Linear systems, coalgebraically

Jan Rutten

CWI & Vrije Universiteit Amsterdam

CALCO 2007

Motivation

Why linear systems, coalgebraically?

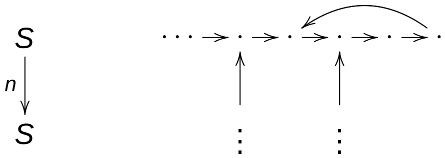
- Very simple state-based systems
- Basic model of memory and feedback
- Uses vector spaces, not just sets
- Central role for the notion of rationality
- Interplay algebra-coalgebra
- Fundamental role (again) for streams
- Teaching!!!

Motivation

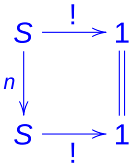
Why linear systems, coalgebraically?

- Very simple state-based systems
- Basic model of memory and feedback
- Uses vector spaces, not just sets
- Central role for the notion of rationality
- Interplay algebra-coalgebra
- Fundamental role (again) for streams
- Teaching!!!

Coalgebras of $id : Set \to Set$

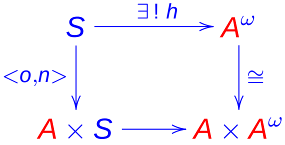


- *Finite S*: not very interesting
- All (S, n) are behaviourally trivial:

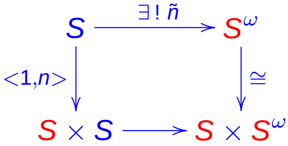


- (But morphisms and bisimulations are *not* trivial.)
  - Cf. Lawvere and Schanuel: Conceptual Mathematics.

Adding observations: $A \times - : Set \rightarrow Set$



Special case:



Note: $\tilde{n}(s) = (1, n, n^2, \ldots)(s) = (s, n(s), n^2(s), \ldots)$

Overview

Today we will . . .

- analyse the situation

$$\begin{array}{ccc} V & \xrightarrow{\exists !\, \tilde{F}} & V^{\omega} \\ {\scriptstyle <1,F>}\downarrow & & \downarrow{\scriptstyle \cong} \\ V \times V & \longrightarrow & V \times V^{\omega} \end{array}$$

  for the category *Vect* of *vector spaces* and *linear maps*
- using, *first*, plain stream calculus and stream circuits, as in
  - J.J.M.M. Rutten. A tutorial on coinductive stream calculus and signal flow graphs. TCS 343, 2005
- and, *next*, stream calculus of linear maps, as in
  - the present CALCO 2007 paper

Coalgebras of $id : Vect \longrightarrow Vect$

- *Vect*: *vector spaces* (here: over $\mathbb{R}$) and *linear maps*.
- Today we concentrate on *finite-dimensional* spaces.
- Our running example is the coalgebra

$$\mathbb{R}^2 \xrightarrow{G} \mathbb{R}^2$$

given, for all $(r_1, r_2) \in \mathbb{R}^2$, by

$$G(r_1, r_2) = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = (\, -r_2, \;\; r_1 + 2r_2 \,)$$

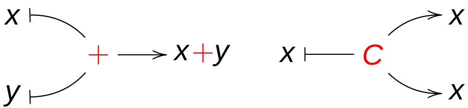
- We'll represent such linear maps by *stream circuits*

Stream circuits

- Simple representations of linear maps
- Four basic types of gates:

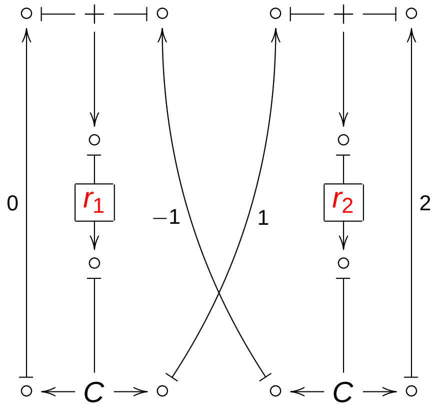
r-multiplier: $x \overset{r}{\longmapsto} r \times x$

register: $x \longmapsto \boxed{r} \longrightarrow$ first r then x



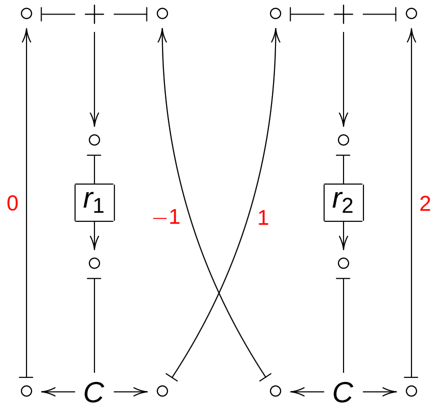
adder copier

$$G(r_1, r_2) = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$



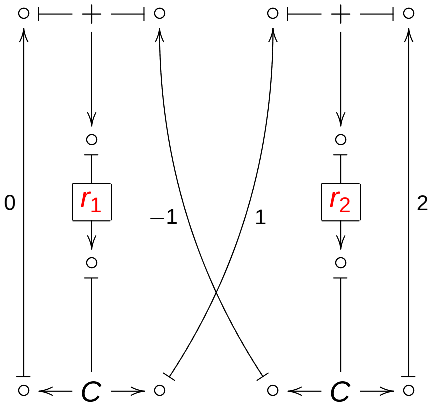
Present state = contents of the registers

$$G(r_1, r_2) = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

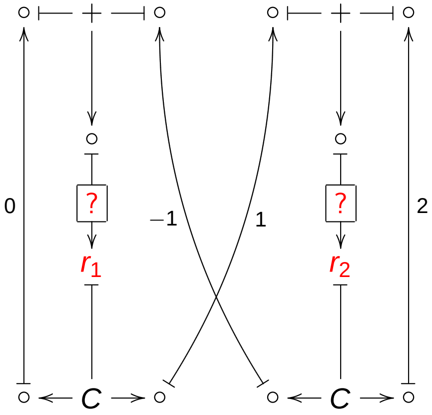


Matrix entries determine values of multipliers (feedback lines)

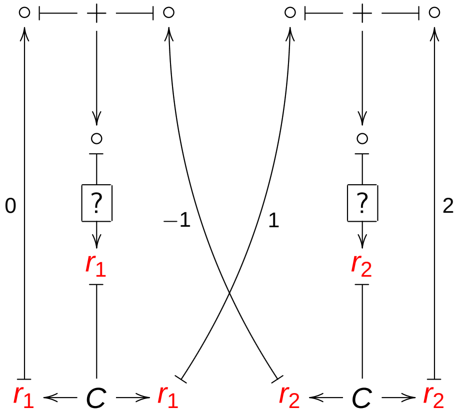
Computing $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$



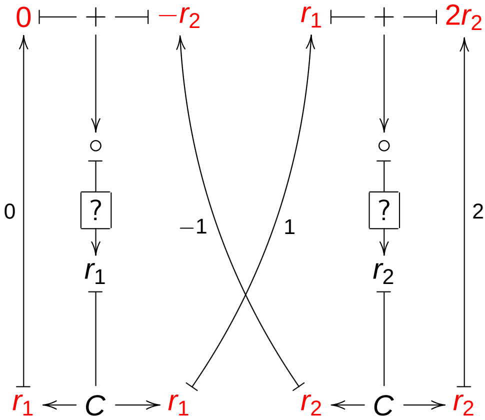
Computing $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$



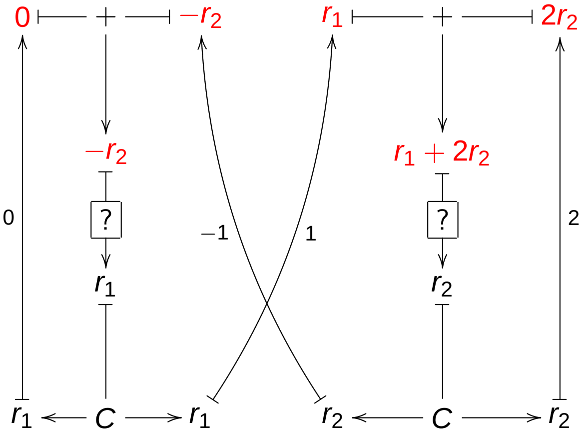
Computing $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$



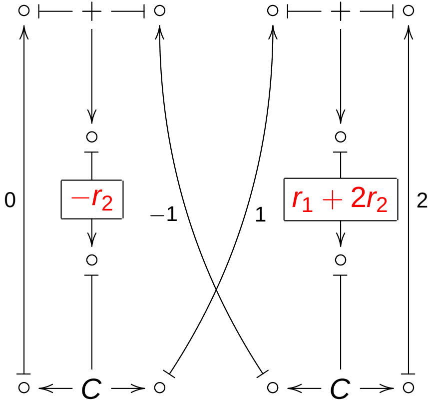
Computing $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$



Computing $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$



$$\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = (-r_2, r_1 + 2r_2)$$

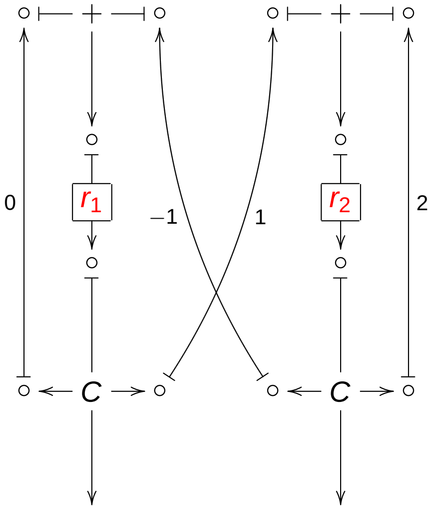


Observing states

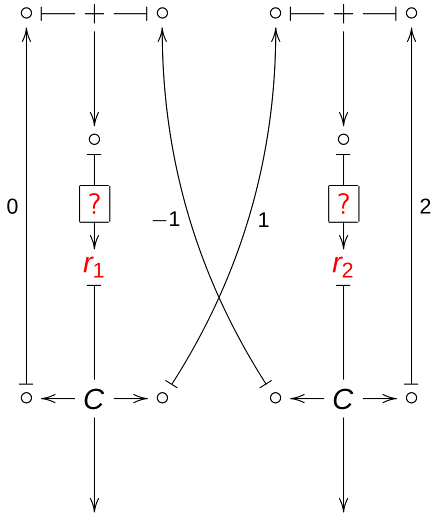
Remember that we want to analyse

$$\mathbb{R}^2 \xrightarrow{\langle 1,G \rangle} \mathbb{R}^2 \times \mathbb{R}^2$$

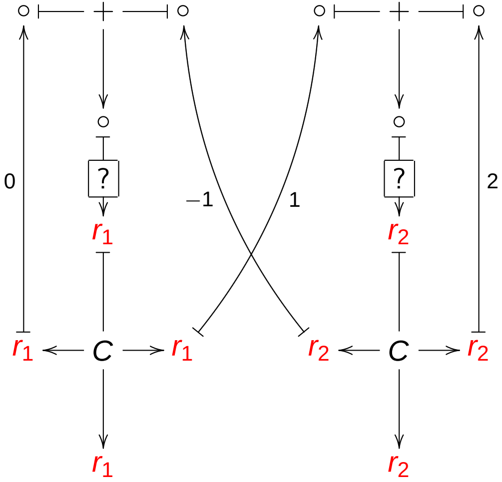
Adding two output ports



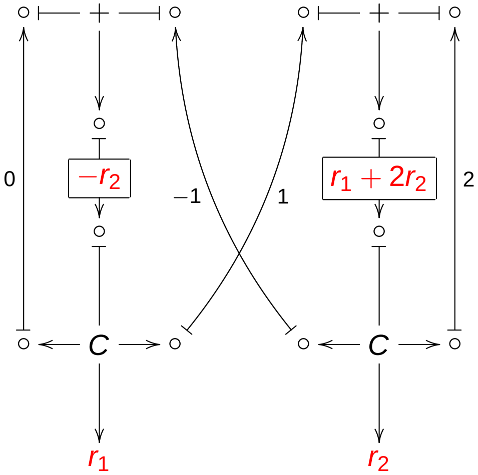
Computing next state: as before



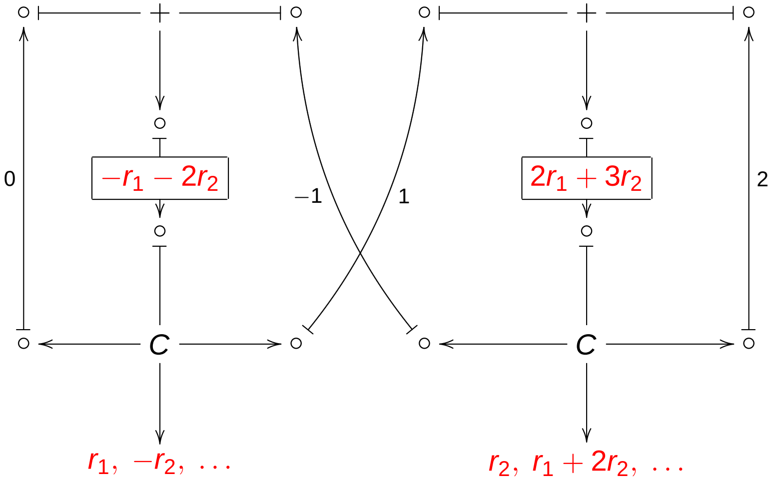
Present state is output



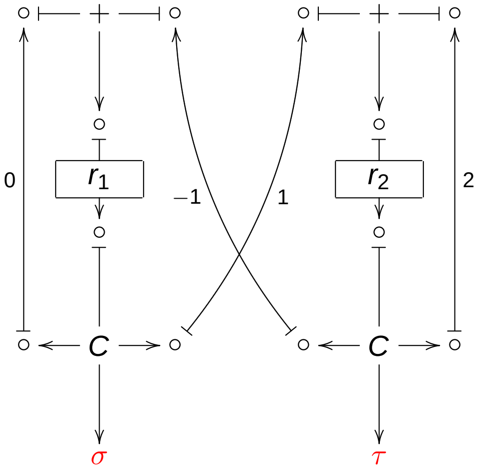
And new state as before



And so on ···



Our next goal: compute streams σ and τ



Stream calculus (on $\mathbb{R}^\omega$)

We use the following constants and operators:

- constants (one for each $r \in \mathbb{R}$): $[r] = (r, 0, 0, 0, \ldots)$
- constant (cf. formal variable): $X = (0, 1, 0, 0, 0, \ldots)$
- sum: $(\sigma + \tau)\,(n) = \sigma(n) + \tau(n)$
- convolution (aka Cauchy) product:

$$(\sigma \times \tau)\,(n) = \sigma(0) \cdot \tau(n) + \cdots + \sigma(n) \cdot \tau(0)$$

- (formal) inverse to product: if $\sigma(0) \neq 0$ then $\exists!\ \frac{1}{\sigma}$ s.t.

$$\sigma \times \frac{1}{\sigma} = (1, 0, 0, 0, \ldots)$$

Stream calculus (on $\mathbb{R}^\omega$)

- Note:

$$X \times \sigma = (0, \sigma(0), \sigma(1), \sigma(2), \ldots)$$

- Convention:

$$3 \times X^2 = [3] \times X \times X \quad (= (0, 0, 3, 0, 0, 0, \ldots))$$

- Polynomial streams: for instance,

$$2 + 3X - 7X^4 \quad (= (2, 3, 0, 0, -7, 0, 0, 0, \ldots))$$

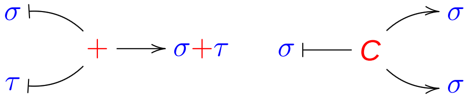
- Rational streams: for instance,

$$\frac{1 + X}{1 - 2X + X^2} \quad (= (1, 3, 5, 7, \ldots))$$

Semantics of stream circuits

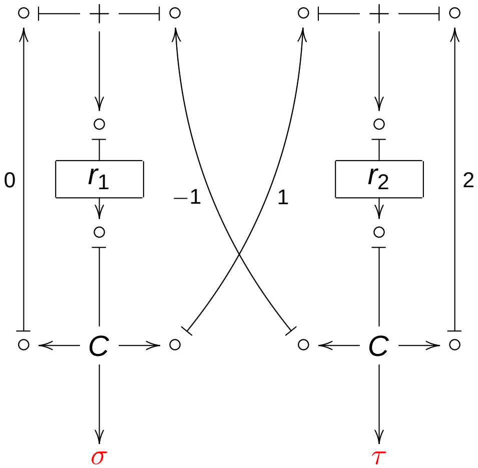
r-multiplier: $\sigma \stackrel{r}{\longmapsto} r \times \sigma$

register: $\sigma \longmapsto \boxed{r} \longrightarrow r + (X \times \sigma) \quad (= (r, \sigma(0), \sigma(1), \ldots))$

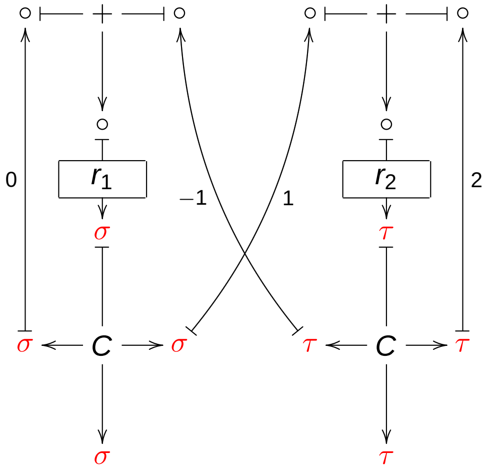


adder copier

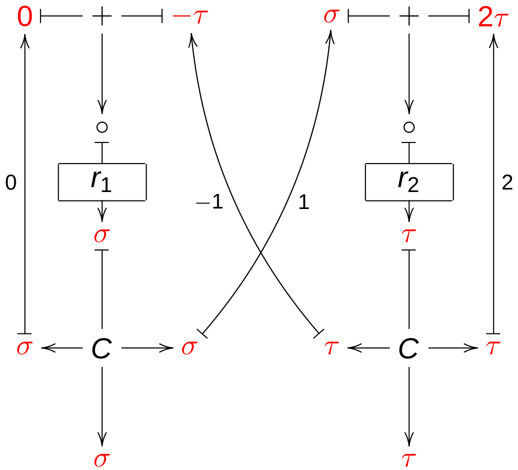
Computing streams σ and τ



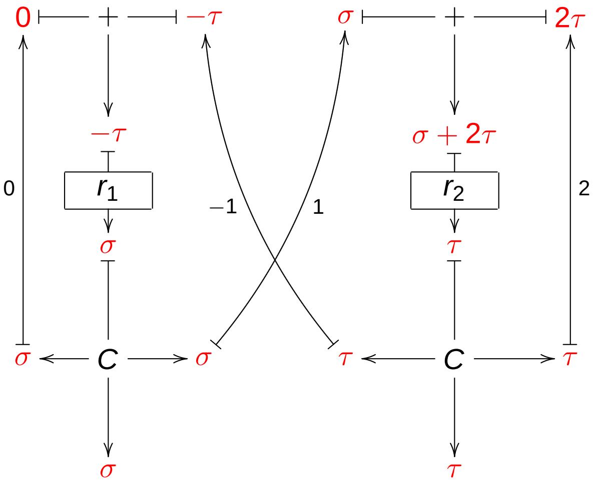
Computing streams σ and τ



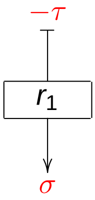
Computing streams σ and τ



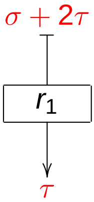
Computing streams σ and τ



Using $\rho \longmapsto \boxed{r} \longrightarrow r + (X \times \rho)$



implies $$\sigma = r_1 + (X \times -\tau)$$



implies $$\tau = r_2 + (X \times (\sigma + 2\tau))$$

Computing streams σ and τ

Two equations:

$$\begin{aligned} \sigma &= r_1 + (X \times -\tau) \\ \tau &= r_2 + (X \times (\sigma + 2\tau)) \end{aligned}$$

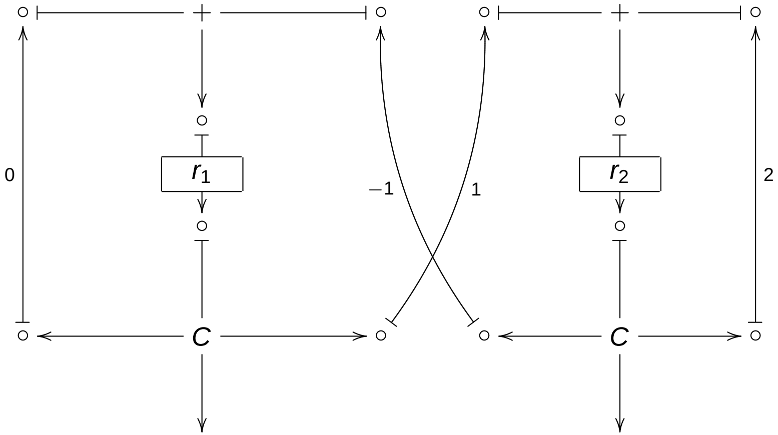
with two unknowns: σ, τ

solved in stream calculus (just compute as usual):

$$\sigma = \frac{r_1 \times (1 - 2X) - (r_2 \times X)}{(1 - X)^2} \qquad \tau = \frac{(r_1 \times X) + r_2}{(1 - X)^2}$$

Note: σ and τ are **rational** streams!

Our present goal completed:



$\sigma = \frac{r_1 \times (1-2X) - (r_2 \times X)}{(1-X)^2}$ $\tau = \frac{(r_1 \times X) + r_2}{(1-X)^2}$

Summary sofar:

We have represented the coalgebra

$$\mathbb{R}^2 \xrightarrow{<1,G>} \mathbb{R}^2 \times \mathbb{R}^2 \qquad \text{with} \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad G = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$$

as a stream circuit which computes, for any $< r_1, r_2 > \in \mathbb{R}^2$,

two *rational* streams $\sigma, \ \tau \in \mathbb{R}^\omega$:

$$\sigma = \frac{r_1 \times (1 - 2X) \ - \ (r_2 \times X)}{(1 - X)^2} \qquad \tau = \frac{(r_1 \times X) \ + \ r_2}{(1 - X)^2}$$

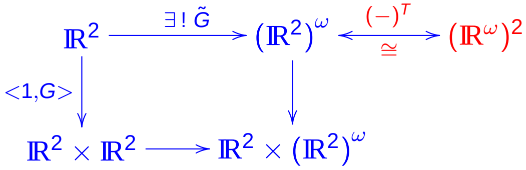
Final semantics

- Stream circuits: computationally intuitive model of linear systems
- Present CALCO paper: same result purely coalgebraically, via *final semantics*:

$$\begin{array}{ccc}
\mathbb{R}^2 & \xrightarrow{\exists !\, \tilde{G}} & (\mathbb{R}^2)^\omega \\
{\scriptstyle <1,G>}\downarrow & & \downarrow{\scriptstyle \cong} \\
\mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \times (\mathbb{R}^2)^\omega
\end{array}$$

Final semantics $\tilde{G}$

- First note that



where $(-)^T$ is the *transpose* function.

- So $(\tilde{G}(r_1, r_2))^T$ *is* a pair of streams, as expected.

Final semantics $\tilde{G}$

$$
\begin{array}{ccc}
\mathbb{R}^2 & \xrightarrow{\exists!\ \tilde{G}} & (\mathbb{R}^2)^\omega \\
{\scriptstyle <1,G>}\downarrow & & \downarrow \\
\mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \times (\mathbb{R}^2)^\omega
\end{array}
$$

Note that

$$\tilde{G}(r_1, r_2) = (\,(r_1, r_2),\ G(r_1, r_2),\ G^2(r_1, r_2),\ \ldots) \ \in (\mathbb{R}^2)^\omega$$

In other words:

$\tilde{G} = (1,\ G,\ G^2,\ \ldots)$ is itself a *stream* !!!

$(\mathbb{R}^2 \to_L \mathbb{R}^2)$ is a ring

- We have a *ring* $\left((\mathbb{R}^2 \to_L \mathbb{R}^2), +, \circ, 0, 1 \right)$ with
  - $F + H$: elementwise sum
  - $F \circ H$: function composition or, equivalently

    matrix multiplication: $M_{F \circ H} = M_F \times M_H$

- Hence we can do *stream calculus* on $(\mathbb{R}^2 \to_L \mathbb{R}^2)^\omega$

  and compute a nice closed formula for

$$\begin{aligned} \tilde{G} &= (1, G, G^2, G^3, \ldots) \\ &= (1, G, G \circ G, G \circ G \circ G, \ldots) \quad \in (\mathbb{R}^2 \to_L \mathbb{R}^2)^\omega \end{aligned}$$

Stream calculus on $(\mathbb{R}^2 \to_L \mathbb{R}^2)^\omega$

- elements $\sigma = (F_0, F_1, F_2, \ldots)$
- constants: $[F] = (F, 0, 0, 0, \ldots)$
- constant (cf. formal variable): $X = (0, 1, 0, 0, 0, \ldots)$
- sum: $(\sigma + \tau)(n) = \sigma(n) + \tau(n)$
- convolution product:
  $(\sigma \times \tau)(n) = \sigma(0) \circ \tau(n) + \cdots + \sigma(n) \circ \tau(0)$
- inverse: if $\sigma(0)$ is *invertible* then $\exists!\ \sigma^{-1}$ s.t.

$$\sigma \times \sigma^{-1} = (1, 0, 0, 0, \ldots)$$

Computing $\tilde{G} = (1, G, G^2, \ldots)$

- Recall from stream calculus on $\mathbb{R}^\omega$:

$$(1 - (c \times X))^{-1} = \frac{1}{1 - (c \times X)} = (1, c, c^2, \ldots)$$

- Similarly, on $(\mathbb{R}^2 \to_L \mathbb{R}^2)^\omega$:

$$(1 - (G \times X))^{-1} = (1, G, G^2, \ldots)$$

- Let's *do* this for our $G = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

Please sit back, relax, and have confidence.

Computing $\tilde{G}$ for $G = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

$$
\begin{aligned}
\tilde{G} &= (1, G, G^2, \ldots) \\
&= (1 - (G \times X))^{-1} \\
&= (1 - (\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \times X))^{-1} \\
&= (1 - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix})^{-1} \\
&= (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix})^{-1} \\
&= \begin{pmatrix} 1 & X \\ -X & 1 - 2X \end{pmatrix}^{-1}
\end{aligned}
$$

Computing $\tilde{G}$ for $G = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

$$\begin{aligned}
\tilde{G} &= (1, G, G^2, \ldots) \\
&= (1 - (G \times X))^{-1} \\
&= \left(1 - \left(\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \times X\right)\right)^{-1} \\
&= \left(1 - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix}\right)^{-1} \\
&= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix}\right)^{-1} \\
&= \begin{pmatrix} 1 & X \\ -X & 1-2X \end{pmatrix}^{-1}
\end{aligned}$$

Computing $\tilde{G}$ for $G = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

$$\begin{aligned}
\tilde{G} &= (1, G, G^2, \ldots) \\
&= (1 - (G \times X))^{-1} \\
&= (1 - (\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \times X))^{-1} \\
&= (1 - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix})^{-1} \\
&= (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix})^{-1} \\
&= \begin{pmatrix} 1 & X \\ -X & 1-2X \end{pmatrix}^{-1}
\end{aligned}$$

Computing $\tilde{G}$ for $G = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

$$\begin{aligned}
\tilde{G} &= (1, G, G^2, \ldots) \\
&= (1 - (G \times X))^{-1} \\
&= \left(1 - \left(\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \times X\right)\right)^{-1} \\
&= \left(1 - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix}\right)^{-1} \\
&= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix}\right)^{-1} \\
&= \begin{pmatrix} 1 & X \\ -X & 1-2X \end{pmatrix}^{-1}
\end{aligned}$$

Computing $\tilde{G}$ for $G = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

$$\begin{aligned}
\tilde{G} &= (1, G, G^2, \ldots) \\
&= (1 - (G \times X))^{-1} \\
&= (1 - (\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \times X))^{-1} \\
&= (1 - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix})^{-1} \\
&= (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix})^{-1} \\
&= \begin{pmatrix} 1 & X \\ -X & 1-2X \end{pmatrix}^{-1}
\end{aligned}$$

Computing $\tilde{G}$ for $G = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

$$
\begin{aligned}
\tilde{G} &= (1, G, G^2, \ldots) \\
&= (1 - (G \times X))^{-1} \\
&= (1 - (\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \times X))^{-1} \\
&= (1 - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix})^{-1} \\
&= (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix})^{-1} \\
&= \begin{pmatrix} 1 & X \\ -X & 1 - 2X \end{pmatrix}^{-1}
\end{aligned}
$$

Computing $\tilde{G} = (1, G, G^2, \ldots)$

$$\tilde{G} = \begin{pmatrix} 1 & X \\ -X & 1-2X \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1-2X}{(1-X)^2} & \frac{-X}{(1-X)^2} \\ \frac{X}{(1-X)^2} & \frac{1}{(1-X)^2} \end{pmatrix} \in (\mathit{Rat}\ \mathbb{R}^\omega)^{2\times 2}$$

(The paper in the proceedings justifies all of the above.)

Computing $\tilde{G} = (1, G, G^2, \ldots)$

$$
\begin{aligned}
\tilde{G} &= \begin{pmatrix} 1 & X \\ -X & 1-2X \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \frac{1-2X}{(1-X)^2} & \frac{-X}{(1-X)^2} \\ \frac{X}{(1-X)^2} & \frac{1}{(1-X)^2} \end{pmatrix} \quad \in (\, Rat\ \mathbb{R}^{\omega}\,)^{2\times 2}
\end{aligned}
$$

(The paper in the proceedings justifies all of the above.)

Final semantics

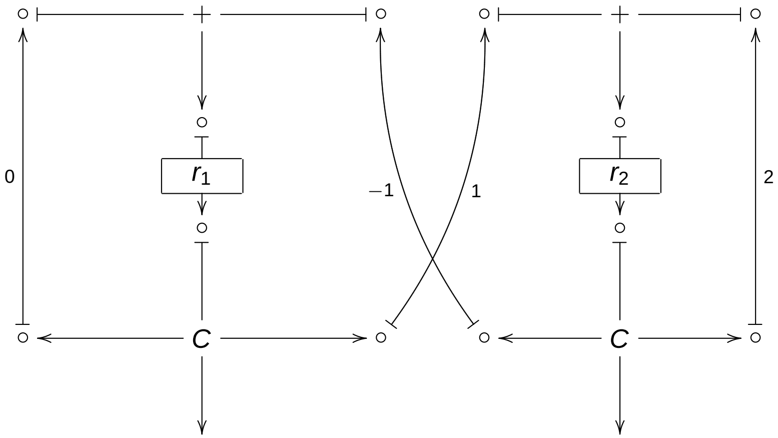
We have computed (omitting transpose):

$$\begin{array}{ccc}
\mathbb{R}^2 & \xrightarrow{\exists !\, \tilde{G}} & (\mathbb{R}^\omega)^2 \\
{\scriptstyle <1,G>}\downarrow & & \downarrow \\
\mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \times (\mathbb{R}^\omega)^2
\end{array}$$

$$\tilde{G}(r_1, r_2) = \begin{pmatrix} \frac{1-2X}{(1-X)^2} & \frac{-X}{(1-X)^2} \\ \frac{X}{(1-X)^2} & \frac{1}{(1-X)^2} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \in Rat(\mathbb{R}^\omega)^2$$

This coincides with our previous solution:

Final semantics coincides with our previous solution:



$$\sigma = \frac{r_1 \times (1-2X) - (r_2 \times X)}{(1-X)^2}$$

$$\tau = \frac{(r_1 \times X) + r_2}{(1-X)^2}$$

Summarizing

- Final semantics of finite dimensional linear system in terms of matrices of rational streams.
- Final semantics is basis for minimization of linear systems (see paper).
- Final semantics is basis for realisation of linear systems (see paper).
- Adding outputs (= non-full observations) is trivial (see paper).
- Adding inputs is (somewhat surprisingly) trivial too (see paper).

In conclusion

- Simplest model of linear systems up to date . . .
- at least for me, that is . . . ; -)
- Central role for the notion of rationality
- Interplay algebra-coalgebra
- Fundamental role (again) for streams
- Teaching!!!

In conclusion

- Simplest model of linear systems up to date . . .
- at least for me, that is . . . ; -)
- Central role for the notion of rationality
- Interplay algebra-coalgebra
- Fundamental role (again) for streams
- Teaching!!!

In conclusion

- Simplest model of linear systems up to date ...
- at least for me, that is ... ; -)
- Central role for the notion of rationality
- Interplay algebra-coalgebra
- Fundamental role (again) for streams
- Teaching!!!

In conclusion

- Simplest model of linear systems up to date . . .
- at least for me, that is . . . ; -)
- Central role for the notion of rationality
- Interplay algebra-coalgebra
- Fundamental role (again) for streams
- Teaching!!!