The positive semidefinite Grothendieck problem with rank constraint

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Abstract. Given a positive integer n and a positive semidefinite matrix $A = (A_{ij}) \in \mathbb{R}^{m \times m}$, the positive semidefinite Grothendieck problem with rank-n-constraint (SDP_n) is

$$\text{maximize } \sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}\, x_i \cdot x_j, \qquad \text{where } x_1, \ldots, x_m \in S^{n-1}.$$

In this paper we design a randomized polynomial-time approximation algorithm for SDP_n achieving an approximation ratio of

$$\gamma(n) = \frac{2}{n}\left(\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\right)^2 = 1 - \Theta(1/n).$$

We show that under the assumption of the unique games conjecture the achieved approximation ratio is optimal: There is no polynomial-time algorithm which approximates SDP_n with a ratio greater than $\gamma(n)$. We improve the approximation ratio of the best known polynomial-time algorithm for SDP_1 from $2/\pi$ to $2/(\pi\gamma(m)) = 2/\pi + \Theta(1/m)$, and we show a tighter approximation ratio for SDP_n when A is the Laplacian matrix of a graph with nonnegative edge weights.

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1 Introduction

Given a positive integer n and a positive semidefinite matrix $A = (A_{ij}) \in \mathbb{R}^{m \times m}$, the *positive semidefinite Grothendieck problem with rank-n-constraint* is defined as

$$\mathrm{SDP}_n(A) = \max\left\{\sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}\ x_i \cdot x_j : x_1, \ldots, x_m \in S^{n-1}\right\},$$

where $S^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}$ is the unit sphere. Note that the inner product matrix of the vectors $x_1, \ldots, x_m$ has rank n. This problem was introduced by Briët, Buhrman, and Toner [5] in the context of quantum nonlocality where they applied it to nonlocal XOR games. The case $n = 1$ is the classical positive semidefinite Grothendieck problem where $x_1, \ldots, x_m \in \{-1, +1\}$. It was introduced by Grothendieck [7] in the study of norms of tensor products of Banach spaces. It is an NP-hard problem: If A is the Laplacian matrix of a graph then $\mathrm{SDP}_1(A)$ coincides with the value of a maximum cut of the graph. The maximum cut problem (MAX CUT) is one of Karp's 21 NP-complete problems. Over the last years, there has been a lot of work on algorithmic applications, interpretations, and generalizations of the Grothendieck problem and the companion Grothendieck inequalities. For instance, Nesterov [18] showed that it has applications to finding and analyzing semidefinite relaxations of nonconvex quadratic optimization problems. Ben-Tal and Nemirovski [4] showed that it has applications to quadratic Lyapunov stability synthesis in system and control theory. Alon and Naor [3] showed that it has applications to constructing Szemerédi partitions of graphs and to estimating the cut norm of matrices. Linial and Shraibman [15] showed that it has applications to finding lower bounds in communication complexity. Khot and Naor [12], [13] showed that it has applications to kernel clustering. For other applications, see also Alon, Makarychev, Makarychev, and Naor [2], and Raghavendra and Steurer [20].

One can reformulate the positive semidefinite Grothendieck problem with rank-n-constraint as a semidefinite program with an additional rank constraint:

$$\begin{aligned}
\text{maximize } & \sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}X_{ij}\\
\text{subject to } & X = (X_{ij}) \in \mathbb{R}^{m\times m} \text{ is positive semidefinite,}\\
& X_{ii} = 1, \quad \text{for } i = 1, \ldots, m,\\
& X \text{ has rank at most } n.
\end{aligned}$$

When n is a constant that does not depend on the matrix size m there is no polynomial-time algorithm known which solves SDP_n. It is also not known if the problem SDP_n is NP-hard when $n \geq 2$. On the other hand the *semidefinite relaxation* of $\mathrm{SDP}_n(A)$ defined by

$$\mathrm{SDP}_\infty(A) = \max\left\{\sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}\ u_i \cdot u_j : u_1, \ldots, u_m \in S^\infty\right\}$$

can be computed in polynomial time to any desired precision by using, e.g., the ellipsoid method. Here S^∞ denotes the unit sphere of the Hilbert space $l^2(\mathbb{R})$ of square summable sequences, which contains $\mathbb{R}^n$ as the subspace of the first n components. Clearly, it would suffice to use unit vectors in $\mathbb{R}^m$ for solving $\mathrm{SDP}_\infty(A)$ when $A \in \mathbb{R}^{m\times m}$, but using S^∞ will simplify many formulations in this paper. Rietz [21] (in the context of the Grothendieck inequality) and Nesterov [18] (in the context of approximation algorithms for NP-hard problems) showed that SDP_1 and SDP_∞ are always within a factor of at most $2/\pi$ from each other. That is, for all positive semidefinite matrices $A \in \mathbb{R}^{m\times m}$ we have

$$1 \geq \frac{\mathrm{SDP}_1(A)}{\mathrm{SDP}_\infty(A)} \geq \frac{2}{\pi}. \tag{1}$$

By exhibiting an explicit series of positive semidefinite matrices, Grothendieck [7] (see also Alon and Naor [3, Section 5.2]) showed that one cannot improve the constant $2/\pi$ to $2/\pi + \varepsilon$ for any positive ε which is independent of m. Nesterov [18] gave a randomized polynomial-time approximation algorithm for SDP_1 with approximation ratio $2/\pi$ which can be derandomized using the techniques presented by Mahajan and Ramesh [16]. This algorithm is optimal in the following sense: Khot and Naor [12] showed that under the assumption of the unique games conjecture (UGC) there is no polynomial-time algorithm which approximates SDP_1 to within a ratio of $2/\pi + \varepsilon$ for any positive ε independent of m. The unique games conjecture was introduced by Khot [10] and by now many tight UGC hardness results are known, see e.g. Khot, Kindler, Mossel, and O'Donnell [11] for the maximum cut problem, Khot and Regev [14] for the minimum vertex cover problem, and Raghavendra [19] for general constrained satisfaction problems. The aim of this paper is to provide a corresponding analysis for SDP_n.

Our results

In Section 2 we start by reviewing our methodological contributions: Our main contribution is the analysis of a rounding scheme which can deal with rank-n-constraints in semidefinite programs. For this we use the Wishart distribution from multivariate statistics (see e.g. Muirhead [17]). We believe this analysis is of independent interest and will turn out to be useful in different contexts, e.g. for approximating low dimensional geometric embeddings. Our second contribution is that we improve the constant in inequality (1) slightly by considering functions of positive type for the unit sphere S^{m-1} and applying a characterization of Schoenberg [22]. This slight improvement is the key for our UGC hardness result of approximating SDP_n given in Theorem 3. We analyze our rounding scheme in Section 3.

Theorem 1. *For all positive semidefinite matrices $A \in \mathbb{R}^{m\times m}$ we have*

$$1 \geq \frac{\mathrm{SDP}_n(A)}{\mathrm{SDP}_\infty(A)} \geq \gamma(n) = \frac{2}{n}\left(\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\right)^2 = 1 - \Theta(1/n),$$

and there is a randomized polynomial-time approximation algorithm for SDP_n *achieving this ratio.*

The first three values of $\gamma(n)$ are:

$$\gamma(1) = 2/\pi = 0.63661\ldots$$
$$\gamma(2) = \pi/4 = 0.78539\ldots$$
$$\gamma(3) = 8/(3\pi) = 0.84882\ldots$$

In Section 4 we show that one can improve inequality (1) slightly:

Theorem 2. *For all positive semidefinite matrices $A \in \mathbb{R}^{m\times m}$ we have*

$$1 \geq \frac{\mathrm{SDP}_1(A)}{\mathrm{SDP}_\infty(A)} \geq \frac{2}{\pi\gamma(m)} = \frac{m}{\pi}\left(\frac{\Gamma(m/2)}{\Gamma((m+1)/2)}\right)^2 = \frac{2}{\pi} + \Theta\left(\frac{1}{m}\right),$$

and there is a polynomial-time approximation algorithm for SDP_1 *achieving this ratio.*

With this, the current complexity status of the problem SDP_1 is similar to the one of the minimum vertex cover problem. Karakostas [9] showed that one can approximate the minimum vertex cover problem for a graph having vertex set V with an approximation ratio of $2-\Theta(1/\sqrt{\log|V|})$ in polynomial time. On the other hand, Khot and Regev [14] showed, assuming the unique games conjecture, that there is no polynomial-time algorithm which approximates the minimum vertex cover problem with an approximation factor of $2-\varepsilon$ for any positive ε which is independent of $|V|$. In Section 5 we show that the approximation ratio $\gamma(n)$ given in Theorem 1 is optimal for SDP_n under the assumption of the unique games conjecture. By using the arguments of the proof of Theorem 2 and by the UGC hardness of approximating SDP_1 due to Khot and Naor [12] we get the following tight UGC hardness result for approximating SDP_n.

Theorem 3. *Under the assumption of the unique games conjecture there is no polynomial-time algorithm which approximates* SDP_n *with an approximation ratio greater than $\gamma(n)+\varepsilon$ for any positive ε which is independent of the matrix size m.*

In Section 6 we show that a better approximation ratio can be achieved when the matrix A is the Laplacian matrix of a graph with nonnegative edge weights.

2 Rounding schemes and functions of positive type

In this section we discuss our rounding scheme which rounds an optimal solution of SDP_∞ to a feasible solution of SDP_n. In the case $n=1$ our rounding scheme is equivalent to the classical scheme of Goemans and Williamson [6]. To analyze the rounding scheme we use functions of positive type for unit spheres. The randomized polynomial-time approximation algorithm which we use in the proofs of the theorems is the following three-step process. The last two steps are our rounding scheme.

1. Solve $\mathrm{SDP}_\infty(A)$, obtaining vectors $u_1, \ldots, u_m \in S^{m-1}$.
2. Choose $X = (X_{ij}) \in \mathbb{R}^{n \times m}$ so that every matrix entry X_{ij} is distributed independently according to the standard normal distribution with mean 0 and variance 1, that is, $X_{ij} \sim N(0,1)$.
3. Set $x_i = Xu_i/\|Xu_i\| \in S^{n-1}$ with $i = 1, \ldots, m$.

The quality of the feasible solution $x_1, \ldots, x_m$ for SDP_n is measured by the expectation

$$\mathbb{E}\left[\sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}\ x_i \cdot x_j\right] = \sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}\mathbb{E}\left[\frac{Xu_i}{\|Xu_i\|} \cdot \frac{Xu_j}{\|Xu_j\|}\right],$$

which we analyze in more detail.

For vectors $u, v \in S^\infty$ we define

$$E_n(u,v) = \mathbb{E}\left[\frac{Xu}{\|Xu\|} \cdot \frac{Xv}{\|Xv\|}\right], \tag{2}$$

where $X = (X_{ij})$ is a matrix with n rows and infinitely many columns whose entries are distributed independently according to the the standard normal distribution. Of course, if $u, v \in S^{m-1}$, then it suffices to work with finite matrices $X \in \mathbb{R}^{n \times m}$.

The first important property of the expectation E_n is that it is *invariant under* $\mathrm{O}(\infty)$, i.e. for every m it is invariant under the orthogonal group $\mathrm{O}(m) = \{T \in \mathbb{R}^{m \times m} : T^{\mathsf{T}}T = I_m\}$, where I_m denotes the identity matrix. More specifically, for every m and every pair of vectors $u, v \in S^{m-1}$ we have

$$E_n(Tu, Tv) = E_n(u,v) \qquad \text{for all } T \in \mathrm{O}(m).$$

If $n = 1$, then

$$E_1(u,v) = \mathbb{E}[\mathrm{sign}(\xi \cdot u)\,\mathrm{sign}(\xi \cdot v)],$$

where $\xi \in \mathbb{R}^m$ is chosen at random from the m-dimensional standard normal distribution. By Grothendieck's identity (see e.g. [8, Lemma 10.2])

$$\mathbb{E}[\mathrm{sign}(\xi \cdot u)\,\mathrm{sign}(\xi \cdot v)] = \frac{2}{\pi}\arcsin u \cdot v.$$

Hence, the expectation E_1 only depends on the inner product $t = u \cdot v$. For general n, the $\mathrm{O}(\infty)$ invariance implies that this is true also for E_n.

The second important property of the expectation E_n (now interpreted as a function of the inner product) is that it is a function of positive type for S^∞, i.e. it is of positive type for any unit sphere S^{m-1}, independent of the dimension m. In general, a continuous function $f : [-1,1] \to \mathbb{R}$ is called *a function of positive type for* S^{m-1} if the matrix $(f(v_i \cdot v_j))_{1 \le i,j \le N}$ is positive semidefinite for every positive integer N and every choice of vectors $v_1, \ldots, v_N \in S^{m-1}$. The expectation E_n is of positive type for S^∞ because one can write it as a sum of squares. Schoenberg [22] characterized the continuous functions $f : [-1, 1] \to \mathbb{R}$ which are of positive type for S^∞: They are of the form

$$f(t) = \sum_{i=0}^{\infty} f_i t^i,$$

with nonnegative f_i and $\sum_{i=0}^{\infty} f_i < \infty$. In the case $n = 1$ we have the series expansion

$$E_1(t) = \frac{2}{\pi} \arcsin t = \frac{2}{\pi} \sum_{i=0}^{\infty} \frac{(2i)!}{2^{2i}(i!)^2(2i+1)} t^{2i+1}.$$

In Section 3 we treat the cases $n \geq 2$.

Suppose we develop the expectation $E_n(t)$ into the series $E_n(t) = \sum_{i=0}^{\infty} f_i t^i$. Then because of Schoenberg's characterization the function $t \mapsto E_n(t) - f_1 t$ is of positive type for S^∞ as well. This together with the inequality $\sum_{i,j} X_{ij} Y_{ij} \geq 0$, which holds for all positive semidefinite matrices $X, Y \in \mathbb{R}^{m \times m}$, implies

$$\mathrm{SDP}_n(A) \geq \sum_{i=1}^{m} \sum_{j=1}^{m} A_{ij} E_n(u_i, u_j) \geq f_1 \sum_{i=1}^{m} \sum_{j=1}^{m} A_{ij}\; u_i \cdot u_j = f_1 \, \mathrm{SDP}_\infty(A). \quad (3)$$

When $n = 1$ the series expansion of E_1 gives $f_1 = 2/\pi$ and the above argument is essentially the one of Nesterov [18]. To improve on this (and in this way to improve the constant $2/\pi$ in inequality (1)) one can refine the analysis by working with functions of positive type which depend on the dimension m. In Section 4 we show that $t \mapsto 2/\pi(\arcsin t - t/\gamma(m))$ is a function of positive type for S^{m-1}. For the cases $n \geq 2$ we show in Section 3 that $f_1 = \gamma(n)$.

3 Analysis of the approximation algorithm

In this section we show that the expectation E_n defined in (2) is a function of positive type for S^∞ and that in the series expansion $E_n(t) = \sum_{i=0}^{\infty} f_i t^i$ one has $f_1 = \gamma(n)$. These two facts combined with the discussion in Section 2 imply Theorem 1. Let $u, v \in S^{m-1}$ be unit vectors and let $X = (X_{ij}) \in \mathbb{R}^{n \times m}$ be a random matrix whose entries are independently sampled from the standard normal distribution. Because of the invariance under the orthogonal group, for computing $E_n(u, v)$ we may assume that u and v are of the form

$$u = (\cos\theta, \sin\theta, 0, \ldots, 0)^{\mathsf{T}}$$
$$v = (\cos\theta, -\sin\theta, 0, \ldots, 0)^{\mathsf{T}}.$$

Then by the double-angle formula $\cos 2\theta = t$ with $t = u \cdot v$.

We have

$$Xu = \begin{pmatrix} X_{11} & X_{12} \\ \vdots & \vdots \\ X_{n1} & X_{n2} \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad Xv = \begin{pmatrix} X_{11} & X_{12} \\ \vdots & \vdots \\ X_{n1} & X_{n2} \end{pmatrix} \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}.$$

Hence,

$$\frac{Xu}{\|Xu\|}\cdot\frac{Xv}{\|Xv\|}=\frac{x^{\mathsf{T}}Yy}{\sqrt{(x^{\mathsf{T}}Yx)(y^{\mathsf{T}}Yy)}},$$

where $x=(\cos\theta,\sin\theta)^{\mathsf{T}}$, $y=(\cos\theta,-\sin\theta)^{\mathsf{T}}$, and $Y\in\mathbb{R}^{2\times 2}$ is the Gram matrix of the two vectors $(X_{11},\ldots,X_{n1})^{\mathsf{T}}$, $(X_{12},\ldots,X_{n2})^{\mathsf{T}}\in\mathbb{R}^n$. By definition, Y is distributed according to the Wishart distribution from multivariate statistics. This distribution is defined as follows (see e.g. Muirhead [17]). Let p and q be positive integers so that $p\geq q$. The (standard) *Wishart distribution* $W_q(p)$ is the probability distribution of random matrices $Y=X^{\mathsf{T}}X\in\mathbb{R}^{q\times q}$, where the entries of the matrix $X=(X_{ij})\in\mathbb{R}^{p\times q}$ are independently chosen from the standard normal distribution $X_{ij}\sim N(0,1)$. The density function of $Y\sim W_q(p)$ is

$$\frac{1}{2^{pq/2}\Gamma_q(p/2)}e^{-\operatorname{Tr}(Y)/2}(\det Y)^{(p-q-1)/2},$$

where Γ_q is the *multivariate gamma function*, defined as

$$\Gamma_q(x)=\pi^{q(q-1)/4}\prod_{i=1}^{q}\Gamma\Big(x-\frac{i-1}{2}\Big).$$

We denote the cone of positive semidefinite matrices of size $q\times q$ by $S^q_{\geq 0}$. In our case $p=n$ and $q=2$. We can write $E_n(t)$ as

$$E_n(t)=\frac{1}{2^n\Gamma_2(n/2)}\int_{S^2_{\geq 0}}\frac{x^{\mathsf{T}}Yy}{\sqrt{(x^{\mathsf{T}}Yx)(y^{\mathsf{T}}Yy)}}e^{-\operatorname{Tr}(Y)/2}(\det Y)^{(n-3)/2}dY,$$

where $t=\cos 2\theta$, and x as well as y depend on θ. The parameterization of the cone $S^2_{\geq 0}$ given by

$$S^2_{\geq 0}=\left\{Y=\begin{pmatrix}\frac{a}{2}+\alpha\cos\phi & \alpha\sin\phi\\ \alpha\sin\phi & \frac{a}{2}-\alpha\cos\phi\end{pmatrix}:\phi\in[0,2\pi],\alpha\in[0,a/2],a\in\mathbb{R}_{\geq 0}\right\}$$

allows us to write the integral in a more explicit form. With this parametrization we have

$$\operatorname{Tr}(Y)=a,\quad \det(Y)=\frac{a^2}{4}-\alpha^2,\quad dY=\alpha\; d\phi d\alpha da,$$

and

$$\begin{aligned}x^{\mathsf{T}}Yy&=\frac{at}{2}+\alpha\cos\phi,\\ x^{\mathsf{T}}Yx&=\frac{a}{2}+\alpha(t\cos\phi+2\sin\theta\cos\theta\sin\phi),\\ y^{\mathsf{T}}Yy&=\frac{a}{2}+\alpha(t\cos\phi-2\sin\theta\cos\theta\sin\phi).\end{aligned}$$

So,

$$\begin{aligned}E_n(t)=\frac{1}{2^n\Gamma_2(n/2)}\int_0^{\infty}\int_0^{a/2}\int_0^{2\pi}&\frac{\frac{at}{2}+\alpha\cos\phi}{\sqrt{(\frac{a}{2}+\alpha t\cos\phi)^2-\alpha^2(1-t^2)(\sin\phi)^2}}\\ &\cdot e^{-a/2}\left(\frac{a^2}{4}-\alpha^2\right)^{(n-3)/2}\alpha\; d\phi d\alpha da.\end{aligned}$$

Substituting $\alpha = (a/2)r$ and integrating over a yields

$$E_n(t) = \frac{\Gamma(n)}{2^{n-1}\Gamma_2(n/2)} \int_0^1 \int_0^{2\pi} \frac{(t + r\cos\phi)r(1-r^2)^{(n-3)/2}}{\sqrt{(1+rt\cos\phi)^2 - r^2(1-t^2)(\sin\phi)^2}} d\phi dr.$$

Using Legendre's duplication formula (see [1, Theorem 1.5.1]) $\Gamma(2x)\Gamma(1/2) = 2^{2x-1}\Gamma(x)\Gamma(x+1/2)$ one can simplify

$$\frac{\Gamma(n)}{2^{n-1}\Gamma_2(n/2)} = \frac{n-1}{2\pi}.$$

Recall from (3) that the approximation ratio is given by the coefficient f_1 in the series expansion $E_n(t) = \sum_{i=0}^{\infty} f_i t^i$. Now we compute f_1:

$$\begin{aligned} f_1 &= \frac{\partial E_n}{\partial t}(0) \\ &= \frac{n-1}{2\pi} \int_0^1 \int_0^{2\pi} \frac{r(1-r^2)^{(n-1)/2}}{(1-r^2(\sin\phi)^2)^{3/2}} d\phi dr. \end{aligned}$$

Using Euler's integral representation of the hypergeometric function [1, Theorem 2.2.1] and by substitution we get

$$\begin{aligned} f_1 &= \frac{n-1}{2\pi} \int_0^{2\pi} \frac{\Gamma(1)\Gamma((n+1)/2)}{2\Gamma((n+3)/2)} {}_2F_1\left(\begin{matrix} 3/2, 1 \\ (n+3)/2 \end{matrix}; \sin^2\phi\right) d\phi \\ &= \frac{n-1}{4\pi} \frac{\Gamma((n+1)/2)}{\Gamma((n+3)/2)} 4 \int_0^1 {}_2F_1\left(\begin{matrix} 3/2, 1 \\ (n+3)/2 \end{matrix}; t^2\right) (1-t^2)^{-1/2} dt \\ &= \frac{n-1}{\pi} \frac{\Gamma((n+1)/2)}{\Gamma((n+3)/2)} \frac{1}{2} \int_0^1 {}_2F_1\left(\begin{matrix} 3/2, 1 \\ (n+3)/2 \end{matrix}; t\right) (1-t)^{-1/2} t^{-1/2} dt. \end{aligned}$$

This simplies futher by Euler's generalized integral [1, (2.2.2)], and Gauss's summation formula [1, Theorem 2.2.2]

$$\begin{aligned} f_1 &= \frac{n-1}{2\pi} \frac{\Gamma((n+1)/2)}{\Gamma((n+3)/2)} \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)} {}_3F_2\left(\begin{matrix} 3/2, 1, 1/2 \\ (n+3)/2, 1 \end{matrix}; 1\right) \\ &= \frac{n-1}{2} \frac{\Gamma((n+1)/2)}{\Gamma((n+3)/2)} {}_2F_1\left(\begin{matrix} 3/2, 1/2 \\ (n+3)/2 \end{matrix}; 1\right) \\ &= \frac{n-1}{2} \frac{\Gamma((n+1)/2)}{\Gamma((n+3)/2)} \frac{\Gamma((n+3)/2)\Gamma((n-1)/2)}{\Gamma(n/2)\Gamma((n+2)/2)} \\ &= \frac{2}{n} \left(\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\right)^2. \end{aligned}$$

4 Improved analysis

Nesterov's proof of inequality (1) relies on the fact that the function $t \mapsto 2/\pi(\arcsin t - t)$ is of positive type for S^∞. Now we determine the largest value $c(m)$ so that the function $t \mapsto 2/\pi(\arcsin t - c(m)t)$ is of positive type for S^{m-1}. By this we improve the approximation ratio of the algorithm given in Section 2 for SDP_1 from $2/\pi$ to $(2/\pi)c(m)$. The following lemma showing $c(m) = 1/\gamma(m)$ implies Theorem 2.

Lemma 1. *The function*

$$t \mapsto \frac{2}{\pi}\left(\arcsin t - \frac{t}{\gamma(m)}\right)$$

is of positive type for S^{m-1}.

Proof. We equip the space of all continuous functions $f : [-1, 1] \to \mathbb{R}$ with the inner product

$$(f, g)_\alpha = \int_{-1}^{1} f(t)g(t)(1-t^2)^\alpha dt,$$

where $\alpha = (m-3)/2$. With this inner product the Jacobi polynomials satisfy the orthogonality relation

$$(P_i^{(\alpha,\alpha)}, P_j^{(\alpha,\alpha)})_\alpha = 0, \quad \text{if } i \neq j,$$

where $P_i^{(\alpha,\alpha)}$ is the Jacobi polynomial of degree i with parameters (α, α), see e.g. Andrews, Askey, and Roy [1]. Schoenberg [22] showed that a continuous function $f : [-1, 1] \to \mathbb{R}$ is of positive type for S^{m-1} if and only if it is of the form

$$f(t) = \sum_{i=0}^{\infty} f_i P_i^{(\alpha,\alpha)}(t),$$

with nonnegative coefficients f_i such that $\sum_{i=0}^{\infty} f_i < \infty$.

Now we interpret arcsin as a function of positive type for S^{m-1} where m is fixed. By the orthogonality relation and because of Schoenberg's result the function $\arcsin t - c(m)t$ is of positive type for S^{m-1} if and only if

$$(\arcsin t - c(m)t, P_i^{(\alpha,\alpha)})_\alpha \geq 0, \qquad \text{for all } i = 0, 1, 2, \ldots.$$

We have $P_1^{(\alpha,\alpha)}(t) = (\alpha+1)t$. By the orthogonality relation and because the arcsin function is of positive type we get, for $i \neq 1$,

$$(\arcsin t - c(m)t, P_i^{(\alpha,\alpha)})_\alpha = (\arcsin t, P_i^{(\alpha,\alpha)})_\alpha \geq 0.$$

This implies that the maximum $c(m)$ such that $\arcsin t - c(m)t$ is of positive type for S^{m-1} is given by $c(m) = (\arcsin t, t)_\alpha / (t, t)_\alpha$.

The numerator of $c(m)$ equals

$$\begin{aligned}(\arcsin t, t)_\alpha &= \int_{-1}^{1} \arcsin(t)t(1-t^2)^\alpha dt \\ &= \int_{-\pi/2}^{\pi/2} \theta \sin\theta(\cos\theta)^{2\alpha+1} d\theta \\ &= \frac{\Gamma(1/2)\Gamma(a+3/2)}{(2\alpha+2)\Gamma(\alpha+2)}.\end{aligned}$$

The denominator of $c(m)$ equals

$$(t,t)_\alpha = \int_{-1}^{1} t^2(1-t^2)^\alpha dt = \frac{\Gamma(3/2)\Gamma(\alpha+1)}{\Gamma(\alpha+5/2)},$$

where we used the beta integral (see e.g. Andrews, Askey, and Roy [1, (1.1.21)])

$$\int_0^1 t^{2x-1}(1-t^2)^{y-1} dr = \int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1} d\theta = \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)},$$

Now, by using the functional equation $x\Gamma(x) = \Gamma(x+1)$, the desired equality $c(m) = 1/\gamma(m)$ follows. □

5 Hardness of approximation

Proof (of Theorem 3). Suppose that ρ is the largest approximation ratio a polynomial-time algorithm can achieve for SDP_n. Let $u_1, \ldots, u_m \in S^{n-1}$ be an approximate solution to $\mathrm{SDP}_n(A)$ coming from such a polynomial-time algorithm. Then,

$$\sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}\, u_i \cdot u_j \geq \rho\, \mathrm{SDP}_n(A).$$

Applying the rounding scheme to $u_1, \ldots, u_m \in S^{n-1}$ gives $x_1, \ldots, x_m \in \{-1, +1\}$ with

$$\begin{aligned} \mathbb{E}\Big[\sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}\, x_i x_j\Big] &= \frac{2}{\pi}\sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij} \arcsin u_i \cdot u_j \\ &\geq \frac{2\rho}{\pi\gamma(n)}\, \mathrm{SDP}_n(A), \end{aligned}$$

where we used that the matrix A and the matrix

$$\left(\frac{2}{\pi}\left(\arcsin u_i \cdot u_j - \frac{u_i \cdot u_j}{\gamma(n)}\right)\right)_{1 \leq i,j \leq m}$$

are both positive semidefinite. The last statement follows from Lemma 1 applied to the vectors $u_1, \ldots, u_m$ lying in S^{n-1}. Since $\mathrm{SDP}_n(A) \geq \mathrm{SDP}_1(A)$, this is a polynomial-time approximation algorithm for SDP_1 with approximation ratio at least $(2\rho)/(\pi\gamma(n))$. The UGC hardness result of Khot and Naor now implies that $\rho \leq \gamma(n)$. □

6 The case of Laplacian matrices

In this section we show that one can improve the approximation ratio of the algorithm if the positive semidefinite matrix $A = (A_{ij}) \in \mathbb{R}^{m \times m}$ has the following special structure:

$$\begin{aligned} A_{ij} &\leq 0, \quad \text{if } i \neq j, \\ \sum_{i=1}^{n} A_{ij} &= 0, \quad \text{for every } j = 1, \ldots, n. \end{aligned}$$

This happens for instance when A is the Laplacian matrix of a weighted graph with nonnegative edge weights. A by now standard argument due to Goemans and Williamson [6] shows that the algorithm has the approximation ratio

$$v(n) = \min \left\{ \frac{1 - E_n(t)}{1 - t} : t \in [-1, 1] \right\}.$$

To see this, we write out the expected value of the approximation and use the properties of A:

$$\begin{aligned} \mathbb{E}\Big[\sum_{i,j=1}^{n} A_{ij} x_i \cdot x_j \Big] &= \sum_{i,j=1}^{n} A_{ij} E_n(u_i \cdot u_j) \\ &= \sum_{i \neq j} (-A_{ij}) \left(\frac{1 - E_k(u_i \cdot u_j)}{1 - u_i \cdot u_j} \right) (1 - u_i \cdot u_j) \\ &\geq v(n) \sum_{i,j=1}^{n} A_{ij} u_i \cdot u_j \\ &= v(n)\, \mathrm{SDP}_{\infty}(A). \end{aligned}$$

The case $n = 1$ corresponds to the MAX CUT approximation algorithm of Goemans and Williamson [6]. For this we have

$$v(1) = 0.8785\ldots, \quad \text{minimum attained at } t_0 = -0.689\ldots.$$

We computed the values $v(2)$ and $v(3)$ numerically and got

$$\begin{aligned} v(2) &= 0.9349\ldots, \quad \text{minimum attained at } t_0 = -0.617\ldots, \\ v(3) &= 0.9563\ldots, \quad \text{minimum attained at } t_0 = -0.584\ldots. \end{aligned}$$

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