrium is reached.
- Open problem: what is the PoS for arbitrary congestion games?

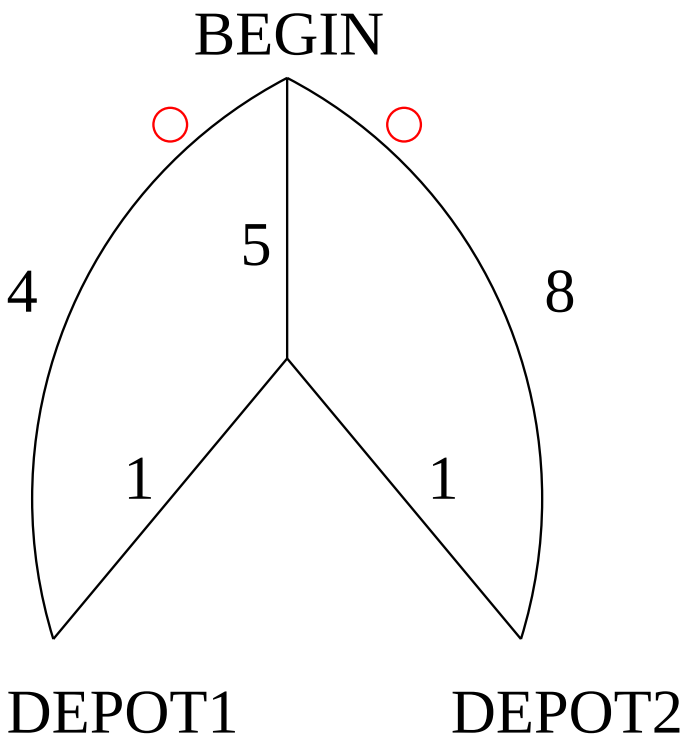
Fair Cost Sharing Games

Example

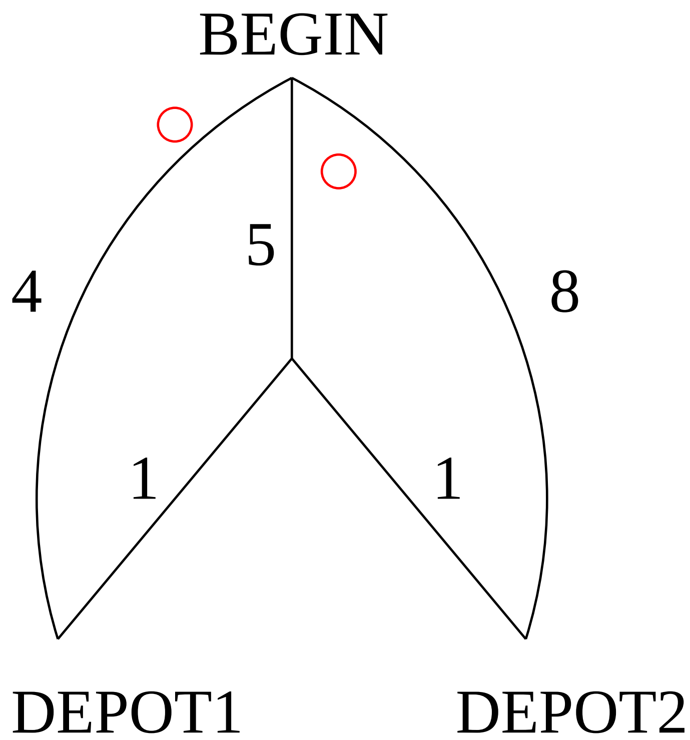
- 2 drivers.
- Each driver chooses a route from BEGIN to his depot.
- More drivers choose the same road segment $\Rightarrow$ the costs are shared.



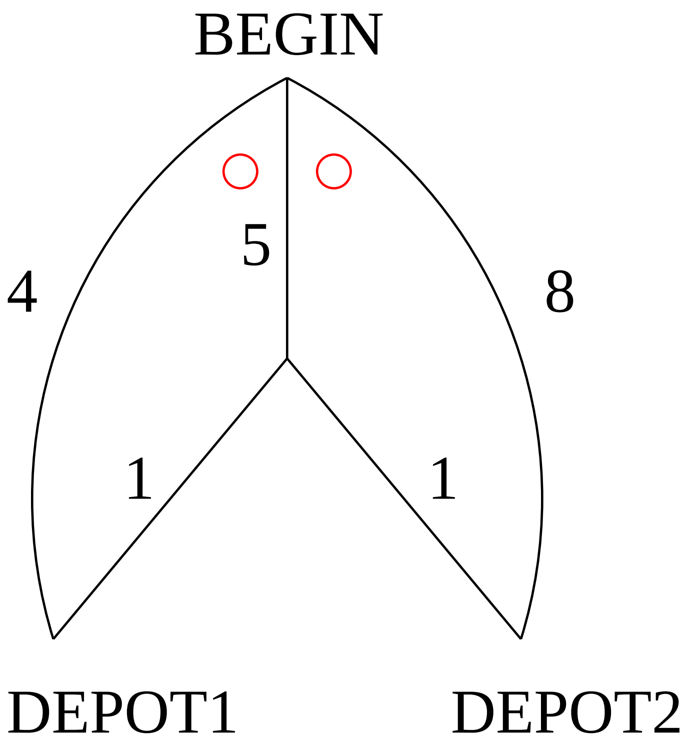
Possible evolution (1)



Possible evolution (2)



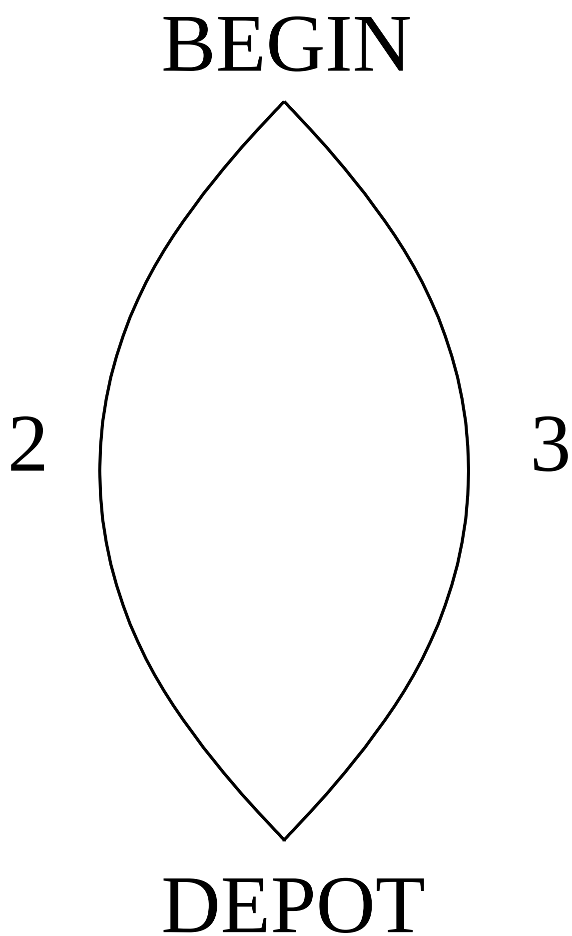
Possible evolution (3)



A Nash equilibrium has been reached.

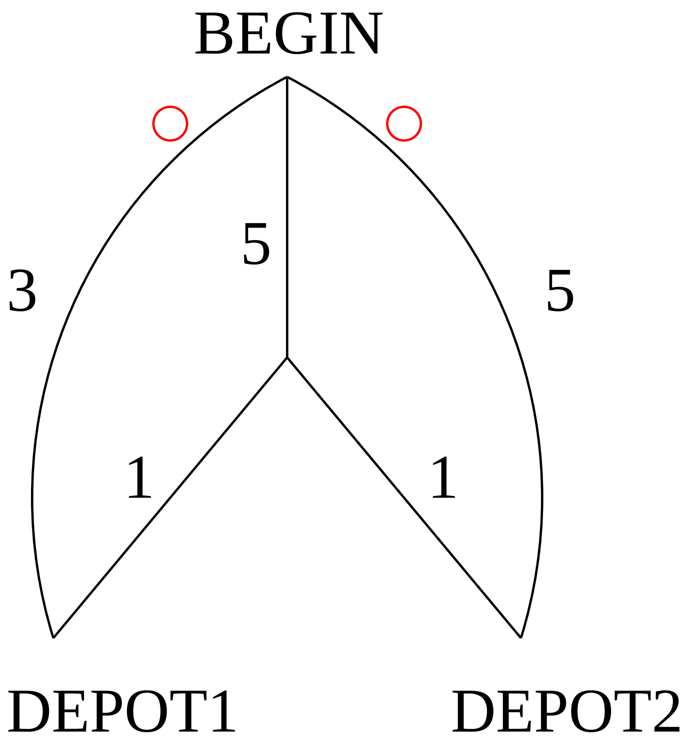
Nash equilibria are not unique

Example



Social Optimum

Example



- Unique Nash equilibrium, with the total cost 8.
- Total cost in social optimum: 7.

General Case

- Assume a finite, non-empty set of resources R.
  Each resource $r \in R$ has a fixed, strictly positive cost *cost*$_r$.
- A strategy is a non-empty set of resources.
  Each player i has a set of strategies S_i, so a set of subsets of R.
- Example: A strategy for player i:
  a path from BEGIN to DEPOT$_i$.
- Recall
  users$(r, s) = |\{i \in \{1, \ldots, n\} \mid r \in s_i\}|$
  is the number of users of resource r in s,
- We define

$$c_i(s) := \sum_{r \in s_i} \textit{cost}_r / \textit{users}(r, s).$$

Fair Cost Sharing Games (2)

- Given a joint strategy s we define $\cup s := \cup_{i=1}^{n} s_i$.
  $\cup s$ is the set of resources used in s.
- Note: Social optimum is a joint strategy s for which
  $\sum_{r \in \cup s} \textit{cost}_r$ is minimal.
  Proof.
  $\sum_{i=1}^{n} c_i(s) = \sum_{r \in \cup s} \textit{cost}_r$.
  That is, the social cost of s is the aggregate cost of the resources used in s.

Theorem
Every fair cost sharing game is a potential game.

Price of Stability (1)

Harmonic numbers

- $H(n) = 1 + 1/2 + \ldots + 1/n$.
- Theorem (Oresme, around 1350)

$$\lim_{n \longrightarrow \infty} H(n) = \infty.$$

Proof

$$\begin{aligned}
& 1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + \ldots \\
= \; & 1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + \ldots \\
> \; & 1 + 1/2 + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) + \ldots \\
= \; & 1 + 1/2 + \qquad 1/2 \qquad + \qquad\qquad 1/2 \qquad\qquad + \ldots
\end{aligned}$$

Price of Stability (2)

Theorem (Anshelevich et al, 2004)
The PoS for the fair cost sharing games for n players is $\leq H(n)$.