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A moving boundary problem motivated by electric breakdown, I: Spectrum of linear perturbations

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ABSTRACT

An interfacial approximation of the streamer stage in the evolution of sparks and lightning can be written as a Laplacian growth model regularized by a 'kinetic undercooling' boundary condition. We study the linear stability of uniformly translating circles that solve the problem in two dimensions. In a space of smooth perturbations of the circular shape, the stability operator is found to have a pure point spectrum. Except for the eigenvalue $\lambda_0 = 0$ for infinitesimal translations, all eigenvalues are shown to have negative real part. Therefore perturbations decay exponentially in time. We calculate the spectrum through a combination of asymptotic and series evaluation. In the limit of vanishing regularization parameter, all eigenvalues are found to approach zero in a singular fashion, and this asymptotic behavior is worked out in detail. A consideration of the eigenfunctions indicates that a strong intermediate growth may occur for generic initial perturbations. Both the linear and the nonlinear initial value problem are considered in a second paper.

1. Introduction

The motion of interfaces in a Laplacian field is of general interest and has been a subject of intense study over many years (see for instance the reviews [1,2]). Such problems arise in many physical contexts, such as viscous fingering in multi-phase fluid flow [3–16], dendritic crystal growth in the quasi-steady small Peclet number limit [17–20], void electromigration [21–25] and a host of other phenomena such as the growth of biological systems like bacterial colonies or corals [26].

More recently, a similar mathematical problem has been discovered in the study of 'streamers' [27–34] which occur during the initial stage of electric breakdown and play an important role both in the natural phenomena of sparks and lightning as well as in numerous technical applications [33]. Streamers are weakly ionized bodies growing into some nonionized medium due to an externally applied electric field. This field is so strong that the drifting electrons very efficiently create additional electron ion pairs by impact ionization, and the nonlinear coupling between ionized body and field further increases this effect.

Models for negative streamers in simple gases like nitrogen or argon are based on a set of partial differential equations for the densities of electrons and of positive ions coupled to the electric field [28–33]. Analysis and numerical solutions of these equations reveal that in the front part of the streamer, a thin surface charge layer develops where the electron density strongly exceeds the ion density. Therefore the electric field $\mathbf{E}$ varies strongly when crossing this layer. Right before the layer, it is enhanced, but in the interior of the streamer, it is screened to such a low level that impact ionization is suppressed and the electron current transporting charge from the interior to the surface charge layer is small. Consequently we may take the interior as being essentially passive, and the growth of the streamer is governed by the surface charge layer which is driven by the strong local field.

If the external field is very strong, the thickness ℓ of the surface charge layer can become small compared to the typical diameter $2R$ of the streamer [34]. This suggests modeling this layer as an interface separating the ionized interior from the nonionized exterior region. In this model the variation of the potential φ across the surface charge layer is replaced by a discontinuity on the interface. Since the interior is considered passive, only the limiting value φ^+ reached by approaching the interface from the outside is relevant for the dynamical evolution, and analysis of results of the PDE-model suggests [34–36] that with an appropriate gauge, φ^+ is coupled to the limiting value $\mathbf{E}^+$ of the electric field by the

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boundary condition

$$\varphi^+ = -\ell \mathbf{n} \cdot \mathbf{E}^+. \quad (1)$$

Here $\mathbf{n}$ is the outward normal on the interface, ℓ is the regularization length corresponding to the interface thickness, and $\mathbf{E}^+ = -\nabla\varphi^+$, where the $^+$ again indicates the limit of approaching the interface from the outside. In the context of dendritic crystal growth, this boundary condition is equivalent to including the kinetic undercooling effect[1] while excluding the usual Gibbs–Thompson surface energy correction to the melting temperature.

As the motion of the interface is caused by the drift of the electrons in the local electric field $\mathbf{v} = -\mathbf{E}$, the interface moves with normal velocity

$$v_n = -\mathbf{n} \cdot \mathbf{E}^+, \quad (2)$$

and outside the streamer the potential obeys the Laplace equation

$$\Delta\varphi = 0. \quad (3)$$

We discuss the problem defined by Eqs. (1)–(3) in infinite two-dimensional space, with the electric field becoming constant, $\mathbf{E} = -\nabla\phi \to \mathbf{E}_\infty$, far from the streamer; such a condition is realized frequently in atmospheric discharges, e.g., inside thunderclouds. This far-field condition on $\nabla\phi$ is different from the usual source/sink condition in viscous fingering in the absence of side-walls, or undercooling specification in the quasi-steady low Peclet number crystal growth problem which have been extensively studied. However, moving bubbles and fingers in a long Hele–Shaw channel are indeed subjected to this type of far-field condition. (For the discussion of the streamer problem equivalent to the Saffman-Taylor finger, we refer to [38].) Further, in the crystal growth or directional solidification problem, a systematic inner-outer analysis for small Peclet number [17,20] shows that this is an appropriate condition at ∞ for the "inner" problem.

A simple steady solution to the streamer equations given above is a circle translating with constant velocity determined by $\mathbf{E}_\infty$ [35,36]. Though such circles differ from proper streamers, which are growing channels of ionized matter [33,38–40], their front half closely resembles the head of the streamer where the growth takes place. It is therefore a question of physical interest whether or not translating circular solutions are stable to small perturbations and this is the subject of the present investigation.

The relevance of this analysis for more realistic streamer shapes is supported by results found in another physical context. Steadily translating circles also arise in viscous fingering in a Hele–Shaw cell when surface tension is included (instead of kinetic undercooling) in the limit when the bubble is small compared to the cell-dimensions [3]. The linear stability of these bubbles, including larger non-circular steadily translating bubbles, has been studied before both for one and two fluids [10,14] and the results largely mimic those obtained for a finger, though the latter calculations are mathematically much more involved.

It has been known for a while that in the absence of any regularization, such as surface tension or kinetic undercooling, the initial value problem in a Laplacian field is ill-posed [9] in any norm that is physically relevant to describing interfacial features. This is reflected in the instability of any steady shape, when the growth rates increase with the wave numbers of the disturbances. Ill-posedness makes idealized model predictions sometimes physically irrelevant (see [15] for a thorough discussion) and regularization becomes essential.

Considering a *planar* front, one finds that regularization does not remove the instability against fluctuations of small wave number. However, for large wave numbers, linear stability analysis exhibits a basic difference between surface tension and kinetic undercooling. All large wave number components of a disturbance decay with surface tension regularization, while for kinetic undercooling the growth rate saturates to a constant that scales as ℓ^{-1} [16]; for streamers, such a saturating dispersion relation is derived and discussed in [41,42].

For *curved* fronts, one can pose the question: how does curvature stabilize, if at all, a disturbance whose wavenumber is in the unstable regime for a flat interface, either with surface tension or with kinetic undercooling regularization? With surface tension regularization, some answers are available in the existing literature. Arguments have been presented [4,6] that suggest that a localized wave packet with wave numbers in the unstable regime[2] advects along the front as it grows; once it reaches the side of the front where the local normal velocity is zero, the disturbance stops growing. If the steady shape is closed, as it is for a circle, the continued advection of the wave-packet towards the receding parts of the interface will cause the disturbance to decay eventually.[3] If regularization is small, there is a large transient growth. Unless the disturbance amplitude is smaller than a threshold that shrinks to zero with regularization, the transient exponential growth causes the interface to enter a nonlinear regime that can destabilize the steady front, even when it is predicted to be linearly stable. Analysis of approximate equations, supported by numerical calculations of the full equations support the above scenario. Similar stabilization should occur for the kinetic undercooling boundary condition as well, though we are not aware of any explicit study affirming this expectation.

Note, however, that stabilization of localized wave packets does not rule out instability to long ranged disturbances. A formal asympotic study for small nonzero surface tension [12,14] as well as numerical studies [7] reveal that surface tension stabilizes precisely one branch of steady solutions for fingers and bubbles in a Hele–Shaw cell. Similar results follow for a needle crystal [18] though in the latter case, convective instability of wave packets caused by significant normal speed along the parabolic front is believed to cause dendritic structures [1]. These conclusions have been challenged at times by alternate scenarios (see for instance [19]) that are based on formal calculations, but with different implicit assumptions. Such controversy affirms the need for more rigorous mathematical studies of the stability problem, even if it is for relatively simple shapes such as the circle in the present study.

For the kinetic undercooling boundary condition, relying merely on a numerical study to understand the long time behavior is fraught with difficulties. One finds a collapsing spatial scale for large time at the rear of the circle. Analytically, this is found for $\epsilon = \ell/R = 1$ in [35,36].[4] As will be argued in the present and the companion paper, the occurrence of this collapsing scale is a general feature for any $\epsilon > 0$. This means that as $t \to \infty$, one must resolve progressively finer scales near the back of the bubble. Further, calculations for small ϵ require resolving a large number of transiently growing modes. All this underscores the need of some progress on the analytical side.

1 Under most natural conditions of crystal growth, kinetic undercooling is important in a limit when the Peclet number is not small enough to justify a Laplacian field approximation; nonetheless, there have been some studies of steady Laplacian crystal growth with kinetic undercooling effects only [37].

2 Localized disturbances refer to those with wavelengths far smaller than the typical radius of curvature of the steady shape. These can be unstable only if the regularization parameter is sufficiently small.

3 Surface tension causes localized disturbances to decay as they advect to the sides even when an interface is not closed but becomes parallel to the direction of motion as is the case for a finger in a Hele–Shaw cell. However, no decay is expected for kinetic regularization. This is where a closed interface is different.

4 We recall that R is a measure of the size of the streamer. The precise definition is given in Eq. (5).

The present paper, which is part I of a two-paper sequence, is devoted to the spectral properties of the linear stability operator, associated with infinitesimal perturbations of a circle. In part II [43], we will consider the initial value problem, presenting analytical and numerical results on the evolution of both infinitesimal and finite perturbations.

The present paper is organized as follows. In Section 2 we reformulate the problem defined by Eqs. (1)–(3) by standard conformal mapping, and we present the PDE governing the time evolution of infinitesimal perturbations of the circle. This material has been presented before in [35,36], where also the general solution of the PDE in the case $\epsilon = 1$ has been discussed in detail. The explicit solution found for $\epsilon = 1$ shows that outside any fixed neighborhood of the rear of the bubble, the long-term behavior of infinitesimal perturbations is described by $\sum_{n=0}^{\infty} e^{\lambda_n t}\beta_{\lambda_n}$, where λ_n is the nth eigenvalue (ordered according to absolute value) of the linear stability operator and β_{λ_n} is the corresponding eigenfunction.

We then study this eigenvalue value problem for arbitrary $\epsilon > 0$. We show in Section 3 that the linear stability operator, defined in an appropriate space of analytic functions, has a pure point spectrum. In Section 4 it is proven that there are no discrete eigenvalues with non-negative real part, except $\lambda = 0$ that corresponds to the trivial translation mode. A set of discrete, purely negative eigenvalues is calculated in Section 5 as a function of ϵ; they smoothly extend the results found previously for $\epsilon = 1$. The results suggests that as $\epsilon \to 0$, the spectrum degenerates to the trivial translation mode and this limit is discussed in detail in Section 6. Section 7 contains a discussion of the eigenfunctions belonging to these eigenvalues, and Section 8 contains the conclusions. Some part of our analysis exploits general results on the asymptotic behavior of the coefficients of Taylor expansions. These results are presented in an Appendix.

2. Reformulation by conformal mapping

In this section, we collect results and notations from [35,36] that will be used in later sections.

2.1. Problem formulation and rescaling

We consider a compact ionized domain $\mathcal{D}$ in the (x, y)-plane. We assume that the net charge on the domain vanishes (i.e., it contains the same number of electrons and positive ions). The domain moves in an external field that far from the domain asymptotically approaches

$$\mathbf{E}_\infty = -|\mathbf{E}_\infty|\,\hat{\mathbf{x}}. \tag{4}$$

Here $\hat{\mathbf{x}}$ is the unit vector in x-direction, and $|\mathbf{E}_\infty|$ sets the scale of $\mathbf{E}$ and thus of the potential φ. As length scale we take

$$R = \sqrt{\frac{|\mathcal{D}|}{\pi}}. \tag{5}$$

where $|\mathcal{D}|$ is the area of $\mathcal{D}$, which is known to be conserved. This follows from the charge neutrality of the streamer since

$$0 = \int_{\mathcal{D}} dx\,dy\,\nabla\cdot\mathbf{E} = \int_{\partial\mathcal{D}} ds\,\mathbf{n}(s)\cdot\mathbf{E} = -\int_{\partial\mathcal{D}} ds\,v_n, \tag{6}$$

where in the last step we inserted Eq. (2) for the normal velocity of the boundary. Since

$$\int_{\partial\mathcal{D}} ds\,v_n = \partial_t|\mathcal{D}|, \tag{7}$$

the area is conserved, irrespective of the precise charge distribution in the interior.[5] Also introducing the time scale $R/|E_\infty|$, we rescale the basic equations to the dimensionless form

$$\Delta\varphi = 0, \quad (x, y) \notin \mathcal{D} \tag{8}$$

$$v_n = \mathbf{n}\cdot(\nabla\varphi)^+ \tag{9}$$

$$\varphi^+ = \epsilon\,\mathbf{n}\cdot(\nabla\varphi)^+. \tag{10}$$

The only remaining parameter in the rescaled problem is

$$\epsilon = \ell/R. \tag{11}$$

The boundary condition at infinity after rescaling takes the form

$$\varphi \to x + \text{const} \quad \text{for } \sqrt{x^2+y^2} \to \infty. \tag{12}$$

2.2. Conformal mapping

We now identify the physical (x, y)-plane with the closed complex plane $z = x + iy$, and we introduce a conformal map $f(\omega, t)$ that maps the unit disk $\mathcal{U}_\omega$ in the ω-plane to the complement of $\mathcal{D}$ in the z-plane, with $\omega = 0$ being mapped on $z = \infty$,

$$z = f(\omega, t) = \frac{a_{-1}(t)}{\omega} + \hat{f}(\omega, t), \quad a_{-1}(t) > 0. \tag{13}$$

We further define a complex potential $\Phi(\omega, t)$ obeying

$$\mathrm{Re}[\Phi(\omega, t)] = \varphi(f(\omega, t)) \quad \text{for } \omega \in \mathcal{U}_\omega. \tag{14}$$

The boundary condition (12) and the Laplace Eq. (8) enforce the form

$$\Phi(\omega, t) = \frac{a_{-1}(t)}{\omega} + \hat{\Phi}(\omega, t) \tag{15}$$

with $\hat{\Phi}$ being holomorphic for $\omega \in \mathcal{U}_\omega$.

The two boundary conditions (9) and (10) take the form

$$\mathrm{Re}\left[\frac{\partial_t f}{\omega\partial_\omega f}\right] = \mathrm{Re}\left[\frac{\omega\partial_\omega\Phi}{|\partial_\omega f|^2}\right] \quad \text{for } \omega \in \partial\,\mathcal{U}_\omega, \tag{16}$$

$$|\partial_\omega f|\,\mathrm{Re}[\Phi] = -\epsilon\,\mathrm{Re}[\omega\partial_\omega\Phi] \quad \text{for } \omega \in \partial\,\mathcal{U}_\omega, \tag{17}$$

which completes the reformulation of the moving boundary problem (8)–(12) by conformal mapping.

We will restrict the analysis here to initial conditions $\hat{f}(\omega, 0)$ holomorphic in some domain $\mathcal{U}'_0 \supset \mathcal{U}_\omega$. In part II [43] of this paper sequence, we will give evidence that analyticity on $\overline{\mathcal{U}}_\omega$ is preserved in time, though the distance of the domain of analyticity $\mathcal{U}'_t$ to $\partial\mathcal{U}_\omega$ shrinks with time. The streamer boundary $\partial\mathcal{D}$, which is the image of boundary $\partial\mathcal{U}_\omega$ under $f(\omega, t)$, will turn out to be analytic and therefore smooth. Similar analytic representations exist for the entire class of 2-D Laplacian growth, with details depending on the type of boundary condition, geometry and asymptotic conditions at infinity. For the classic viscous fingering problem, Polubarinova-Kochina [44] and Galin [45] use a representation that coincides with the one given above in the unregularized case $\epsilon = 0$.

2.3. Linear perturbation of moving circles

It is easily seen that Eqs. (16) and (17) allow for the simple solution

$$f^{(0)}(\omega, t) = \frac{1}{\omega} + \frac{2t}{1+\epsilon}, \qquad \Phi^{(0)}(\omega, t) = \frac{1}{\omega} - \frac{1-\epsilon}{1+\epsilon}\,\omega, \tag{18}$$

which in physical space describes circles of radius 1 moving with constant velocity $2/(1+\epsilon)$ in x direction. (We recall that the radius was scaled to unity in Section 2.1.) We note that relaxing

[5] We remark that the argument is straight forward to generalize to three spatial dimensions. Therefore the volume of a charge neutral object with surface velocity $\mathbf{v} \propto \mathbf{E}^+$ in three spatial dimensions is conserved as well.

the analyticity conditions on $f(\omega, t)$ on $|\omega| = 1$, one can obtain another set of uniformly translating solutions, as recently discovered [46]. The present paper is restricted to perturbations of the steady circle that retain the imposed analyticity of the streamer shapes, and hence analyticity of f (as well as Φ) on $|\omega| = 1$.

As the area is conserved (as shown in Section 2.1), the residue $a_{-1} = 1$ does not change to linear order in the perturbation. We therefore can use the ansatz

$$f(\omega, t) = f^{(0)}(\omega, t) + \eta\, \beta(\omega, t),$$
$$\Phi(\omega, t) = \Phi^{(0)}(\omega, t) + \eta\, \frac{2}{1+\epsilon}\chi(\omega, t), \tag{19}$$

where η is a small parameter, and $\beta(\omega, t)$, $\chi(\omega, t)$ are holomorphic in $\mathcal{U}_\omega$. A first order expansion of Eqs. (16) and (17) in η yields the following boundary conditions for the analytic functions $\beta(\omega, t)$ and $\chi(\omega, t)$ on $|\omega| = 1$:

$$\mathrm{Re}[\omega\partial_\tau\beta - \omega\partial_\omega\beta] = \mathrm{Re}[-\omega\partial_\omega\chi],$$
$$\frac{\epsilon}{2}\mathrm{Re}\left[\left(\omega + \frac{1}{\omega}\right)\omega^2\partial_\omega\beta\right] = \mathrm{Re}[\epsilon\omega\partial_\omega\chi + \chi], \tag{20}$$

where we rescaled time as

$$\tau = \frac{2}{1+\epsilon}\, t. \tag{21}$$

Since the left and right sides of each of the two equations in (20) are real parts of analytic functions and each is assumed *a priori* continuous up to the boundary, they can differ everywhere in ω by at most an imaginary constant. Evaluation at $\omega = 0$ shows this constant to be zero for the first of the two equations. Elimination of χ results in the linear PDE:

$$\mathcal{L}_\epsilon\, \beta = 0 \tag{22}$$

with the operator

$$\mathcal{L}_\epsilon = \frac{\epsilon}{2}\,\partial_\omega\,(\omega^2 - 1)\,\omega\,\partial_\omega + \epsilon\,\partial_\omega\,\omega\,\partial_\tau + \partial_\tau - \partial_\omega. \tag{23}$$

We note that $\mathcal{L}_\epsilon$ is of similar structure as the operator resulting from a linear stability analysis of translating circles in the context of void electromigration [22,24]. The main difference here is the occurrence of the mixed derivative $\partial_\omega\,\omega\,\partial_\tau$.

2.4. Formulation of the eigenvalue problem

To motivate our formulation of the eigenvalue problem, we note some results on the temporal evolution of infinitesimal perturbations. In [36], the equation $\mathcal{L}_\epsilon\,\beta = 0$ was solved as an initial value problem for the special value $\epsilon = 1$. It was found that any initial perturbation $\beta(\omega, 0)$ holomorphic in $\mathcal{U}' \supset \mathcal{U}_\omega$ for $\tau \to \infty$ is exponentially convergent to some constant. This results from the expansion

$$\beta(\omega, \tau) = \sum_{n=0}^{\infty} g_n\, \beta^{(1)}_{\lambda_n}(\omega)\, \mathrm{e}^{\lambda_n\tau}, \tag{24}$$

with

$$\lambda_n = -n, \quad n \in \mathbb{N}_0, \text{ for } \epsilon = 1. \tag{25}$$

The coefficients g_n and the eigenfunctions[6]

$$\beta^{(1)}_{\lambda_n}(\omega) = \int_0^\omega \frac{x\,\mathrm{d}x}{\omega^2}\left(\frac{x-1}{x+1}\right)^{-\lambda_n} \tag{26}$$

[6] Note that the exponent $-\lambda_n$ in (26) is correct while $+\lambda_n$ in Eq. (4.20) in [36] is a typo.

are determined by an expansion of $(2 + \omega\partial_\omega)\beta(\omega, 0)$ in powers of $(1-\omega)/(1+\omega)$. For $n > 0$ the eigenfunctions (26) are singular at $\omega = -1$, though $\beta(\omega, \tau)$ is not. The expansion (24) is convergent in a domain $\mathcal{D}_\tau$ expanding in time that eventually includes every point in $\bar{\mathcal{U}}_\omega \setminus \{-1\}$. For large τ the region where the expansion is invalid, shrinks to $\omega = -1$ exponentially. This region is measured by the new scale $\eta_1(\omega, \tau) = (1+\omega)\mathrm{e}^\tau$, and the expansion (24) is valid if η_1 is large. For $\eta_1 \le O(1)$ the perturbation for $\tau \to \infty$ behaves as $\beta(\omega, \tau) \to F_0(\eta_1) + O(\mathrm{e}^{-\tau})$ where F_0 is some analytic function of its argument, depending on $\beta(\omega, 0)$.

For an analytic initial condition on $\mathcal{U}'$, with a lone branch point singularity ω_s in $|\omega| > 1$ not on the positive real axis, the emergence of this new scale near $\omega = -1$ can be related to the approach of this complex singularity towards -1 exponentially in τ for large τ. Asymptotic arguments that will be presented in part II [43] suggest that this behavior is generic for all $\epsilon > 0$. The analysis is based on the linear the evolution equations for b_k, where

$$\beta(\omega, \tau) = \sum_{k=0}^{\infty} b_k(\tau)\omega^k.$$

For $k \gg \mathrm{e}^\tau$, we find the asymptotic relation

$$b_k \sim (-1)^k k^{-\alpha} h(\tau) \exp\left[-kf(\tau)\right],$$

where

$$f(\tau) = \log\left[\frac{1 + C\mathrm{e}^{-\tau}}{1 - C\mathrm{e}^{-\tau}}\right], \quad \text{with } C = \frac{\omega_s + 1}{\omega_s - 1}.$$

For $\omega_s \notin (1, \infty)$, $f(\tau)$ stays finite and approaches 0 exponentially in τ for large τ. If $\omega_s \in (1, \infty)$, $f(\tau)$ increases monotonically to ∞ for $\tau \in (0, \tau_c)$ where $\mathrm{e}^{-\tau_c} = 1/C$. For $\tau > \tau_c$, $f(\tau)$ decreases monotonically and approaches 0 exponentially in τ as $\tau \to \infty$. In either case, from the known relation between Taylor series coefficients and the location of the closest singularity of an analytic function (see the Appendix), it follows that $f(\tau) \sim \mathrm{e}^{-\tau}$ as $\tau \to \infty$ implies that β has a singularity approaching $\omega = -1$ exponentially in τ for large τ. This feature is retained for any other isolated initial singularities as well, though $k^{-\alpha}$ is replaced by a more complicated dependence in k. Since the problem is linear, the evolution of a distribution of initial singularities can be understood from the linear superposition principle.

This suggests that for any $\epsilon > 0$, as for $\epsilon = 1$, $\beta(\omega, \tau)$ has a collapsing scale $(1+\omega)\mathrm{e}^\tau$, and an expansion of the type (24) cannot be valid in this neighborhood of $\omega = -1$.

Thus, in seeking an eigenfunction by substituting

$$\beta(\omega, \tau) = \beta^{(\epsilon)}_\lambda(\omega)\,\mathrm{e}^{\lambda\tau}, \tag{27}$$

into (22) and (23), it is appropriate to allow $\beta^{(\epsilon)}_\lambda$ to be singular at $\omega = -1$. Indeed, substituting the form (27) reduces Eqs. (22) and (23) to the eigenvalue problem

$$L(\epsilon, \lambda)\;\; \beta^{(\epsilon)}_\lambda(\omega) = 0, \tag{28}$$

$$L(\epsilon, \lambda) = \frac{\epsilon\,(\omega^2-1)\omega}{2}\,\partial^2_\omega + \left(\frac{\epsilon\,(3\omega^2-1)}{2} - 1\right)\partial_\omega + \lambda(1 + \epsilon + \epsilon\omega\partial_\omega). \tag{29}$$

Evidently this ODE has three regular singular points, namely $\omega = 0$ and $\omega = \pm 1$. The independent solutions at these points for $\epsilon > 0$ are in leading order

$$\beta^{(\epsilon)}_\lambda(\omega) \sim \begin{cases} \omega^0 \\ \omega^{-2/\epsilon} \end{cases} \text{for } \omega \to 0, \tag{30}$$

$$\beta^{(\epsilon)}_\lambda(\omega) \sim \begin{cases} (1 \mp \omega)^0 \\ (1 \mp \omega)^{1/\epsilon \mp \lambda} \end{cases} \text{for } \omega \to \pm 1, \tag{31}$$

We require the eigenfunctions β_λ^ϵ to be solutions of (28) that are analytic in $\omega = 0$ and $\omega = 1$. This is also the natural choice from a physical point of view since it is the right half of the circle, $\mathrm{Re}[\omega] > 0$, that corresponds to the physically interesting tip of the streamer. In general, eigenfunctions cannot be expected to be regular at all three points. Starting with a function regular at $\omega = 0$, we cannot generally require regularity at both points $\omega = \pm 1$ by adjusting the single parameter λ. As shown in Section 4.3, the only eigenfunction regular at all three points is the trivial translation mode

$$\lambda_0 = 0, \qquad \beta_0^{(\epsilon)}(\omega) = \text{const.} \tag{32}$$

As noted above, an operator similar to $\mathcal{L}_\epsilon$ (23) occurs in the problem of void electromigration, see section 4.1.3 in [24]. Again an eigenvalue analysis would yield a second order linear operator with three singular points at $\omega = 0$ and ± 1 and therefore the eigenmodes in general cannot be regular at all three singular points. It is interesting to note that the authors [24] conclude that their problem is unstable because the initial value problem for large time is singular at $\omega = -1$. In the current problem, the solution [43] of the initial value problem is not singular at $\omega = -1$; the singularity of the eigenfunctions does not reflect the true behavior of solution since, as has been pointed out earlier, there is an anomalous contracting scale $e^\tau(1+\omega)$ near the back of the bubble. Whether or not there is an analogous contracting scale for the void electromigration problem [24] remains an interesting question. This anomalous scale shows up when the limiting processes $\lim_{\omega\to-1}$ and $\lim_{\tau\to+\infty}$ do not commute for the solution of the initial value problem.

3. Discreteness of the spectrum

We define λ to be in the spectrum, if the linear operator $L(\epsilon,\lambda)$ does not have a bounded inverse in the class of functions f that are analytic in an arbitrary compact connected set $\mathcal{V} \subset \mathcal{U}' \setminus \{-1\}$ that contains the whole line $[0,1]$ in its interior. λ is in the discrete spectrum if (28) has a nonzero solution $\beta_\lambda^{(\epsilon)}(\omega)$ that is analytic in any such domain $\mathcal{V}$. We now argue that if λ is not in the discrete spectrum, then $L(\epsilon,\lambda)$ has a bounded inverse, i.e. there is only a discrete spectrum in this problem.

To determine L^{-1}, we solve the equation

$$L(\epsilon,\lambda)g = h \tag{33}$$

for a given h analytic in $\mathcal{V}$, imposing the condition that also g is analytic in $\mathcal{V}$. The solutions of the homogeneous equation $L(\epsilon,\lambda)f = 0$ that are regular at $\omega = 0$ or $\omega = 1$ will be denoted by $f_1(\omega)$ or $f_2(\omega)$, respectively. It follows from Eqs. (30) and (31) that these functions are determined uniquely up to a multiplicative constant. In the exceptional case where both independent solutions are regular at $\omega = 1$, λ belongs to the discrete spectrum, see Section 5. A standard calculation shows that Eq. (33) is solved by

$$g(\omega) = \frac{1}{C(\epsilon,\lambda)}\int_0^\omega d\omega'\, G(\omega,\omega')\, h(\omega') + a_1[h]\, f_1(\omega), \tag{34}$$

where

$$G(\omega,\omega') = \frac{\omega'^{2/\epsilon}}{(1-\omega')^{1/\epsilon-\lambda}(1+\omega')^{1/\epsilon+\lambda}} \times \left[f_2(\omega)f_1(\omega') - f_1(\omega)f_2(\omega')\right], \tag{35}$$

and the coefficient $a_1[h]$ is a functional of $h(\omega')$. $C(\lambda,\epsilon)$ does not vanish since otherwise the Wronskian $f_1\,\partial_\omega f_2 - f_2\,\partial_\omega f_1$ vanishes identically and λ is part of the discrete spectrum. It is easily seen that Eqs. (34) and (35) render $g(\omega)$ analytic in $\omega = 0$, and this condition eliminates any contribution of the form $a_2[h]\, f_2(\omega)$.

Analyticity at $\omega = 1$ is enforced by a proper choice of $a_1[h]$. To make the analysis explicit, in addition to $f_2(\omega)$, we introduce another solution to $L[\epsilon,\lambda]f = 0$ by requiring

$$f_3(\omega) = (1-\omega)^{1/\epsilon-\lambda}\,\hat f_3(\omega), \tag{36}$$

where $\hat f_3(\omega)$ is analytic at $\omega = 1$. Using this form of $f_3(\omega)$, we exclude the case $\frac{1}{\epsilon} - \lambda \in \mathbb{Z}^+$, that will be discussed later. Writing $f_1(\omega)$ as

$$f_1(\omega) = c_2 f_2(\omega) + c_3 f_3(\omega), \tag{37}$$

we find that $G(\omega,\omega')$ from Eq. (35) takes the form

$$G(\omega,\omega') = \frac{c_3\omega'^{\,2/\epsilon}}{(1+\omega')^{1/\epsilon+\lambda}} \times \left[f_2(\omega)\hat f_3(\omega') - \left(\frac{1-\omega'}{1-\omega}\right)^{\lambda-1/\epsilon}\hat f_3(\omega)f_2(\omega')\right].$$

Evidently the first part in the square brackets for $\omega \to 1$ yields a regular contribution to $g(\omega)$ from Eq. (34). The contribution to $\int G\, h$ that is singular in $\omega = 1$ has the form

$$-c_3 f_3(\omega)\int_0^\omega d\omega'\,(1-\omega')^{\lambda-1/\epsilon} H(\omega'),$$

where

$$H(\omega') = \frac{\omega'^{\,2/\epsilon}}{(1+\omega')^{1/\epsilon+\lambda}} f_2(\omega')\, h(\omega') \tag{38}$$

is regular at $\omega' = 1$. If $\mathrm{Re}\,\lambda - \frac{1}{\epsilon} > -1$, we can write

$$\begin{aligned} &- c_3 f_3(\omega)\int_0^\omega d\omega'\,(1-\omega')^{\lambda-1/\epsilon} H(\omega') \\ &= -c_3 f_3(\omega)\int_0^1 d\omega'\,(1-\omega')^{\lambda-1/\epsilon} H(\omega') \\ &\quad + c_3\hat f_3(\omega)\int_\omega^1 d\omega'\left(\frac{1-\omega'}{1-\omega}\right)^{\lambda-1/\epsilon} H(\omega'). \end{aligned} \tag{39}$$

The second part is regular at $\omega = 1$ and the singular first part is canceled by the choice

$$a_1[h] = \int_0^1 d\omega'\,(1-\omega')^{\lambda-1/\epsilon} H(\omega'). \tag{40}$$

We note that this result is valid also for $\lambda = \frac{1}{\epsilon} + n$, $n \in \mathbb{N}$, where $f_3(\omega)$ instead of being of the form (36) shows a logarithmic singularity.

If $-n > \mathrm{Re}\,\lambda - \frac{1}{\epsilon} > -n-1$, $n \in \mathbb{N}$, we carry through n subtractions of $H(\omega')$ at $\omega' = 1$, defining

$$\left[H(\omega')\right]_n = H(\omega') - \sum_{j=0}^{n-1} H_j\,(1-\omega')^j, \tag{41}$$

so that $\left[H(\omega')\right]_n \sim \text{const}\,(1-\omega')^n$. A short calculation shows that the singular part of $\int G\, h$ is canceled by the choice

$$a_1[h] = \int_0^1 d\omega'\,(1-\omega')^{\lambda-1/\epsilon}\left[H(\omega')\right]_n + \sum_{j=0}^{n-1}\frac{H_j}{\lambda - \frac{1}{\epsilon} + j + 1}. \tag{42}$$

The expressions above clearly remain valid when $\frac{1}{\epsilon} - \mathrm{Re}\,\lambda = n$, except when $\frac{1}{\epsilon} - \lambda = n$, a positive integer.

When $\frac{1}{\epsilon} - \lambda = n$ is a positive integer, from well-known theory [47] for regular singular points, instead of (31), the solutions f_1 and f_2 as defined earlier must have the following local representation near $\omega = 1$:

$$f_1(\omega) = C_1(1-\omega)^n B_1(\omega)\log(1-\omega) + B_2(\omega), \tag{43}$$

$$f_2(\omega) = (1-\omega)^n B_1(\omega), \tag{44}$$

where B_1 and B_2 are analytic at $\omega = 1$. If f_1 and f_2 are independent, as they are when λ is not in the discrete spectrum, then $C_1(\epsilon, \lambda) \neq 0$.

We now define

$$H(\omega) = B_2(\omega)(1+\omega)^{n-2/\epsilon}\omega^{2/\epsilon}h(\omega), \tag{45}$$

while H_j is still defined by the expression (41). It is also convenient to define

$$Q(\omega) = B_1(\omega)(1+\omega)^{n-2/\epsilon}\omega^{2/\epsilon}h(\omega), \tag{46}$$

Note that each of H and Q are analytic at $\omega = 1$. Straight forward calculation based on (34) shows that the possibly singular part of $g(\omega)$ at $\omega = 1$ is given by

$$-\frac{H_{n-1}}{C}\ln(1-\omega)B_1(\omega)(1-\omega)^n + f_1(\omega)\left(a_1 - \int_0^1 d\omega' \frac{\omega'^{2/\epsilon} f_2(\omega')h(\omega')}{C(\epsilon,\lambda)(1-\omega')^n(1+\omega')^{2/\epsilon-n}}\right) + \frac{C_1 f_2(\omega)}{C}\int_1^\omega d\omega'\, Q(\omega')\ln\frac{1-\omega'}{1-\omega}.$$

The last term is analytic at $\omega = 1$. The singularity vanishes if we choose

$$a_1 = \frac{H_{n-1}}{C_1 C} + \int_0^1 d\omega' \frac{\omega'^{2/\epsilon} f_2(\omega')h(\omega')}{C(\epsilon,\lambda)(1-\omega')^n(1+\omega')^{2/\epsilon-n}}. \tag{47}$$

For any λ for which $C(\lambda, \epsilon) \neq 0$, using the explicit expression (34) with a_1 determined from (40), (42) or (47), whatever the case may be, we have in the domain $\mathcal{V}$,

$$\|g\|_\infty \leq C\|h\|_\infty, \tag{48}$$

This conclusion follows from observing the properties of the integrand and noting that the H_j, $j = 1, \ldots n$ are bounded by some multiples of $\sup_{\omega\in\mathcal{V}}|h(\omega)|$, since they involve only a finite number of derivatives of h at $\omega = 1$. Since the boundary $\partial\mathcal{V}$ is at a finite distance from $\omega = 1$, the derivatives $\partial_\omega^j h|_{\omega=1}$ by Cauchy's theorem are bounded by $b_j \sup_{\omega\in\mathcal{V}}|h(\omega)|$ where b_j is independent of h. Hence, we have shown[7] that λ is in the spectrum only if $C(\lambda, \epsilon) = 0$, i.e. we can only have a discrete spectrum in this problem.

4. Absence of eigenvalues with positive real part and of purely imaginary eigenvalues

4.1. Purely positive eigenvalues

Real eigenvalues $\lambda > 0$ easily are excluded. Substituting into Eq. (28) the power series

$$\beta_\lambda^{(\epsilon)}(\omega) = \sum_{k=0}^{\infty} b_k\,\omega^k, \tag{49}$$

which converges for $|\omega| < 1$ due to the location of the regular singular points, we find the recursion relation

$$b_k = 2\lambda\,\frac{1+\epsilon k}{k(2+\epsilon k)}\,b_{k-1} + \epsilon\,\frac{k-2}{2+\epsilon k}\,b_{k-2} \quad \text{for } k \geq 2, \tag{50}$$

and

$$b_1 = 2\lambda\,\frac{1+\epsilon}{2+\epsilon}b_0. \tag{51}$$

We choose $b_0 = 1$ as initial value.

For $\lambda > 0$, evidently all b_k are positive, and b_k obeys the bound

$$b_k > \epsilon\frac{k-2}{2+\epsilon k}\,b_{k-2},$$

and therefore

$$b_k > \frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(1+\frac{1}{\epsilon}+\frac{k}{2}\right)}\,\text{const.} > 0. \tag{52}$$

For $k \gg 1/\epsilon$, this yields the lower bound

$$b_k > \text{const}\,k^{-1-1/\epsilon},$$

which shows that a sufficiently high derivative of $\beta_\lambda^{(\epsilon)}(\omega)$ (49) diverges for $\omega = 1$, which contradicts the regularity requirement.

4.2. Eigenvalues with positive real part

To eliminate eigenvalues $\lambda = \mu + i\nu$ with $\mu > 0$ needs more refined arguments. We first derive an inequality replacing (52) above. Motivated by (52), we rewrite the recursion relation (50) in terms of

$$c_k = \frac{\Gamma\left(1+\frac{1}{\epsilon}+\frac{k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\,b_k, \quad k \geq 1. \tag{53}$$

This yields

$$c_k = \lambda\,g_k\,c_{k-1} + c_{k-2}, \quad k \geq 3, \tag{54}$$

where

$$g_k \equiv \frac{2(1+\epsilon k)\,\Gamma\left(\frac{k-1}{2}\right)\,\Gamma\left(1+\frac{1}{\epsilon}+\frac{k}{2}\right)}{k(2+\epsilon k)\,\Gamma\left(\frac{k}{2}\right)\,\Gamma\left(\frac{1}{2}+\frac{1}{\epsilon}+\frac{k}{2}\right)} = \frac{2}{k}\left(1+\mathcal{O}\left(\frac{1}{k}\right)\right). \tag{55}$$

We now multiply (54) by c_{k-1}^* and take the real part. With the notation

$$r_k = \text{Re}\left\{c_k c_{k-1}^*\right\}, \tag{56}$$

we get the relation

$$r_k = \mu\,g_k\,|c_{k-1}|^2 + r_{k-1}, \quad k \geq 3. \tag{57}$$

Since $\mu > 0$ and

$$r_2 = \text{Re}\left\{c_2 c_1^*\right\} = 2\mu|\lambda|^2\,\frac{(1+\epsilon)(1+2\epsilon)}{(2+\epsilon)^2}\,\frac{\Gamma\left(\frac{3}{2}+\frac{1}{\epsilon}\right)\,\Gamma\left(2+\frac{1}{\epsilon}\right)}{\Gamma\left(\frac{1}{2}\right)} > 0,$$

the r_k form an increasing series of positive numbers bounded by

$$r_k \geq r_2 > 0. \tag{58}$$

The recursion relation (57) formally is solved as

$$r_k = \mu\sum_{j=3}^{k} g_j\,|c_{j-1}|^2 + r_2, \quad k \geq 3. \tag{59}$$

Using now the relation

$$|c_k|^2 + |c_{k-1}|^2 = 2r_k + |c_k - c_{k-1}|^2 \geq 2r_k,$$

we find the bound

$$\frac{|c_k|^2 + |c_{k-1}|^2}{2} \geq \mu\sum_{j=3}^{k} g_k\,|c_{j-1}|^2 + r_2. \tag{60}$$

[7] Note that by definition, λ is not in the spectrum if the resolvent L^{-1} is bounded.