MULTIPLAYER XOR GAMES AND QUANTUM COMMUNICATION COMPLEXITY WITH CLIQUE-WISE ENTANGLEMENT

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Abstract. XOR games are a simple computational model with connections to many areas of complexity theory including interactive proof systems, hardness of approximation, and communication complexity. Perhaps the earliest use of XOR games was in the study of quantum correlations, as an experimentally realizable setup which could demonstrate non-local effects as predicted by quantum mechanics. XOR games also have an interesting connection to Grothendieck's inequality, a fundamental theorem of analysis—Grothendieck's inequality shows that two players sharing entanglement can achieve at most a constant factor advantage over players following classical strategies in an XOR game.

The case of multiplayer XOR games is much less well understood. Pérez-García et al. show the existence of entangled states which allow an unbounded advantage for players in a three-party XOR game over their classical counterparts. On the other hand, they show that when the players share GHZ states, a well studied multiparty entangled state, this advantage is bounded by a constant.

We use a multilinear generalization of Grothendieck's inequality due to Blei and Tonge to simplify the proof of the second result and extend it to the case of so-called Schmidt states, answering an open problem of Pérez-García et al. Via a reduction given in that paper, this answers a 35-year-old problem in operator algebras due to Varopoulos, showing that the space of compact operators on a Hilbert space is a Q-algebra under Schur product.

A further generalization of Grothendieck's inequality due to Carne lets us show that the gap between the entangled and classical value is at most a constant in any multiplayer XOR game in which the players are allowed to share combinations of GHZ states and EPR pairs of any dimension. Based on a result by Bravyi et al. this implies that in a three-party XOR game, players sharing an arbitrary stabilizer state cannot achieve more than a constant factor advantage over unentangled players.

Finally, we discuss applications of our results to communication complexity. We show that the discrepancy method in communication complexity remains a lower bound in the multiparty model where the players have quantum communication and any of the kinds of entanglement discussed above. This answers an open question of Lee, Schechtman, and Shraibman who showed that discrepancy was a lower bound on multiparty communication complexity but were unable to handle the case of entanglement.

Date: November 20, 2009.
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1. Introduction

In an XOR game $G = (f, \pi)$, with probability $\pi(x, y)$ two parties Alice and Bob are given inputs x and y respectively. Without communicating, their task is for Alice to output a bit a, and Bob a bit b, such that $a \oplus b = f(x, y)$. Performance in an XOR game (f, π) is usually measured as the largest *bias* $\beta(G)$ achievable by a protocol, where the bias of a protocol is the probability under π that the output of the protocol is correct minus the probability under π that the protocol is incorrect.

XOR games are a simple and natural model of computation and have been studied in the context of interactive proofs (the complexity class $\oplus$MIP), hardness of approximation (results for the simplest CSPs, such as MAX2SAT [Hås01], can be phrased as results on the inapproximability of the bias of XOR games), and communication complexity (where they are tightly related to the discrepancy method). Perhaps more importantly, they have also proved to be an excellent framework in which to study the relationship between classical and quantum computation. For this, one can consider allowing Alice and Bob to share various resources, such as an entangled state or another non-local resource, in order to help them win the game by better coordinating their answers. As such, XOR games provide the simplest setting in which the non-local properties of quantum mechanics manifest themselves.

The case of two-player XOR games is reasonably well understood [CHTW04]. In this setting, Grothendieck's inequality, a fundamental inequality in Banach-space theory, plays a surprising role: in conjunction with Tsirelson's characterization [Tsi93] of entangled XOR games, it essentially says that, in any XOR game, the entangled bias $\beta^*(G)$, which is defined as the maximum bias achievable by two players sharing any entangled state, can be at most a constant factor larger than the classical bias $\beta(G)$. For convenience, we will refer to the ratio $\beta^*(G)/\beta(G)$ as the QC-gap.

The case of entangled XOR games with more than two players is much less well understood. This is in part a reflection of the fact that multipartite entanglement has proved to be much more unwieldy than its bipartite counterpart. Among the few results known in this setting, one of the most striking is due to Pérez-García et al. [PGWP$^+$08], who show that for every n, there is a game G_n such that $\beta^*(G_n) \geq C\sqrt{n}\,\beta(G_n)$ for some constant C. Here n refers to the smallest local dimension of the entangled state for any of the three players. On the other hand, Pérez-García et al. also show that if the entanglement of the players is restricted to a GHZ state, the QC-gap is at most a constant depending only on the number of players. These results raise the following question: what kinds of entanglement allow for unbounded QC-gaps?

In this paper, we initiate the systematic study of multiplayer XOR games in which the players are allowed to share specific patterns of entanglement. The patterns that we consider are quite general, and in fact include many of the states known to be easily prepared in the lab, such as stabilizer states. We now describe our results and techniques, together with their applications.

1.1. **Our results.** We study two main types of N-partite entanglement. The first is a generalization of GHZ states which we call *Schmidt states*: states of the form $|\Psi\rangle = \sum_i \alpha_i |i\rangle^{\otimes N}$. The second is formed of any combination of EPR pairs or GHZ states shared among any subsets of the N parties. We call such entanglement *clique-wise entanglement*. This is quite a general form of entanglement, as for instance it includes 3-partite stabilizer states [BFG06]. We denote by $\beta^*_{\mathrm{S}}(G)$ (resp. $\beta^*_{\mathrm{C}}(G)$) the maximal bias achievable in game G by players who are restricted to sharing a Schmidt state of arbitrary dimension (resp. arbitrary clique-wise entanglement). Our results can be seen as proving constant upper bounds on the QC-gaps of these quantities.

The main technical contribution of this paper is the expansion of Tsirelson's connection between QC-gaps of two-player XOR games and Grothendieck's inequality, to connections between QC-gaps for N-player XOR games with the above mentioned patterns of entanglement and certain multi-linear extensions of Grothendieck's inequality. The unifying idea underlying the proofs of our results is that, if the players share a specific type of entangled state, then the QC-gap can be upper-bounded by a constant appearing in a related Grothendieck-type inequality.

We now explain our results in more detail. Concerning Schmidt states we prove the following theorem.

Theorem 1. *Let G be an N-player game. Then the maximum bias achievable by players sharing a Schmidt state $|\Psi\rangle := \sum_{i=1}^{d} \alpha_i |i\rangle^{\otimes N}$, for an arbitrary dimension d, is at most a constant times the classical bias. Formally:*

$$\beta^*_{\mathrm{S}}(G) \;\leq\; 2^{(3N-5)/2}\, K_G^{\mathbb{C}}\, \beta(G),$$

where $K_G^{\mathbb{C}} \lesssim 1.40491$ is the complex Grothendieck constant.

This generalizes – with slightly improved constants – a result of Pérez-García et al. who show a constant QC-gap for the case of GHZ states. The main tool in this result is an extension of Grothendieck's inequality due to Blei [Ble79] and later simplified and improved by Tonge [Ton78] while studying certain extensions of the von Neumann inequality. The exponential dependence on the number of players is necessary in this theorem as Zukowski [Zuk93] has given an explicit sequence of N-player XOR games G_N where players sharing a GHZ state can a achieve a bias $2^{-1}(\pi/2)^N$ times that of the classical bias.

Theorem 1 answers an open question of Pérez-García et al., who were particularly interested in the case of Schmidt states because of a connection to a 35-year-old open problem of Varopoulos on operator algebras. We are able to resolve this question via a reduction given in Pérez-García et al. This is described below in Section 1.5.

Our second result further exploits the connection between QC-gaps and Grothendieck-type inequalities and deals with the case where the players share clique-wise entanglement. Here, we consider the general setting where the N players are organized in k coalitions of r players each (a given player can take part in any number of coalitions). The members of each of the coalitions are allowed to share a GHZ state of arbitrary dimension among themselves. We relate the QC-gap in this setting to an inequality initially proved by Carne [Car80] in the context of Banach lattices. This lets us prove that, even in this complex setting, the QC-gap is bounded by a constant depending only on the number of coalitions, and the number of players taking part in each of them, but independent of the dimension of the various states shared among the parties.

Theorem 2. *Let G be an N-player game. Then the maximum bias achievable by players sharing clique-wise entanglement, in which the players are organized in k coalitions of r players each, is at most a constant depending only on k and r times the classical bias. Formally:*

$$\beta^*_{\mathrm{C}}(G) \;\leq\; 2^{k(3r-5)/2}\, (K_G^{\mathbb{C}})^k\, \beta(G).$$

Based on a result by Bravyi et al. [BFG06], we obtain the following corollary, which provides a good example of the applicability of our results.

Corollary 3. *Let G be a 3-player game, and let $|\Psi\rangle$ be an arbitrary stabilizer state shared among the three players. Then the following inequality holds:*

$$\beta^*_{|\Psi\rangle}(G) \leq 8\,(K_G^{\mathbb{C}})^4\, \beta(G).$$

1.2. **Application to parallel repetition.** For two-party non-local games the parallel repetition theorem [Raz98, Hol07] states that the bias decreases exponentially in the number of parallel repetitions of the game. Closely related to parallel repetition are XOR lemmas. The ℓ-fold XOR repetition of an XOR game $G = (f, \pi)$ is again an XOR game defined as $G^{\otimes \ell} = (f^{\otimes \ell}, \pi^{\otimes \ell})$ whose game matrix $f^{\otimes l}$ is the ℓ-fold tensor product of the matrix $[f(x,y)]_{x,y}$ and similarly whose distribution is the ℓ-fold tensor product of $[\pi(x,y)]_{x,y}$. In other words, in this game ℓ input pairs

$(x_i, y_i)_{i=1...l}$ are picked independently with respect to π, and all x_i are sent to Alice, y_i to Bob. They should answer bits a and b respectively such that $a \oplus b = f(x_1, y_1) \oplus \cdots \oplus f(x_\ell, y_\ell)$.

Cleve et al. [CSUU08] show that for any XOR game G, the game $G^{\otimes \ell}$ has entangled bias $\beta^*(G)^\ell$. Since the classical and quantum biases are within a constant factor of each other, this also implies that if $\beta^*(G) < 1$, then $\beta(G^{\otimes \ell})$ must go down exponentially with ℓ (although it does not behave as nicely as the quantum bias with respect to taking XORs). Cleve et al. further use this XOR lemma to give a strong parallel repetition theorem for XOR games with entanglement. In fact, quite generally XOR lemmas imply parallel repetition theorems [Ung09].

Surprisingly, our results (as well as the previous results by Pérez-García et al. [PGWP+08]) imply that there is no such XOR lemma for classical XOR games in the N-party setting for $N > 2$. This can be seen as follows. Suppose that $\beta^*_S(G) = 1$ and $\beta(G) < 1$ for some game G. Then clearly $\beta^*_S(G^{\otimes \ell}) = 1$, and so by Theorem 1 it must be the case that the classical bias of $G^{\otimes \ell}$ is bounded from below by the constant $(2^{(3N-5)/2}\, K_G^{\mathbb{C}})^{-1}$, which is independent of ℓ. Mermin [Mer90] gave an example of such a game, constructing a 3-party XOR game G with the property that $\beta^*_S(G) = 1$ (in fact the players can always win just by sharing a GHZ state) and $\beta(G) = 1/2$.

1.3. **Application to communication complexity.** There is a close connection between the bias of XOR games and communication complexity, at least in one direction. The classical bias of a game $G = (f, \pi)$ is equivalent to the *discrepancy* of f under the distribution π, up to a constant factor. The discrepancy method is a common way to show lower bounds in communication complexity, and is especially valuable in the multiparty model where fewer alternatives are available.

A simple argument shows that, if a function f has communication complexity c, then for any distribution π the XOR game $G = (f, \pi)$ has bias at least 2^{-c}. This is the content of the discrepancy lower bound. Viewing things in terms of XOR games, however, has certain advantages. The same argument gives that if f has a c-bit communication protocol where the players share entanglement $|\Psi\rangle$, then for any distribution π, the bias of $G = (f, \pi)$, for players using entanglement $|\Psi\rangle$, is at least 2^{-c}. As Grothendieck's inequality implies that the classical bias and the bias with entanglement are related by a constant factor, this gives that the discrepancy method also lower bounds the communication complexity with entanglement. Using teleportation, classical communication and entanglement can simulate quantum communication, and so this argument already shows that the discrepancy method is a lower bound on quantum communication complexity with entanglement. This was an open question resolved relatively recently by Linial and Shraibman [LS07], who take a point of view dual to the one presented here.

These connections readily extend to the multiparty case, for both the number-in-the-hand and number-on-the-forehead models of multiparty communication complexity. By showing a constant QC-gap for clique-wise entanglement, we get that the discrepancy method lower bounds multiparty communication complexity where the players share GHZ states and EPR-pairs. Allowing shared EPR-pairs enables teleportation and so we get the same result allowing quantum communication. This answers an open question from [LSS09] who show that the discrepancy method is a lower bound on multiparty quantum communication, but are not able to handle the case of entanglement.

Once we can show the discrepancy method is a lower bound on a certain model of communication complexity, a by now standard argument leverages this to show that the *generalized* discrepancy method is as well [Kla07, Raz03, She08]. The generalized discrepancy method says that to show a lower bound on a function f, it suffices to find a function g and probability distribution π such that g has low discrepancy with respect to π and f has constant correlation with g under π. In other words, the dual norm of discrepancy can be used to lower bound communication complexity. Disjointness is the canonical example of a function which itself has large discrepancy, yet for which the generalized discrepancy method can show good lower bounds.

We can summarize our results on communication complexity in the following theorem.

Theorem 4. *For a sign N-tensor A, let $Q_\epsilon^C(A)$ denote the N-party number-on-the-forehead quantum communication complexity of A where the players share clique-wise entanglement involving at most k subsets of players. Then*

$$Q_\epsilon^C(A) \geq \frac{1}{2} \max_{B:\ell_1(B)=1} \log\left(\frac{\langle A, B\rangle - 2\epsilon}{\beta(B)}\right) - O(kN^3).$$

A proof of this theorem is given in Section 5.

1.4. **Application to hardness of approximation.** Tsirelson's characterization of two-player entangled XOR games gives a means to efficiently compute the bias $\beta^*(G)$ to high accuracy via semidefinite programming. It is also known that approximating the *classical* bias of two-player XOR games within a sufficiently small constant is NP-Hard [Hås01]. Hence the natural relaxation that corresponds to allowing the players to share entanglement marks the transition from a hard optimization problem to a tractable one, and Alon and Naor [AN06] have given a constant factor approximation algorithm for the classical bias based on this idea. This fact has important consequences when one considers games as interactive proof systems, showing the collapse of the class $\oplus$MIP(2) from NEXP to EXP (for specific values of completeness and soundness parameters) when the provers are allowed to share entanglement [CHTW04].

As our results show, for multi-player XOR games, generalized Grothendieck inequalities can be used to bound the QC-gap when the provers share specific forms of entanglement, so it is interesting to ask whether the quantum bias can again be efficiently approximated. It turns out, however, that the situation in this case is quite different. In fact, our results imply the following:

Theorem 5. *For any constant $c > 1$, unless P=NP there is no polynomial-time algorithm which approximates the entangled biases $\beta_{\mathrm{S}}^*(G)$ or $\beta_{\mathrm{C}}^*(G)$ to within a multiplicative factor c. Equivalently, for any integer N and any $\varepsilon > 0$, unless P=NP there is no polynomial-time algorithm which gives a factor $2-\varepsilon$ approximation to the maximum success probability of an entangled N-player game in which the players are restricted to sharing either an arbitrary Schmidt state or any type of clique-wise entanglement.*

In particular this implies that, while for 2-player games it is known that $\beta_{\mathrm{S}}^*(G)$ can be efficiently approximated using semidefinite programming [CHTW04] (indeed Schmidt states constitute the most general kind of bipartite entanglement), in the case of three or more players, unless P=NP there is no polynomial-time constant-factor approximation algorithm for $\beta_{\mathrm{S}}^*(G)$. Note however that our results only hold for the specific types of entanglement that we consider, and it could still very well be the case that, for general entanglement, $\beta^*(G)$ can be computed or approximated in polynomial-time.

The proof of Theorem 5 follows from a hardness of approximation result for Max-E3-Lin2 due to Håstad and Venkatesh [HV04], and we give it in Section 6.

1.5. **Application to operator algebras.** A *Banach algebra* $\mathcal{X} = (X, \cdot)$ is a complex Banach space X equipped with a continuous multiplication operation $X \times X \to X : (x, y) \mapsto x \cdot y$ which is associative and distributive. For a Hilbert space $\mathcal{H}$, let S_∞ denote the Banach space of compact operators on $\mathcal{H}$. The Spectral Theorem characterizes compact operators on a Hilbert space as follows:

Theorem 6 (Spectral Theorem). *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and let $\mathcal{H}_1$ have associated inner product $\langle\cdot,\cdot\rangle$. An operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is compact if and only if it has a representation of the form*

$$T = \sum_i \lambda_i \langle\cdot, e_i\rangle f_i,$$

where $(e_i)_i, (f_i)_i$ are orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $\lim_{i\to\infty} \lambda_i = 0$.

For $1 \le p < \infty$, let S_p denote the space of operators $T \in S_\infty$ that satisfy $\mathrm{Tr}(|T|^p) < \infty$, equipped with the Schatten p-norm, defined by $\|T\|_p = \big(\mathrm{Tr}(|T|^p)\big)^{1/p}$. This is called the p-Schatten space.

Varopoulos [Var75] asked if for all $1 \le p \le \infty$, the Banach algebra formed by S_p under the Schur product (the entry-wise product) is a so-called *Q-algebra.* A Banach algebra $\mathcal{X}$ with underlying field $\mathbb{K}$ is a Q-algebra if there exists a constant C such that for any $N \in \mathbb{N}$, and elements $x_1, \dots, x_N \in \mathcal{X}$ with $\|x_i\|_{\mathcal{X}} \le C$, any polynomial q in N variables without constant term satisfies the inequality

$$\|q(x_1, \dots, x_N)\|_X \le \|q\|_{\infty,\mathbb{K}}.$$

It was proved that for every $p \in [1, 4]$, the Banach algebra formed by S_p together with the Schur product, is a Q-algebra. The cases $1 \le p \le 2$ and $2 \le p \le 4$ were proved by Pérez-García [PG06] and Le Merdy [LM98], respectively.

Varopoulos' question is also stated as Open Question 2' in [PGWP$^+$08], where a connection is made with the QC-gap in the case the players share a Schmidt state. Using this connection, we complete the answer to the question.

Theorem 7. *The Banach algebra formed by S_∞ with the Schur product is a Q-algebra.*

We elaborate on this theorem and give a self-contained proof in Section 7. It is a simple consequence of the connection already made in [PGWP$^+$08], together with our Theorem 1.

1.6. **Application to Grothendieck-type inequalities.** The constant upper bounds on the QC-gaps that we consider are proved by using our expanded connection between QC-gaps and known Grothendieck-type inequalities that emerged in Banach-space theory. In some cases, this connection also allows us to prove results in the opposite direction.

Grothendieck's inequality (cf. Section 2.3) may be interpreted as a statement about the inner-product function, which is a linear functional on the tensor product of two Hilbert spaces. Similarly, the extensions of this inequality we consider may be seen as statements about linear functionals on tensor products of multiple Hilbert spaces. Our main technique in proving upper bounds on QC-gaps is to relate entangled states to linear functionals for which Grothendieck-type inequalities are known to hold. The linear functionals that correspond to clique-wise entanglement are exactly those which Carne [Car80] considered and proved to satisfy Grothendieck-type inequalities.

Bravyi et al. [BFG06] proved that any stabilizer state shared among three parties is equivalent—up to local unitary transformations applied by the parties on their respective systems—to collections of shared GHZ and Bell states (i.e., to 3-partite clique-wise entanglement). Based on our connection, this result and Carne's inequalities imply that the linear functionals that correspond to 3-partite stabilizer states satisfy Grothendieck-type inequalities. We show this more formally in Section 8.

Organization of the paper. We start with some preliminaries in the next section. We then give a proof of our two main results in Sections 3 and 4 respectively. In Section 5 we elaborate on the connection to communication complexity, while in Section 6 we explain the implications of our results to hardness of approximation. Section 7 is devoted to the proof of our positive answer to the question by Varopoulos, and finally in Section 8 we explain how our results can lead to new Grothendieck-type inequalities.

2. Preliminaries

2.1. **Notation.** We manipulate finite-dimensional complex Hilbert spaces, usually denoted by $\mathcal{H}$. Given vectors $x_1, \dots, x_k \in \mathcal{H}$, their *generalized inner product* is defined as

$$\langle x_1, \dots, x_k \rangle = \sum_{i=1}^{d} x_1(i) \cdots x_k(i)$$

where d is the dimension of the ambient space $\mathcal{H}$, and $x_l(i)$ refers to the i-th coordinate of x_l in the canonical basis.

If V is a normed vector space, then $\mathcal{S}(V)$ denotes the set of all vectors with norm at most 1. For example, if ℓ_∞^n is the space of all sequences of n real numbers equipped with the supremum norm, then $\mathcal{S}(\ell_\infty^n)$ is the set of all sequences of n reals that all have absolute value at most 1.

An *N-tensor* $A \in \mathbb{K}^{d^N}$ is a tensor with entries specified by N coordinates $(i_1, \ldots, i_N)$ where $i_j \in [d]$. If $\mathbb{K} = \{-1, 1\}$ then A is called a *sign* N-tensor. For an N-tuple of coordinates $I \in [d]^N$, we will also write $A[I]$ for the corresponding entry of A. If A, B are two tensors of the same dimension, we denote their entry-wise product by $A \circ B$: $A \circ B[I] = A[I] \cdot B[I]$ and their inner product by $\langle A, B \rangle = \sum_{I \in [d]^N} A[I] \cdot B[I]$.

2.2. **XOR games.** An N-player *XOR game* can be described as follows. Let n be a positive integer, π a probability distribution on $[n]^N$, and $A : [n]^N \to \{\pm 1\}$ a sign N-tensor. At the start of the game $G = (A, \pi)$, a verifier picks an N-tuple $(i_1, \ldots, i_N)$ of questions according to π and distributes these among the players. The players, who may agree on a strategy beforehand but are not allowed to communicate after receiving their questions, must answer the verifier with bits $x_1(i_1), \ldots, x_N(i_N) \in \{\pm 1\}$. They win the game if the product of their answers equals $A[i_1, \ldots, i_N]$. The player's success is given in terms of the *bias* of the game:

Definition 1. *Let M be a real or complex N-tensor. Define*

$$\beta(M) := \max_{x_1, \ldots, x_N : [n] \to \{\pm 1\}} \Big| \sum_{i_1, \ldots, i_N = 1}^{n} M[i_1, \ldots, i_N]\, x_1(i_1) \cdots x_N(i_N) \Big|$$

The classical bias *of a game $G = (A, \pi)$ is $\beta(A \circ \pi)$.*

With some abuse of notation, we often denote the classical bias by $\beta(G)$. With this definition, the maximum probability under π with which the players can win the game is simply $1/2 + \beta(G)/2$.

We also consider XOR games where the players share entanglement. In this setting, the players are allowed to have quantum systems described by Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_N$, respectively. Before the game begins, the players put the overall system in an entangled state $|\psi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and decide on sequences of $\{\pm 1\}$-valued measurement observables $X_1(i_1), \ldots, X_N(i_N)$ on their respective Hilbert spaces. Upon receiving their questions $i_1, \ldots, i_N$, the players answer with the measurement outcomes $x_1(i_1), \ldots, x_N(i_N)$ of their respective observables, when performed on their share of $|\psi\rangle$. The expectation of the product of their answers is exactly $\langle \psi | X_1(i_1) \otimes \cdots \otimes X_N(i_N) | \psi \rangle$. We can now define an entangled version of the bias.

Definition 2. *Let M be a real or complex N-tensor. Define*

$$\beta^*(M) := \max_{|\psi\rangle, X_1, \ldots, X_N} \Big| \sum_{i_1, \ldots, i_N = 1}^{n} M[i_1, \ldots, i_N]\, \langle \psi | X_1(i_1) \otimes \cdots \otimes X_N(i_N) | \psi \rangle \Big|.$$

The entangled bias *of a game $G = (A, \pi)$ is $\beta^*(A \circ \pi)$.*

2.3. **Grothendieck inequalities.** Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Then the (real or complex) Grothendieck constant *of order d*, denoted by $K_G^{\mathbb{K}}(d)$, is the smallest positive constant such that for every $n_1, n_2 \in \mathbb{N}$, for every matrix $M = (M_{ij}) \in \mathbb{K}^{n_1 \times n_2}$ and vectors $u_1, \ldots, u_{n_1}, v_1, \ldots, v_{n_2} \in \mathcal{S}(\mathbb{K}^d)$, the following inequality holds:

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} M_{ij} \langle u_i, v_j \rangle \leq K_G^{\mathbb{K}}(d) \max_{\substack{x : [n_1] \to \mathcal{S}(\mathbb{K}) \\ y : [n_2] \to \mathcal{S}(\mathbb{K})}} \Big| \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} M_{ij} x(i) y(j) \Big|. \tag{1}$$

The real and complex Grothendieck constants are defined as $K_G^{\mathbb{K}} := \sup_d K_G^{\mathbb{K}}(d)$. The exact values of $K_G^{\mathbb{R}}$ and $K_G^{\mathbb{C}}$ are still unknown, but they have been shown to be bounded as follows: $1.6770 \lesssim K_G^{\mathbb{R}} \lesssim 1.7822$ and $1.3380 \lesssim K_G^{\mathbb{C}} \lesssim 1.4049$.

Tsirelson [Tsi93] gave a very elegant characterization of the bias of two-player XOR games with entanglement. Namely, for an n-by-n sign matrix A and probability distribution π he showed

$$\beta^*(A \circ \pi) = \max_{u_i, v_j \in \mathcal{S}(\mathbb{R}^n)} \sum_{i,j=1}^{n} A[i,j]\pi(i,j)\, \langle u_i, v_j \rangle.$$

Combined with Grothendieck's inequality (1), this yields $\beta^*(A \circ \pi) \le K_G^{\mathbb{R}} \beta(A \circ \pi)$, for any sign matrix A and probability distribution π. This shows that for two-player XOR games the QC-gap is bounded by a constant.

To facilitate our discussion on extensions of Grothendieck's inequality, we will use the following notation.

Definition 3. *Let $\mathbb{K}, \mathbb{K}' = \mathbb{R}$ or $\mathbb{C}$. For an N-tensor $A : [n]^N \to \mathbb{K}$, define its norm*

$$\|A\|_{\infty,\mathbb{K}'} := \max_{\phi_1,\dots,\phi_N \in \mathcal{S}(\ell_\infty^n)} \Big| \sum_{i_1,\dots,i_N=1}^{n} A[i_1,\dots,i_N]\phi_1(i_1)\cdots\phi_N(i_N) \Big|,$$

where the underlying scalar field for ℓ_∞^n is $\mathbb{K}'$. Also, define

$$\gamma^*(A) = \sup_d \sup_{\phi_1,\dots,\phi_N : [n] \to \mathcal{S}(\mathbb{C}^d)} \Big| \sum_{i_1,\dots,i_N=1}^{n} A[i_1,\dots,i_N]\langle \phi_1(i_1),\dots,\phi_N(i_N)\rangle \Big|.$$

If $\mathbb{K}' = \mathbb{R}$ and A is a tensor that was constructed from a game G, then $\|A\|_{\infty,\mathbb{R}}$ is exactly the classical bias $\beta(G)$ of the game.

2.4. **Multipartite entanglement.** The following definition will be useful in studying the different biases achievable by players who are restricted to sharing a specific type of entanglement.

Definition 4. *Let A be a N-dimensional tensor, and $|\Psi\rangle \in \mathcal{H}^{\otimes N}$ a fixed entangled state shared by N players. Then the* bias restricted to $|\Psi\rangle$, *denoted $\beta^*_{|\Psi\rangle}(A)$, is defined as*

$$\beta^*_{|\Psi\rangle}(A) \;=\; \max_{M_1,\dots,M_N} \Big| \sum_{i_1,\dots,i_N} A[i_1,\dots,i_N]\langle\Psi|M_1(i_1)\otimes\cdots\otimes M_k(i_N)|\Psi\rangle \Big|$$

where the maximum is taken over all sets of $\{\pm1\}$-valued observables $M_\ell(i_\ell)$ on $\mathcal{H}$. For a game $G = (A,\pi)$, we will also write $\beta^*_{|\Psi\rangle}(G)$ for $\beta^*_{|\Psi\rangle}(A \circ \pi)$.

The following setups are the ones that we will encounter most frequently, and for each we introduce a special notation for the bias. For the case of GHZ states $|\Psi\rangle = d^{-1/2}\sum_{i=1}^d |i\rangle_1 \cdots |i\rangle_N$ (of arbitrary dimension d) we will denote the maximum bias by $\beta^*_{\mathrm{Z}}(G)$, while for Schmidt states $|\Psi\rangle = \sum_{i=1}^d \alpha_i |i\rangle_1 \cdots |i\rangle_N$ (with arbitrary dimension d and coefficients α_i) we will use the notation $\beta^*_{\mathrm{S}}(G)$. Finally, clique-wise entanglement is any type of entanglement that can be obtained by grouping the N players into k coalitions of r players each (a given player can take part in any number of coalitions), and allowing the members of each of the coalitions to share a GHZ state of arbitrary dimension. In that case, we denote the maximal bias by $\beta^*_{\mathrm{C}}(G)$. This may depend on the parameters k and r, which are kept implicit so as not to overload the notation, but will always be clear in context.

We have the following obvious relationships between the biases:

$$\beta(G) \le \beta^*_{\mathrm{Z}}(G) \le \beta^*_{\mathrm{S}}(G) \le \beta^*(G) \qquad \text{and} \qquad \beta(G) \le \beta^*_{\mathrm{Z}}(G) \le \beta^*_{\mathrm{C}}(G) \le \beta^*(G)$$

3. Proof of Theorem 1

In this section we give a proof of Theorem 1. First, in Section 3.1 we analyze the maximum bias $\beta^*_{\mathrm{Z}}(G)$ achievable by strategies that are limited to sharing a GHZ state, and show that it is upper-bounded by the maximum of an expression involving generalized inner products of n-dimensional vectors. This section can be seen as a warm-up to give the reader the general idea of our proofs. Then, in Section 3.2, we adapt a theorem by Tonge [Ton78] to show that the optimum of this problem is upper-bounded by a constant times the classical bias of the original game, thus showing a constant QC-gap for the case of GHZ states. Finally, in Section 3.3, we extend our proof to cover the case where the players are allowed to share a Schmidt state.

3.1. Strategies with GHZ states.

We prove the following lemma:

Lemma 8. *Let $G = (A, \pi)$ be an N-player game. Assume that the players are restricted to sharing a state of the form $|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle^{\otimes N}$. Then the maximum bias the players can achieve is upper-bounded as follows:*

$$\beta^*_{\mathrm{Z}}(G) \;\leq\; \gamma^*(A \circ \pi).$$

Proof: Fix any strategy of the players using entanglement $|\Psi\rangle$, and let $M_\ell(i)$ be the $\{\pm 1\}$ valued observable of player ℓ on question i in that strategy. Let $B = A \circ \pi$. The players' bias is given by

$$\begin{aligned}
\beta^*_{\mathrm{Z}}(G) &= \Big| \sum_{i_1,\dots,i_N} B[i_1,\dots,i_N] \langle \Psi | M_1(i_1) \otimes \cdots \otimes M_N(i_N) | \Psi \rangle \Big| \\
&= \frac{1}{d} \Big| \sum_{i_1,\dots,i_N} B[i_1,\dots,i_N] \sum_{i,j=1}^d \langle i | M_1(i_1) | j \rangle \cdots \langle i | M_N(i_N) | j \rangle \Big| \\
&\leq \frac{1}{d} \sum_{j=1}^d \Big| \sum_{i_1,\dots,i_N} B[i_1,\dots,i_N] \sum_{i=1}^d \langle i | M_1(i_1) | j \rangle \cdots \langle i | M_N(i_N) | j \rangle \Big| \\
&\leq \frac{1}{d} \sum_{j=1}^d \gamma^*(B) = \gamma^*(B).
\end{aligned}$$

The inequality holds as the inner sum on the third line is a generalized inner product of the j^{th} columns of the $M_k(i_k)$'s, which are unit vectors since these matrices are unitary. $\square$

3.2. Tonge's theorem.

In this section we introduce a slight generalization of a result originally due to Blei [Ble79] and later improved and simplified by Tonge [Ton78]. This theorem allows us to relate the maximization over generalized inner products from Lemma 8 to the classical bias of the game and thus prove Theorem 1 for the case of GHZ states.

Theorem 9. *Let $n, N \geq 2$ and d be positive integers, and $\mathcal{H}$ be a d-dimensional complex Hilbert space. Then, for every N-tensor $B : [n]^N \to \mathbb{R}$ and $f_1, \dots, f_N : [n] \to \mathcal{S}(\mathcal{H})$, the following inequality holds:*

$$\Big| \sum_{i_1,\dots,i_N=1}^n B[i_1,\dots,i_N] \langle f_1(i_1), \dots, f_N(i_N) \rangle \Big| \leq 2^{(3N-5)/2} K_G^{\mathbb{C}} \|B\|_{\infty,\mathbb{R}} \tag{2}$$

This theorem is proved in Appendix A. Since $\|A \circ \pi\|_{\infty,\mathbb{R}}$ is exactly the classical bias of $G = (A, \pi)$, combining Lemma 8 with Theorem 9 leads to the bound $\beta^*_{\mathrm{Z}}(G) \leq 2^{(3N-5)/2}\, K_G^{\mathbb{C}}\, \beta(G)$, thus proving Theorem 1 for the special case of GHZ states.

3.3. **Extension to Schmidt states.** We extend the result of Section 3.1 to the case of Schmidt states, thus proving Theorem 1 in full generality. For this, analoguously to Lemma 8, it is suffient to show that, if $|\Psi\rangle = \sum_{i=1}^{d} \alpha_i |i\rangle^{\otimes N}$ is a Schmidt state and B an N-tensor, then

$$\beta^*_{|\Psi\rangle}(B) \;\le\; \gamma^*(B)$$

The theorem will follow by setting $B = A \circ \pi$ for a N-player game $G = (A, \pi)$, and applying Theorem 9.

Proof of Theorem 1: Let B be an N-tensor, and $\{M_j(x_j)\}_{j,x_j}$ a choice of ± 1 valued observables which achieve the maximal bias $\beta^*_{|\Psi\rangle}(B)$. By absorbing any complex phases into the strategy of player one, we can assume that all coefficients α_i are positive reals. The following claim shows that $|\Psi\rangle$ can be expressed as a weighted sum of GHZ-type states.

Claim 10. *Let $|\Psi\rangle = \sum_{i=1}^{d} \alpha_i |i\rangle^{\otimes N}$ be a (normalized) state such that the α_i are positive reals. Then there exists positive reals $\beta_1, \ldots, \beta_d$ such that $|\Psi\rangle = \sum_{\ell=1}^{d} \beta_\ell |\phi_\ell\rangle$, where $|\phi_\ell\rangle = \sum_{i=1}^{\ell} |i\rangle^{\otimes N}$ for $\ell = 1, \ldots, d$ is a "partial" (un-normalized) GHZ state. Moreover, the β_i satisfy the following equation:*

$$\sum_{i,j=1}^{d} \beta_i \beta_j \cdot \min\{i,j\} = 1 \tag{3}$$

Proof: Renaming the basis vectors as necessary, we can assume that $\alpha_d \le \cdots \le \alpha_1$. Let $\beta_d = \alpha_d$ and $\beta_i = \alpha_i - \alpha_{i+1}$ for $i = 1, \ldots, d-1$. Then we have

$$|\Psi\rangle = \sum_{i=1}^{d} \beta_i |\phi_i\rangle.$$

Moreover, Eq. (3) is immediate from the fact that $|\langle\Psi|\Psi\rangle| = 1$ and $\langle\phi_i|\phi_j\rangle = \min\{i,j\}$ (recall that $|\phi_\ell\rangle$ itself was not normalized). □

This reformulation of $|\Psi\rangle$ reduces the task of showing an upper bound on $\beta^*_{|\Psi\rangle}(B)$ to a form similar to what we had before. Namely,

$$\beta^*_{|\Psi\rangle}(B) = \sum_{i,j} \beta_i \beta_j \sum_{x_1,\ldots,x_N} B[x_1,\ldots,x_N] \langle\phi_i| M_1(x_1) \otimes \cdots \otimes M_N(x_N) |\phi_j\rangle.$$

For fixed i, j, each term of the sum involves unnormalized "partial" GHZ states, which can be handled in the same fashion as Lemma 8.

Claim 11. *Let B be an N-tensor and $\{M_k(x_k)\}$ be ± 1 valued observables. Then*

$$\sum_{x_1,\ldots,x_N} B[x_1,\ldots,x_N] \langle\phi_i| M_1(x_1) \otimes \cdots \otimes M_N(x_N) |\phi_j\rangle \le \min\{i,j\} \gamma^*(B).$$

Proof:

$$\sum_{x_1,\ldots,x_N} B[x_1,\ldots,x_N] \langle\phi_i| M_1(x_1) \otimes \cdots \otimes M_N(x_N) |\phi_j\rangle =$$

$$\sum_{x_1,\ldots,x_N} B[x_1,\ldots,x_N] \sum_{s=1}^{i} \sum_{t=1}^{j} \langle s| M_1(x_1) |t\rangle \cdots \langle s| M_N(x_N) |t\rangle$$

We will order the double sum over s, t depending on whether i or j is smaller—we want the outer sum to be over the smaller one. Suppose that $i \leq j$. The other case is completely analogous. Then

$$\sum_{x_1,\dots,x_N} B[x_1,\dots,x_N] \sum_{s=1}^{i}\sum_{t=1}^{j} \langle s|M_1(x_1)|t\rangle \cdots \langle s|M_N(x_N)|t\rangle =$$
$$\sum_{s=1}^{i}\sum_{x_1,\dots,x_N} B[x_1,\dots,x_N] \sum_{t=1}^{j} \langle s|M_1(x_1)|t\rangle \cdots \langle s|M_N(x_N)|t\rangle$$

For each fixed s, the inner sum is now a generalized inner product of the first j entries of the s^{th} row of the $M_k(x_k)$'s. Since these have norm at most one, we have

$$\sum_{x_1,\dots,x_N} B[x_1,\dots,x_N]\langle \phi_i|M_1(x_1)\otimes\cdots\otimes M_N(x_N)|\phi_j\rangle \leq \min\{i,j\}\gamma^*(B).$$

□

We can now finish the proof.

$$\begin{aligned}
\beta^*_{|\Psi\rangle}(B) &= \sum_{i,j}\beta_i\beta_j \sum_{x_1,\dots,x_N} \langle \phi_i|M_1(x_1)\otimes\cdots\otimes M_N(x_N)|\phi_j\rangle \\
&\leq \sum_{i,j}\beta_i\beta_j \min\{i,j\}\gamma^*(B) \\
&= \gamma^*(B).
\end{aligned}$$

The first inequality follows from Claim 11 and the second by Claim 10. □

4. Proof of Theorem 2

The proof of Theorem 2 is based on a result by Carne [Car80], which essentially shows how Grothendieck-type inequalities can be composed in order to prove new inequalities of the same type. This will let us prove bounds on the entangled bias when the players are allowed to share any combination of EPR pairs and GHZ states. We first explain Carne's theorem in Section 4.1, for which we give a self-contained proof in Appendix B. We explain how it is applied to prove Theorem 2 in Section 4.2. We will end this section with a proof of Corollary 3.

4.1. **Carne's theorem.** Carne [Car80] showed that inequalities such as the one by Blei and Tonge (Theorem 9), could be composed in order to prove more general inequalities. His result also shows how Tonge's inequality can be re-derived as a consequence of the original Grothendieck inequality, though Tonge's version of inequality (2) is tighter.

To describe Carne's theorem, consider a hypergraph $H = (V, E)$, and associate with each edge $e \in E$ and vertex $x \in e$ a complex Hilbert space $\mathcal{H}(x, e)$. Define $\mathcal{H}_x := \bigotimes_{e\in E(x)} \mathcal{H}(x, e)$. Assume that, with each edge $e \in E$ is associated a multi-linear continuous functional $\psi_e : \bigotimes_{x\in e} \mathcal{H}(x, e) \to \mathbb{C}$ that satisfies a Grothendieck-type inequality, i.e. for every $|e|$-tensor $D : [n]^e \to \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and set of functions $f_x : [n] \to \mathcal{S}(\mathcal{H}_x)$, for each $x \in e$, the inequality

$$\sum_{K\in[n]^{|e|}} D[K]\psi_e\Big(\bigotimes_{x\in e} f_x(i_x)\Big) \leq C_e^{\mathbb{K}'} \|D\|_{\infty,\mathbb{K}'} \tag{4}$$

holds for a field $\mathbb{K}' = \mathbb{R}$ or $\mathbb{C}$, and some constant $C_e^{\mathbb{K}'}$ independent of the tensor D. Carne's theorem then states that the natural combination of the linear functionals ψ_e in a general multi-linear functional Φ defined over the whole Hilbert space $\mathcal{H} = \otimes_{x\in V}\mathcal{H}(x)$ also satisfies a Grothendieck-type inequality, with underlying constant the product of the $C_e^{\mathbb{K}'}$. Note that, since a vertex x can be part of many edges, there can be many functionals ψ_e which act on the same space $\mathcal{H}(x)$. This is what makes Carne's theorem non-trivial. To state it we need to define the linear *re-arranging map* σ as

$$\sigma : \bigotimes_{x\in V}\Big(\bigotimes_{e\in E(x)}\mathcal{H}(x,e)\Big) \to \bigotimes_{e\in E}\Big(\bigotimes_{x\in e}\mathcal{H}(x,e)\Big),$$

which simply permutes the factors of a vector $v \in \bigotimes_{x\in V}\mathcal{H}_x$.

Theorem 12 (Slight extension of Carne 1980)**.** *Let $\mathbb{K},\mathbb{K}' \in \{\mathbb{R},\mathbb{C}\}$. The linear functional defined by $\Phi := \Big(\bigotimes_{e\in E}\psi_e\Big)\circ\sigma$ satisfies that for every $|V|$-tensor $A : [n]^V \to \mathbb{K}$ and set of functions $f_x : [n] \to \mathcal{S}\left(\mathcal{H}_x\right)$, for $x \in V$, the following inequality holds:*

$$\sum_{I\in[n]^V} A[I]\cdot\Phi\Big(\bigotimes_{x\in V} f_x(i_x)\Big) \le \Big(\prod_{e\in E} C_e^{\mathbb{K}'}\Big)\|A\|_{\infty,\mathbb{K}'}. \tag{5}$$

where the constants $C_e^{\mathbb{K}'}$ are such that (4) *holds.*

In particular, if $\mathbb{K} = \mathbb{R}$ and each ψ_e is the generalized inner product function on $\bigotimes_{x\in e}\mathcal{H}(x,e)$, then it follows from Theorem 9 that the constant in (5) is upper bounded by $\Big(\prod_{e\in E}2^{(3|e|-5)/2}\Big)(K_G^{\mathbb{C}})^{|E|}$.

4.2. **Bounding the bias achievable by strategies with clique-wise entanglement.** Consider an N-player game $G = (A,\pi)$. Let the players be organized in k coalitions of r players each[1], where a given player can take part in any number of coalitions. Each coalition of players is allowed to share a GHZ state between its members.

To model this setup, associate a hypergraph $H = (V,E)$ to the coalition structure, with $V = [N]$ and there is a hyperedge for every coalition. For every edge e we introduce a Hilbert space $\mathcal{H}(e) = \otimes_{x\in e}\mathcal{H}(x,e)$, where $\mathcal{H}(x,e)$ is the local space of player x corresponding to edge e. The state of the players in this space is initialized in a GHZ state $|\Psi_e\rangle = d^{-1/2}\sum_{i=1}^d |i\rangle^{\otimes|e|}$. The global entangled state shared by the players at the start of the game is then

$$|\tilde{\Psi}\rangle = \otimes_{e\in E}|\Psi_e\rangle \in \bigotimes_{e\in E}\Big(\bigotimes_{x\in e}\mathcal{H}(x,e)\Big) \tag{6}$$

Finally, each player x has observables $M_x(i)$ corresponding to question i. These act on player x's local space $\mathcal{H}(x) = \otimes_{e\ni x}\mathcal{H}(x,e)$.

Theorem 2 states that the maximum bias achievable by a strategy of the form that we have just described is at most a constant times the classical bias of the game. In order to prove it, we first relate the bias achieved by any strategy to an expression similar to the one appearing on the left-hand side of (5) in Carne's theorem, where ψ_e will be the linear functional that is associated with the GHZ state, i.e. the generalized inner product function. Applying Theorem 12 will conclude the argument.

Proof of Theorem 2: Fix observables M_x and an entangled state $|\Psi\rangle \in \otimes_{x\in V}\mathcal{H}_x$ of the form described above. Note that $|\Psi\rangle = \sigma^{-1}(|\tilde{\Psi}\rangle)$, where $|\tilde{\Psi}\rangle$ is described in Eq. (6). This is because we need to re-arrange the terms in the definition of $|\tilde{\Psi}\rangle$ to correspond to the decomposition of space $\otimes_{x\in V}\mathcal{H}_x$.

[1] The organization of these coalitions is independent of the game itself; rather it is used to define the structure of the entanglement that is shared between the players.

We begin by expanding $\langle\Psi|\bigotimes_{x\in V} M_x|\Psi\rangle$, with the goal of relating it to the map Φ of Theorem 12. Let $[d]^E$ denote the set of $|E|$-tuples $(j_e)_{e\in E}$. We have

$$\begin{aligned}
|\Psi\rangle &= \sigma^{-1}\left(\frac{1}{\sqrt{d^{|E|}}}\bigotimes_{e\in E}\Big(\sum_{e_i=1}^{d}\bigotimes_{x\in e}|j_{e_i}\rangle\Big)\right)\\
&= \frac{1}{\sqrt{d^{|E|}}}\sum_{J\in[d]^E}\bigotimes_{x\in V}|J_{|E(x)}\rangle
\end{aligned}$$

where $J_{|E(x)}$ denotes the restriction of the tuple $J\in[d]^E$ to those edges that contain the vertex x. Since observables are Hermitian, the expected value $\langle\Psi|\bigotimes_{x\in V} M_x|\Psi\rangle$ is given by

$$\begin{aligned}
\langle\Psi|\bigotimes_{x\in V} M_x|\Psi\rangle &= \frac{1}{2\cdot d^{|E|}}\sum_{J',J\in[d]^E}\left(\prod_{x\in V}\langle J'_{|E(x)}|M_x|J_{|E(x)}\rangle+\prod_{x\in V}\langle J_{|E(x)}|M_x|J'_{|E(x)}\rangle\right)\\
&= \frac{1}{2\cdot d^{|E|}}\sum_{J',J\in[d]^E}\left(\prod_{x\in V}\langle J'_{|E(x)}|M_x|J_{|E(x)}\rangle+\prod_{x'\in V}\langle J'_{|E(x')}|M^*_{x'}|J_{|E(x')}\rangle\right)\\
&= \frac{1}{d^{|E|}}\sum_{J'\in[d]^E}\left(\sum_{J\in[d]^E}\Re\Big(\prod_{x\in V}\langle J'_{|E(x)}|M_x|J_{|E(x)}\rangle\Big)\right)\\
&= \frac{1}{d^{|E|}}\sum_{J'\in[d]^E}\Re\left(\sum_{J\in[d]^E}\prod_{x\in V}\left[M_x\right]_{J_{|E(x)},\,J'_{|E(x)}}\right), \qquad (7)
\end{aligned}$$

where the subscript $(J_{|E(x)}, J'_{|E(x)})$ indicates a row-column pair of the matrix M_x. Note that, since the expression on the left-hand side is real, the one on the right is too, and we can safely ignore the $\Re$ symbol on the right. Since the M_x are unitary matrices, their columns are unit vectors. This implies that there exists unit vectors $v_x\in\bigotimes_{e\in E(x)}\mathcal{H}(x,e)$ (depending on J') such that the expression between the brackets in equation (7) is of the form

$$\sum_{J\in[d]^E}\prod_{x\in V} v_x(J_{|E(x)})$$

where as usual $v_x(J_{|E(x)})$ denotes the restriction of the vector v_x to those indices in $J_{|E(x)}$.

Claim 13. *For $\Phi := \Big(\bigotimes_{e\in E}\psi_e\Big)\circ\sigma$ with ψ_e the generalized inner product function on $\bigotimes_{x\in e}\mathcal{H}(x,e)$, we have*

$$\sum_{J\in[d]^E}\prod_{x\in V} v_x(J_{|E(x)}) = \Phi\Big(\bigotimes_{x\in V} v_x\Big).$$

Proof: Since Φ is linear, it suffices to prove the claim for vectors of the form $v_x = \bigotimes_{e\in E(x)} v_{x,e}$, where each $v_{x,e} \in \mathcal{H}(x,e)$. In this case, we have

$$\begin{aligned}
\Big(\bigotimes_{e\in E}\psi_e\Big)\circ\sigma\Big(\bigotimes_{x\in V}\big(\bigotimes_{e\in E(x)} v_{x,e}\big)\Big) &= \bigotimes_{e\in E}\Big(\psi_e\big(\bigotimes_{x\in e} v_{x,e}\big)\Big)\\
&= \prod_{e\in E}\Big(\sum_{j_e=1}^{d}\Big(\prod_{x\in e} v_{x,e}(j_e)\Big)\Big)\\
&= \sum_{J\in[d]^E}\prod_{e\in E}\Big(\prod_{x\in e} v_{x,e}(j_e)\Big)\\
&= \sum_{J\in[d]^E}\prod_{x\in V}\Big(\prod_{e\in E(x)} v_{x,e}(j_e)\Big),
\end{aligned}$$

where the last product is $\prod_{e\in E(x)} v_{x,e}(j_e) = v_x(J_{|E(x)})$. □

Let $M_x(i)$ be the observable used by player x on question i, so that the bias achieved by this strategy in the game $G = (A,\pi)$ is

$$\Big|\sum_{I\in[n]^V} B[I]\,\langle\Psi|\bigotimes_{x\in V} M_x(i_x)|\Psi\rangle\Big|$$

where $B = A\circ\pi$. We can bound this expression by

$$\begin{aligned}
(8)\quad &\Big|\sum_{I\in[n]^V} B[I]\left(\frac{1}{d^{|E|}}\sum_{J'\in[d]^E}\sum_{J\in[d]^E}\prod_{x\in V}\big[M_x(i_x)\big]_{J_{|E(x)},\,J'_{|E(x)}}\right)\Big|\\
&\leq \frac{1}{d^{|E|}}\sum_{J'\in[d]^E}\Big|\sum_{I\in[n]^V} B[I]\cdot\sum_{J\in[d]^E}\prod_{x\in V}\big[M_x(i_x)\big]_{J_{|E(x)},\,J'_{|E(x)}}\Big|\\
&\leq \max_{J'\in[d]^E}\Big|\sum_{I\in[n]^V} B[I]\cdot\sum_{J\in[d]^E}\prod_{x\in V}\big[M_x(i_x)\big]_{J_{|E(x)},\,J'_{|E(x)}}\Big|\\
&\leq \max_{f_x:[n]\to\mathcal{S}(\mathcal{H}_x):\,x\in V}\Big|\sum_{I\in[n]^V} B[I]\cdot\Phi\Big(\bigotimes_{x\in V} f_x(i_x)\Big)\Big|,
\end{aligned}$$

where the first equality is (7), and the last inequality follows from Claim 13. The result then follows directly from Theorem 12 combined with the bound in Theorem 9, giving the last part of the theorem. □

We end this section with a proof of Corollary 3.

Proof of Corollary 3: Theorem 5 in [BFG06] states that, if $|\Psi\rangle$ is any stabilizer state shared in an arbitrary way among three parties, then $|\Psi\rangle$ is local-unitarily equivalent to a number of EPR pairs shared between each of the three pairs of players, together with a GHZ state shared in common. They even give the number of such states, based on the structure of the initial stabilizer state. It now suffices to consider the hypergraph G with vertex set $V = \{1,2,3\}$, and edge set $E = \{\{1,2\},\{2,3\},\{1,3\},\{1,2,3\}\}$. In the notation of Theorem 2, this hypergraph has $k = 4$ and $r \leq 3$, which gives the bound $2^8(K_G^{\mathbb{C}})^4$. However, a careful examination of the proof of Theorem 2 easily reveals that the inequality holds with the smaller constant $8(K_G^{\mathbb{C}})^4$. □

5. Application to communication complexity

In this section we give a proof of Theorem 4, showing a lower bound on quantum multiparty communication complexity for clique-wise entanglement.

For a sign N-tensor A, let $R_\epsilon(A)$ denote the multiparty randomized communication complexity of A with error at most ϵ, and let $R_\epsilon^{|\psi\rangle}(A)$ denote the minimal cost of an ϵ bounded-error protocol where the players share a state $|\psi\rangle$, and communicate classical bits. Finally, let $Q_\epsilon^{|\psi\rangle}(A)$ denote the minimal cost of a protocol in the strongest model we will consider—where the players share entanglement $|\psi\rangle$ and use quantum communication. We refer the reader to [LSS09] for a description of the multiparty quantum model of communication. The reader should think of all these measures in the number-in-the-hand (NIH) model of communication; at the end we will explain why the results also hold in the number-on-the-forehead (NOF) model.

The generalized discrepancy method is a very useful lower bound method for randomized communication complexity and still essentially the only lower bound method available in the NOF model of multiparty complexity. It was developed over a sequence of works for the two-party and multiparty models [Kla07, Raz03, She08, LS09, CA08].

Theorem 14. *Let A be a sign N-tensor. Then*

$$2^{R_\epsilon(A)} \geq \max_{B,\pi} \frac{\langle A, B \circ \pi\rangle - 2\epsilon}{\beta(B \circ \pi)}$$

where the maximization is over all sign N-tensors B and probability distributions π.

We start by proving a result exactly analogous to Theorem 14 for protocols with entanglement.

Proposition 15. *Let A be a sign N-tensor. Then for any state $|\psi\rangle$*

$$2^{R_\epsilon^{|\psi\rangle}(A)} \geq \max_{B,\pi} \frac{\langle A, B \circ \pi\rangle - 2\epsilon}{\beta^*_{|\psi\rangle}(B \circ \pi)}$$

where the maximization is over all sign N-tensors B and probability distributions π.

Proof: Consider a communication protocol with entanglement $|\psi\rangle$ for A of minimal cost c and error at most ϵ. Let R be the N-tensor such that $R[x_1, \dots, x_N]$ is the expectation of the output of this protocol on input $(x_1, \dots, x_N)$. By assumption of the correctness of the protocol, if $A[x_1, \dots, x_N] = 1$ then $1 - 2\varepsilon \leq R[x_1, \dots, x_N] \leq 1$ and if $A[x_1, \dots, x_N] = -1$ then $-1 \leq R[x_1, \dots, x_N] \leq -1 + 2\varepsilon$.

Fix a probability distribution π and let B be an arbitrary sign tensor of the same dimensions as A. We will see how the communication protocol for A can be used to design a XOR protocol for B. The bias of this protocol will be related to the amount of communication c and the correlation $\langle A, B \circ \pi\rangle$.

The strategy in the XOR game is as follows. We may assume that the players have access to a shared random string r. A convexity argument shows that shared randomness cannot increase the bias. On input $(x_1, \dots, x_k)$ the players look at the shared random string r of length c. The players interpret r as a "guess" for the transcript of the communication protocol on input $(x_1, \dots, x_N)$. Their goal is to discover if this transcript is correct. The point is that if it is not, at least one player will notice it.

Suppose that the first player speaks first. She makes a measurement on the entangled state and determines that in the communication protocol she would speak a bit b_1. She then checks if b_1 agrees with r_1, the first bit of r. Say that the second player speaks next. Assuming that r_1 is the bit communicated by the first player, he then makes a measurement and determines a bit b_2 that he would communicate in the protocol. He then checks if b_2 agrees with r_2, the second bit of the random string. This process continues in this fashion as the players simulate the entire communication protocol.

If at any time player i notices that a bit r_t does not agree with what he would communicate, assuming that the communication thus far has been given by $r_1 \cdots r_{t-1}$, we say that r is inconsistent with player i. Otherwise it is consistent.

Now we define the output conditions

- If r is inconsistent with the first player, then she outputs a random bit in $\{-1,+1\}$. Otherwise, she outputs a bit $\{-1,+1\}$ with expectation $R[x_1,\ldots,x_N]$.
- If r is inconsistent with player i for $i>1$, then they output a random bit. Otherwise, they output 1.

Let $P[x_1,\ldots,x_N]$ be the expected output of this protocol on input $x_1,\ldots,x_N$. Let us now compute the correlation of this protocol with B under π:

$$\begin{aligned}\beta^*(B\circ\pi) &\geq \langle B\circ\pi,P\rangle\\ &= \frac{1}{2^c}\sum_{x_1,\ldots,x_N}\pi(x_1,\ldots,x_N)B[x_1,\ldots,x_N]R[x_1,\ldots,x_N]\\ &\geq \frac{1}{2^c}\left(\sum_{x_1,\ldots,x_N}\pi(x_1,\ldots,x_N)B[x_1,\ldots,x_N]A[x_1,\ldots,x_N]-2\varepsilon\right)\end{aligned}$$

Rearranging, this gives the desired result:

$$2^c \geq \max_{B,\pi}\frac{\langle A,B\circ\pi\rangle-2\varepsilon}{\beta^*(B\circ\pi)}$$

□

In the two-party case, it is known that the model of shared entanglement and classical communication can simulate the model of shared entanglement and quantum communication with a factor of two overhead. The key idea is that if the parties share EPR-pairs, they can use these to pass quantum messages via teleportation with a cost of two classical bits per qubit. We can also use this trick in the multiparty setting.

Claim 16. *Let A be a sign N-tensor. Let $|\psi\rangle$ be an entangled state, and let $R_\epsilon^{|\psi\rangle,E}(A)$ be the minimum of $R_\epsilon^{|\psi\rangle'}(A)$ over all entangled states $|\psi\rangle'$ constituted of $|\psi\rangle$ together with an arbitrary number of EPR pairs. Then*

$$Q_\epsilon^{|\psi\rangle}(A)\geq\frac{R_\epsilon^{|\psi\rangle,E}(A)}{2}.$$

We now can prove Theorem 4.

Proof of Theorem 4: Let $|\psi\rangle$ be a clique-wise entangled state containing k coalitions. We augment $|\psi\rangle$ to a state $|\psi\rangle'$ which additionally includes an arbitrary number of shared EPR-pairs between each pair of players. By Theorem 15 and Claim 16 we have

$$Q_\epsilon^{|\psi\rangle}(A)\geq\frac{1}{2}\max_{B,\pi,|\psi\rangle'}\log\left(\frac{\langle A,B\circ\pi\rangle-2\epsilon}{\beta^*_{|\psi\rangle'}(B\circ\pi)}\right)$$

where the maximum is over all sign N-tensors B, probability distributions π, and states $|\psi\rangle'$ constituted of $|\psi\rangle$ together with an arbitrary number of EPR pairs. By Theorem 2, for any such B, π and $|\psi\rangle'$ we have

$$\beta^*_{|\psi\rangle'}(B\circ\pi)\leq 2^{3(k+N^2)N/2}\beta(B\circ\pi).$$

Thus we obtain

$$Q_\epsilon^{|\psi\rangle}(A)\geq\frac{1}{2}\max_{B,\pi}\log\left(\frac{\langle A,B\circ\pi\rangle-2\epsilon}{\beta(B\circ\pi)}\right)-O(kN^3).$$

□

Thus far we have phrased things for the NIH model of multiparty communication complexity. We can transfer this reasoning to the NOF model as follows. For a function $f(x_1, \ldots, x_N)$ we can define a new function f' that takes as arguments N many $N-1$-tuples of strings. We say that these tuples are consistent if they are a valid input to the NOF problem, that is if their union is exactly N distinct strings. When the arguments are consistent the value of f' is the same as f, otherwise it is zero. In the same way, for a probability distribution π on f we can define a distribution π' on f'. It can now be seen that

$$\beta(f' \circ \pi') = \max_{x_1 \ldots, x_N} \sum_{i_1, \ldots, i_N} (f \circ \pi)(i_1, \ldots, i_N) x_1(i_2, \ldots, i_N) \cdots x_N(i_1, \ldots, i_{N-1}),$$

where the maximum is over functions $x_i : [n]^{N-1} \to \{-1, +1\}$. The right-hand side is the standard definition of discrepancy in the number-on-the-forehead model (up to a constant $O(2^N)$ as discrepancy is usually defined in terms of 0/1 vectors). All our arguments carry through considering the function f'. The fact that f' is a much larger tensor than f is immaterial as Grothendieck's inequality is independent of the size of the tensor.

Finally, we conclude this section by giving some examples of bounds that can be shown by the generalized discrepancy method. Let $\mathrm{GIP}_n(x_1, \ldots, x_N)$ be the generalized inner product function, which returns the parity of the intersection size of the x_i. Here the x_i are n bit strings. Babai, Nisan, and Szegedy showed a lower bound of $\frac{n}{2^{2N}}$ on the NOF complexity of GIP_n using the discrepancy method [BNS92]. The generalized discrepancy method can be used to show a bound of $\frac{n^{1/(N+1)}}{2^{2^N}}$ on the NOF complexity of the set intersection problem [LS09, CA08].

6. Hardness of approximation of the entangled bias

As noted by Khot and Naor [KN08], hardness of approximation results for Max-E3-Lin2 due to Håstad and Venkatesh [HV04] can be extended to show that:

- Unless P=NP, for any constant $c > 1$ there is no polynomial-time algorithm which approximates the classical bias of a three-party XOR game to within a multiplicative factor c.
- Unless NP$\subseteq$DTIME$(n^{(\log n)^{O(1)}})$, for any $\varepsilon > 0$ the classical bias of a three-party XOR game cannot be approximated to within a multiplicative factor $2^{(\log n)^{1-\varepsilon}}$ in time $2^{(\log n)^{O(1)}}$.

The inapproximability results in [HV04] only hold for *symmetric* strategies, in which the players all share the same strategy. However, Khot and Naor show that the inapproximability result holds even when restricted to games $G = (A, \pi)$ that are invariant under permutations of the three players (i.e. $B[i,j,k] = B[i,k,j] = B[j,i,k] = B[j,k,i] = B[k,i,j] = B[k,j,i]$, where $B = A \circ \pi$) and are such that the same question is never asked to two players simultaneously (i.e. $B[i,j,j] = B[j,i,j] = B[j,j,i] = 0$). In this case Lemma 2.1 in [KN08] shows that the optimum with respect to symmetric strategies is within a factor 10 of the general optimum.

Combining this result with Theorems 1 and 2 immediately gives a proof of Theorem 5. Indeed, Theorem 1 (resp. Theorem 2) shows that, as long as the players are restricted to using an arbitrary Schmidt state (resp. clique-wise entanglement), the quantum bias is at most a constant times the classical bias. Hence any constant approximation to the quantum bias would give a constant approximation to the classical bias, which is ruled out by the hardness result from [HV04].

7. Proof of Theorem 7

Here, we prove Theorem 7, which says that the Banach algebra formed by S_∞, the space of compact operators on a Hilbert space $\mathcal{H}$, together with the Schur product (the entry-wise product), is a Q-algebra. The following theorem gives a simple characterization of a Q-algebra. It is a slight reformulation of a result by Davie [Dav73], and is taken from Theorem 23 of Pérez-García et al. [PGWP+08].

Theorem 17. *Let $\mathcal{X} = (X, \cdot)$ be a commutative Banach algebra. Then $\mathcal{X}$ is a Q-algebra if and only if there exists a universal constant K, such that for every choice of positive integers N and n, N-tensor $A : [n]^N \to \mathbb{R}$, and functions $f_1, \dots, f_N : [n] \to \mathcal{S}(X)$, the following inequality holds:*

$$\Big\| \sum_{I\in[n]^N} A[I] f_1(i_1)\cdots f_N(i_N) \Big\|_X \le K^N \|A\|_{\infty,\mathbb{R}}, \tag{9}$$

where $\|\cdot\|_X$ denotes the norm associated with the Banach space X.

Theorem 17 follows from the more standard characterization of Q-algebras of [DJT95, Theorem 18.7] by using the inequality $\|A\|_{\infty,\mathbb{C}} \le 2^N \|A\|_{\infty,\mathbb{R}}$, and the fact that without loss of generality, we may decouple the variables and consider N-linear forms instead of general polynomials [DJT95, Lemma 18.5 and Proposition 18.6].

Proof of Theorem 7: We will show that Equation (9) holds for $\mathcal{X} = (S_\infty, \circ)$. It follows from the Spectral Theorem that for any $\epsilon > 0$, we can approximate any $T \in \mathcal{S}(S_\infty)$ by a finite-rank operator $T' \in S_\infty$. Since both N and n are finite, we only need to deal with a finite number of finite rank operators in S_∞. All-together these operators act only on a finite-dimensional subspace of the original Hilbert space $\mathcal{H}$. Hence, it will suffice to prove the statement for the case where $\mathcal{H}$ is finite dimensional and the operators $f_l(i_l)$ are finite dimensional matrices. Setting $\epsilon = 1/(4N)$ introduces at most an extra factor of 2 on the right-hand side of Equation (9). Further notice that it suffices to show this for Hermitian matrices $f_l(i_l)$, since for $T \in M_d$, we have that the matrix

$$\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$$

has the same norm as T and is Hermitian. We have

$$\begin{aligned} \Big\| \sum_{I\in[n]^N} A[I] f_1(i_1)\circ\cdots\circ f_N(i_N) \Big\|_{S_\infty} &= \max_{\alpha\in\mathcal{S}(\mathcal{H})} \Big| \bar{\alpha}\Big(\sum_{I\in[n]^N} A[I] f_1(i_1)\circ\cdots\circ f_N(i_N)\Big)\alpha \Big| \\ &= \max_{\alpha\in\mathcal{S}(\mathcal{H})} \Big| \sum_{I\in[n]^N} A[I] \sum_{i,j} \bar{\alpha}_i \alpha_j \langle i|^{\otimes N} f_1(i_1)\otimes\cdots\otimes f_N(i_N) |j\rangle^{\otimes N} \Big|, \end{aligned} \tag{10}$$

where we used the fact that $\|\cdot\|_{S_\infty}$ simply denotes the spectral norm and wrote $\alpha = (\alpha_1, \alpha_2, \dots)$ using some orthonormal basis for $\mathcal{H}$. Fix the $\alpha = (\alpha_1, \alpha_2, \dots)$ which maximizes this sum.

Let $|\Psi\rangle = \sum_{i=1}^d \alpha_i |i\rangle^{\otimes N}$. Then we can succinctly write the last expression in (10) as

$$\left| \sum_{I\in[n]^N} A[I] \langle\Psi| f_1(i_1)\otimes\cdots\otimes f_N(i_N) |\Psi\rangle \right|, \tag{11}$$

where $|\Psi\rangle = \sum_i \alpha_i |i\rangle^{\otimes N}$. By the triangle inequality, replacing the $f_l(i_l)$ by $\{\pm 1\}$-valued observables (Hermitian unitary matrices) which maximize the quantity (11) can only increase its value since these observables are the extreme points in the convex set of Hermitian matrices of norm at most 1. Hence by definition, (11) is bounded by the bias $\beta^*_{\mathrm{S}}(A)$. Theorem 1 then implies that Equation (9) holds with a constant $K = 2^{3/2}$. ∎

For further information on this problem, we refer to [Var75, LM98, PG06] and for information on Q-algebras, we refer to [DJT95, Chapter 18].

8. Grothendieck-type inequalities

We prove the following tri-linear extension of Grothendieck's inequality:

Theorem 18. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $G = (V, E)$ be a simple undirected graph and (V_1, V_2, V_3) be a partitioning of V. For each $x \in V_l$ let $\mathcal{H}(x, l)$ be a two-dimensional Hilbert space with underlying field $\mathbb{K}$ and $\mathcal{H}_l = \bigotimes_{x \in V_l} \mathcal{H}(x, l)$. Define the linear functional $\Phi_G : \bigotimes_{l=1}^3 \mathcal{H}_l \to \mathbb{K}$ by*

$$\Phi_G : \bigotimes_{l=1}^{3} v_l \mapsto \sum_{S_1 \subseteq V_1} \sum_{S_2 \subseteq V_2} \sum_{S_3 \subseteq V_3} (-1)^{|E(S_1 \cup S_2 \cup S_3)|} \prod_{l=1}^{3} v_l\big(S_l\big),$$

where $v_l \in \mathcal{H}_l$ and we index the $2^{|I_l|}$ coordinates of v_l by the subsets $S_l \subseteq I_l$. Then Φ_G satisfies that for every 3-tensor $A : [n]^3 \to \mathbb{K}$ and set of functions $f_l : [n] \to \mathcal{S}(\mathcal{H}_l)$, the following inequality holds

$$\Big| \sum_{i,j,k=1}^{n} A[i, j, k] \Phi_G\big(f_1(i) \otimes f_2(j) \otimes f_3(k)\big) \Big| \leq C \, \|A\|_{\infty, \mathbb{K}}$$

where $C = O(2^{|V|/2})$.

To prove this, we use the following theorem of Bravyi et al. [BFG06].

Theorem 19. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be complex Hilbert spaces and $|\Psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ a stabilizer state. Then there exist unitary operators U_1, U_2, U_3 on $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, respectively, such that the state $U_1 \otimes U_2 \otimes U_3 |\Psi\rangle$ is equal to a collection of GHZ and Bell states.*

The linear functionals appearing in Theorem 18 are derived from a special class of stabilizer states known as graph states. A q-qubit graph state is a unit vector in $\mathbb{C}^{2^q}$ which is uniquely defined by a simple undirected graph $G = (V, E)$ on q vertices. The graph state associated with G is given by

$$|\Psi\rangle = \frac{1}{\sqrt{2^q}} \sum_{S \subseteq V} (-1)^{|E(S)|} |S\rangle, \tag{12}$$

where $|E(S)|$ denotes the number of edges in the subgraph of G induced by the vertices in S and $|S\rangle$ denotes the computational basis state corresponding to the length-q characteristic vector of the set S.

Proof of Theorem 18: Let $|V| = q$ and let $|\Psi\rangle$ be the unique graph state associated with G, as given by (12). Let (V_1, V_2, V_3) be a partitioning of V such that party l has the qubits indexed by the labels in V_l and denote the respective Hilbert spaces by $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$. Then, using an appropriate arrangement of the Hilbert spaces, we may write $|\Psi\rangle$ as

$$|\Psi\rangle = \frac{1}{\sqrt{2^q}} \sum_{S_1 \subseteq V_1} \sum_{S_2 \subseteq V_2} \sum_{S_3 \subseteq V_3} (-1)^{|E(S_1 \cup S_2 \cup S_3)|} |S_1\rangle \otimes |S_2\rangle \otimes |S_3\rangle,$$

It is easily seen that for vectors $v_l \in \mathcal{H}_l$ and linear functional Φ_G as defined in the theorem, we have $(v_1 \otimes v_2 \otimes v_3) \cdot |\Psi\rangle = 2^{-q/2} \Phi_G(v_1 \otimes v_2 \otimes v_3)$.

Now let U_1, U_2 and U_3 be the unitary operators on $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, respectively, from Theorem 19. Then, by the unitary invariance of the sets $\mathcal{S}(\mathcal{H}_l)$, we have that, for any 3-tensor $A : [n]^3 \to \mathbb{R}$ and $f_l : [n] \to \mathcal{S}(\mathcal{H}_l)$,

$$\Big|\sum_{i,j,k} A[i,j,k]\,\Phi_G\big(f_1(i)\otimes f_2(j)\otimes f_3(k)\big)\Big| = 2^{q/2}\Big|\sum_{i,j,k} A[i,j,k]\,(f_1(i)\otimes f_2(j)\otimes f_3(k))\cdot|\Psi\rangle\Big|$$

$$\le 2^{q/2}\max_{g_l:[n]\to\mathcal{S}(\mathcal{H}_l)}\Big|\sum_{i,j,k} A[i,j,k]\,(g_1(i)\otimes g_2(j)\otimes g_3(k))\cdot|\Psi\rangle\Big|$$

$$\le 2^{q/2}\max_{g'_l:[n]\to\mathcal{S}(\mathcal{H}_l)}\Big|\sum_{i,j,k} A[i,j,k]\,(U_1g'_1(i)\otimes U_1g'_2(j)\otimes U_3g'_3(k))\cdot|\Psi\rangle\Big|$$

$$= 2^{q/2}\max_{g'_l:[n]\to\mathcal{S}(\mathcal{H}_l)}\Big|\sum_{i,j,k} A[i,j,k]\,g'_1(i)\otimes g'_2(j)\otimes g'_3(k)\cdot(U_1\otimes U_2\otimes U_3)|\Psi\rangle\Big|$$

$$\le 2^{q/2}\max_{|\Psi_C\rangle}\max_{h_l:[n]\to\mathcal{S}(\mathcal{H}_l)}\Big|\sum_{i,j,k} A[i,j,k]\,h_1(i)\otimes h_2(j)\otimes h_3(k)\cdot|\Psi_C\rangle\Big|, \tag{13}$$

where $|\Psi_C\rangle$ is a clique-wise entangled state, shared among the three parties. For a hypergraph $H=(T,E')$ with vertex set $T=[3]$ on three vertices and some dimension d, such a state has the form

$$|\Psi_C\rangle = \frac{1}{\sqrt{d^{|E'|}}}\sum_{J\in[d]^{E'}}\bigotimes_{l=1}^{3}|J_{|E'(l)}\rangle$$

as we saw in the proof of Theorem 2. Now, by Claim 13, we have that for vectors $a\in\mathcal{H}_1, b\in\mathcal{H}_2, c\in\mathcal{H}_3$, the following equalities hold:

$$(a\otimes b\otimes c)\cdot|\Psi_C\rangle = \frac{1}{\sqrt{d^{|E'|}}}\sum_{J\in[d]^{E'}} a(J_{|E'(1)})\cdot b(J_{|E'(2)})\cdot c(J_{|E'(3)}) = \frac{1}{\sqrt{d^{|E'|}}}\widetilde{\Phi}(a\otimes b\otimes c)$$

where $\widetilde{\Phi} = \big(\bigotimes_{e\in E'}\psi_e\big)\circ\sigma$ for generalized inner-product functions ψ_e on each of the edges. Hence, the last expression in (13) is equal to

$$\sqrt{\frac{2^q}{d^{|E'|}}}\max_{h_l:[n]\to\mathcal{S}(\mathcal{H}_l)}\Big|\sum_{i,j,k} A[i,j,k]\,\widetilde{\Phi}\big(h_1(i)\otimes h_2(j)\otimes h_3(k)\big)\Big|.$$

The result now follows directly from Theorems 9 and 12 and the fact that $d\ge 1$. □

Acknowledgements

JB thanks Peter Høyer, TL thanks Gideon Schechtman, and TV thanks Falk Unger for useful discussions.

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Appendix A. Proof of Tonge's theorem

In this section we prove an extension of Theorem 9 to the case where the base Hilbert space is not necessarily complex:

Theorem 20. *Let $n, N \geq 2$ and d be positive integers, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $\mathcal{H}$ be a d-dimensional Hilbert space with underlying field $\mathbb{K}$. Then, for every N-tensor $A : [n]^N \to \mathbb{K}$ and $f_1, \ldots, f_N : [n] \to \mathcal{S}(\mathcal{H})$, the following inequality holds:*

$$\Big| \sum_{i_1,\ldots,i_N=1}^{n} A[i_1,\ldots,i_N]\langle f_1(i_1),\ldots,f_k(i_N)\rangle \Big| \leq 2^{(N-2)/2} K_G^{\mathbb{K}} \|A\|_{\infty,\mathbb{K}}, \tag{14}$$

where $K_G^{\mathbb{R}}$ and $K_G^{\mathbb{C}}$ are the real and complex Grothendieck constant, respectively. Moreover, if the underlying field for A and the scalars on the right-hand side is $\mathbb{R}$, but the underlying field for $\mathcal{H}$ is $\mathbb{C}$, then the inequality

$$\Big| \sum_{i_1,\ldots,i_N=1}^{n} A[i_1,\ldots,i_N]\langle f_1(i_1),\ldots,f_k(i_N)\rangle \Big| \leq 2^{(3N-5)/2} K_G^{\mathbb{C}} \|A\|_{\infty,\mathbb{R}} \tag{15}$$

holds.

Tonge proved inequality (14) and Theorem 9 is inequality (15).

Remark 1. *Note that from this extension, it follows that if the players use a real state and observables which can be represented by real matrices, then Theorems 1 and 2 are valid with the constants on the right-hand side replaced by $2^{(N-2)/2}K_G^{\mathbb{R}}$ and $2^{k(r-2)/2}(K_G^{\mathbb{R}})^k$, respectively. In particular, this implies that Zukowski's QC-gap for the games given in* [Zuk93] *cannot be achieved with a strategy that involves a real Schmidt state and only real observables.*

The proof of Theorem 20 is by induction on N and uses the following slight modification of a Theorem by Littlewood [Lit30] (see also [Pie72, page 43] and [Sza76]) in the inductive step.

Lemma 21 (Slight extension of Littlewood 1964)**.** *Let n, d be positive integers and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then, for every $n \times d$ matrix $M : [n] \times [d] \to \mathbb{K}$, the following inequality holds:*

$$\sum_{i=1}^{n} \Big(\sum_{j=1}^{d} |M[i,j]|^2 \Big)^{1/2} \leq \sqrt{2} \max_{\substack{\phi \in \mathcal{S}(\ell_\infty^n) \\ \chi \in \mathcal{S}(\ell_\infty^d)}} \Big| \sum_{i=1}^{n} \sum_{j=1}^{d} M[i,j]\phi(i)\chi(j) \Big|, \tag{16}$$

where the underlying field for ϕ and χ is $\mathbb{K}$. Moreover, if the underlying field for M and χ is $\mathbb{C}$, but that for ϕ is $\mathbb{R}$, then the inequality

$$\sum_{i=1}^{n} \Big(\sum_{j=1}^{d} |M[i,j]|^2 \Big)^{1/2} \leq 2\sqrt{2} \max_{\substack{\phi \in \{\pm 1\}^n \\ \chi \in \mathcal{S}(\ell_\infty^d)}} \Big| \sum_{i=1}^{n} \sum_{j=1}^{d} M[i,j]\phi(i)\chi(j) \Big|, \tag{17}$$

holds.

Inequality (16) is Littlewood's inequality. These ineqaulities, in turn, can be derived from Khintchine's inequality, which states that for $0 < p < \infty$, there exist constants A_p and B_p such that for every finite sequence of real or complex scalars $(c_j)_{i=1}^n$ the following inequality holds:

$$A_p \left(\sum_{i=1}^{n} |c_i|^2 \right)^{1/2} \leq \left(\int_{t=0}^{1} \Big| \sum_{i=1}^{n} c_i r_i(t) \Big|^p dt \right)^{1/p} \leq B_p \left(\sum_{i=1}^{n} |c_i|^2 \right)^{1/2}, \tag{18}$$

where $r_i(t) = \text{sign}\big(\sin(2^i \pi t)\big)$ denotes the i'th Rademacher function. Haagerup [Haa82] found all values of A_p and B_p for sequences of real numbers c_i. The best value of A_1 is due to Szarek [Sza76], who proved that $A_1 = 1/\sqrt{2}$ for sequences of real scalars (see [Tom87] for an elementary proof). He also has an argument attributed to Tomaszewski which implies that for sequences of complex scalars and $p = 1$, the left-hand side of (18) holds with $A_1 \geq 1/\sqrt{2}$.

Proof of Lemma 21: If we set $p = 1$ and use the left side of Khintchine's inequality (18) for every $i = 1 \dots n$, we get

$$\begin{aligned} \sum_{i=1}^{n} \Big(\sum_{j=1}^{d} |M[i,j]|^2 \Big)^{1/2} &\leq \sqrt{2} \int_{t=0}^{1} \sum_{i=1}^{n} \Big| \sum_{j=1}^{d} M[i,j] r_j(t) \Big| dt \\ &\leq \sqrt{2} \sup_{x:[d]\to\{\pm 1\}} \Big\{ \sum_{i=1}^{n} \Big| \sum_{j=1}^{d} M[i,j] x(j) \Big| \Big\}. \end{aligned}$$

Inequality (16) now follows from the fact that there exists $\phi \in \mathcal{S}(\ell_\infty^n)$ and $\chi \in \mathcal{S}(\ell_\infty^d)$ such that

$$\Big| \sum_{i=1}^{n} \sum_{j=1}^{d} M[i,j] \phi(i) \chi(j) \Big| = \sum_{j=1}^{d} \Big| \sum_{i=1}^{n} M[i,j] x(j) \Big|.$$

To see this, set $\chi(j) = x(j)$ and $\phi(i) = \big(\sum_{j=1}^{d} M[i,j] x(j) \big)^* / \big| \sum_{j=1}^{d} M[i,j] x(j) \big|$ for every $i \in [n]$ and $j \in [d]$.

To prove inequality (17), consider the case $\mathbb{K} = \mathbb{C}$ and let χ and ϕ be the complex sequences that maximize the right-hand side of inequality (16). We write this quantity as the inner product $|\phi \cdot a|$, where $a_i = \sum_{j=1}^{d} M[i,j] \chi(j)$. By the triangle inequality, we have

$$\begin{aligned} |\phi \cdot a| &= \big| \Re(\phi) \cdot a + i \Im(\phi) \cdot a \big| \\ &\leq \big| \Re(\phi) \cdot a \big| + \big| \Im(\phi) \cdot a \big| \\ &\leq 2 \max \big\{ \big| \Re(\phi) \cdot a \big|, \big| \Im(\phi) \cdot a \big| \big\}. \end{aligned} \tag{19}$$

Without loss of generality, we may assume that this maximum is achieved with $\phi' = \Re(\phi)$. This is a vector in $[-1,1]^n$ for which the inequality

$$\sum_{i=1}^{n} \Big(\sum_{j=1}^{d} |M[i,j]|^2 \Big)^{1/2} \leq 2\sqrt{2} \Big| \sum_{i=1}^{n} \sum_{j=1}^{d} M[i,j] \chi(j) \phi'(i) \Big|$$

holds. By convexity, we have that the maximum is achieved at one of the extreme points of $[-1,1]^n$. This completes the proof. $\square$

We now turn to the proof of Theorem 20.

Proof of Theorem 20: (By induction on N.) For the base case, $N = 2$, we only need to prove something for inequality (15), since the base cases for inequality (14) are the real and complex Grothendieck inequality. We use the following simplified version of [MST99, Proposition 15]:

Claim 22. *For a real matrix $A[i,j]$, we have*

$$\max_{\alpha_i, \beta_j \in \mathcal{S}(\mathbb{C})} \Big| \sum_{i,j} A[i,j] \alpha_i \beta_j \Big| \leq \max_{\alpha_i', \beta_j' \in \mathcal{S}(\mathbb{C})} \sum_{i,j} A[i,j] \Re(\alpha_i' \beta_j'), \tag{20}$$

Proof: For a complex number $\gamma := re^{i\phi}$ (with $r \geq 0$) we have that $e^{-i\phi}\gamma = r = |\gamma|$. Trivially, we have that $\Re(e^{-i\phi}\gamma) = e^{-i\phi}\gamma$. Hence, for complex α_i, β_j and real A_{ij}, we have that there exists a ϕ such that

$$\Re\Big(e^{-i\phi} \sum_{i,j} A[i,j] \alpha_i \beta_j \Big) = e^{-i\phi} \sum_{i,j} A[i,j] \alpha_i \beta_j = \Big| \sum_{i,j} A[i,j] \alpha_i \beta_j \Big|.$$

Let $\alpha_i, \beta_j \in \mathcal{S}(\mathbb{C})$ that maximize the left-hand side of (20). We have that for some ϕ:

$$\begin{aligned}\Big|\sum_{i,j} A[i,j]\alpha_i\beta_j\Big| &= \Re\Big(e^{-i\phi}\sum_{i,j} A[i,j]\alpha_i\beta_j\Big)\\ &= \sum_{i,j} A[i,j]\Re\Big((\alpha_i e^{-i\phi/2})(\beta_j e^{-i\phi/2})\Big)\\ &\leq \max_{\alpha_i',\beta_j'\in\mathcal{S}(\mathbb{C})} \sum_{i,j} A[i,j]\Re(\alpha_i'\beta_j').\end{aligned}$$

□

We can write the real part $\Re(\alpha_i\beta_j)$ of two complex numbers α_i, β_j as the inner product between real vectors $a_i = \big(\Re(\alpha_i), \Im(\alpha_i)\big)^T$ and $b_j = \big(\Re(\beta_j), -\Im(\beta_j)\big)^T$. Using this and Claim 22, we get that for every sequence of unit vectors $u_i, v_j \in \mathcal{S}(\mathbb{C}^d)$,

$$\begin{aligned}\Big|\sum_{i,j=1}^n A[i,j]u_i\cdot v_j\Big| &\leq K_G^{\mathbb{C}} \max_{\alpha_i,\beta_j\in\mathcal{S}(\mathbb{C})}\Big|\sum_{i,j=1}^n A[i,j]\alpha_i\beta_j\Big|\\ &\leq K_G^{\mathbb{C}} \max_{\alpha_i,\beta_j\in\mathcal{S}(\mathbb{C})}\sum_{i,j=1}^n A[i,j]\Re(\alpha_i\beta_j)\\ &\leq K_G^{\mathbb{C}} \max_{a_i,b_j\in\mathcal{S}(\mathbb{R}^2)}\sum_{i,j=1}^n A[i,j]a_i\cdot b_j\\ &\leq K_G^{\mathbb{R}}(2)K_G^{\mathbb{C}}\|A\|_{\infty,\mathbb{R}} = \sqrt{2}K_G^{\mathbb{C}}\|A\|_{\infty,\mathbb{R}},\end{aligned}$$

where we used the fact that $K_G^{\mathbb{R}}(2) = \sqrt{2}$, as Krivine showed. This proves the base case.

Define the $n \times d$ matrix:

$$B[i_N, j] := \sum_{i_1,\ldots,i_{N-1}=1}^n A[i_1,\ldots,i_{N-1},i_k]f_1(i_1)_j\cdots f_{N-1}(i_{N-1})_j.$$

By the triangle inequality, the Cauchy-Schwarz inequality and inequality (17), we have

$$\begin{aligned}\Big|\sum_{i_1,\ldots,i_N=1}^n A[i_1,\ldots,i_N]\langle f_1(i_1),\ldots,f_N(i_N)\rangle\Big| &= \Big|\sum_{i_N=1}^n\sum_{j=1}^d B[i_N,j]f_N(i_N)_j\Big|\\ &\leq \sum_{i_N=1}^n\Big|\sum_{j=1}^d B[i_N,j]f_N(i_N)_j\Big|\\ &\leq \sum_{i_N=1}^n\Big(\sum_{j=1}^d\big|B[i_N,j]\big|^2\Big)^{1/2}\\ &\leq 2\sqrt{2}\max_{\substack{\phi:[n]\to\{\pm1\}\\ \chi:[d]\to\mathcal{S}(\mathbb{C})}}\Big|\sum_{i_N=1}^n\sum_{j=1}^d B[i_N,j]\phi(i_N)\chi(j)\Big| \qquad (21)\end{aligned}$$

Let $\phi^* : [n] \to \{\pm1\}$ be the function that maximizes (21). Applying the induction hypothesis to the real $(N-1)$-tensor

$$C[i_1,\ldots,i_{N-1}] := \sum_{i_N=1}^n A[i_1,\ldots,i_{N-1},i_N]\phi^*(i_N)$$

allows us to bound (21) as follows:

$$
\begin{aligned}
&2\sqrt{2} \max_{\chi \in \mathcal{S}(\ell_\infty^d)} \Big| \sum_{i_N=1}^n \sum_{j=1}^d B[i_N, j] \phi^*(i_N) \chi(j) \Big| \\
&\quad = 2\sqrt{2} \max_{\chi \in \mathcal{S}(\ell_\infty^d)} \Big| \sum_{i_N=1}^n \sum_{j=1}^d \Big(\sum_{i_1,\ldots,i_{N-1}=1}^n A[i_1,\ldots,i_{N-1},i_N] f_1(i_1)_j \cdots f_{N-1}(i_{N-1})_j \Big) \phi^*(i_N)\chi(j) \Big| \\
&\qquad = 2\sqrt{2} \max_{\chi \in \mathcal{S}(\ell_\infty^d)} \Big| \sum_{i_1,\ldots,i_{N-1}=1}^n C[i_1,\ldots,i_{N-1}] \cdot \Big(\sum_{j=1}^d \chi(j) f_1(i_1)_j \cdots f_{N-1}(i_{N-1})_j \Big) \Big| \\
&\leq 2\sqrt{2} \cdot \big((\sqrt{2}\,(2\sqrt{2})^{((N-1)-2)} \big) K_G^{\mathbb{C}} \max_{\phi_1,\ldots,\phi_{N-1} \in \{\pm 1\}^n} \Big| \sum_{i_1,\ldots,i_{N-1}=1}^n C[i_1,\ldots,i_{N-1}] \phi_1(i_1) \cdots \phi_{N-1}(i_{N-1}) \Big| \\
&= \sqrt{2}\, 2^{3(N-2)/2} K_G^{\mathbb{C}} \max_{\phi_1,\ldots,\phi_{N-1} \in \{\pm 1\}^n} \Big| \sum_{i_1,\ldots,i_{N-1}=1}^n \Big(\sum_{i_N=1}^n A[i_1,\ldots,i_{N-1},i_N] \phi^*(i_N) \Big) \phi_1(i_1) \cdots \phi_{N-1}(i_{N-1}) \Big| \\
&\qquad\qquad \leq \sqrt{2}\,(2\sqrt{2})^{3(N-2)/2} K_G^{\mathbb{C}} \max_{\phi_1,\ldots,\phi_N \in \{\pm 1\}^n} \Big| \sum_{i_1,\ldots,i_N=1}^n A[i_1,\ldots,i_{N-1},i_N] \phi_1(i_1) \cdots \phi_N(i_N) \Big|.
\end{aligned}
$$

The final term is $2^{(3N-5)/2} K_G^{\mathbb{C}} \|A\|_{\infty,\mathbb{R}}$. This proves inequality (15). Inequality (14) is proved in the same way, except with the original Grothendieck inequality for the base case and Littlewood's inequality (17) in Equation (21), giving the factor $2^{(N-2)/2} K_G^{\mathbb{K}}$. This completes the proof. ☐

Appendix B. Proof of Carne's theorem

In this section we prove Theorem 12.

Proof of Theorem 12: The proof is by induction on the number of edges $|E|$. If the edge set is empty, then there is nothing to prove. Let e_0 be any edge in G, and consider the graph $G_0 = (V, E \backslash \{e_0\})$. To re-write the expression, first assume that each vector $f_x(i_x) \in \mathcal{H}_x = \bigotimes_{e \in E(x)} \mathcal{H}(x,e)$ has the following tensor structure:

$$f_x(i_x) = f_x^0(i_x) \otimes f_x^1(i_x)$$

where $f_x^0(i_x) \in \otimes_{e \in E \backslash \{e_0\}} \mathcal{H}(x,e)$ and $f_x^1(i_x) \in \mathcal{H}(x,e_0)$. Define $\Phi_{G_0} = \Big(\bigotimes_{e \in E \backslash \{e_0\}} \psi_e \Big) \circ \sigma_{G_0}$, where σ_{G_0} is the re-arranging map for G_0. With this notation we have

$$
\begin{aligned}
\Phi\Big(\bigotimes_{x \in V} f_x(i_x) \Big) &= \Phi\Big(\bigotimes_{x \in V} f_x^0(i_x) \otimes f_x^1(i_x) \Big) \\
&= \Phi_{G_0}\Big(\bigotimes_{x \in V} f_x^0(i_x) \Big) \cdot \psi_{e_0}\Big(\bigotimes_{x \in e_0} f_x^1(i_x) \Big)
\end{aligned}
$$

Define the tensor $B[I] = A[I] \cdot \psi_{e_0} \left(\bigotimes_{x \in e_0} f_x^1(i_x) \right)$. Applying the induction hypothesis to $B[I]$ and the graph G_0 (note that the $\psi_{e_0}(\cdots)$ term is simply a number, dependent on I) gives

$$\sum_{I \in [n]^V} B[I] \cdot \Phi_{G_0}\Big(\bigotimes_{x \in V} f_x^0(i_x) \Big) \leq \Big(\prod_{e \in E \backslash \{e_0\}} C_e^{\mathbb{K}'} \Big) \|B\|_{\infty,\mathbb{K}'} \tag{22}$$

By definition,

$$
\begin{aligned}
\|B\|_{\infty,\mathbb{K}'} &= \max_{\phi_1,\dots,\phi_N\in\mathcal{S}(l_\infty^n)} \left|\sum_I B[I]\,\phi_1(i_1)\cdots\phi_N(i_N)\right| \\
&= \max_{\phi_1,\dots,\phi_N\in\mathcal{S}(l_\infty^n)} \left|\sum_I A[I]\,\phi_1(i_1)\cdots\phi_N(i_N)\,\psi_{e_0}\left(\bigotimes_{x\in e_0} f_x^1(i_x)\right)\right|
\end{aligned}
$$

Fix $\phi_1,\dots,\phi_N$ that achieve this maximum, and define the tensor $C[I]=A[I]\phi_1(i_1)\cdots\phi_N(i_N)$. By hypothesis, the function ψ_e enjoys a Grothendieck-type inequality, hence the expression above can be bounded by

$$
\|B\|_{\infty,\mathbb{K}'} = \sum_I C[I]\cdot\psi_{e_0}\left(\bigotimes_{x\in e_0} f_x^1(i_x)\right) \le C_{e_0}^{\mathbb{K}'}\|C\|_{\infty,\mathbb{K}'} \tag{23}
$$

To conclude, we can relate $\|C\|_{\infty,\mathbb{K}'}$ to $\|A\|_{\infty,\mathbb{K}'}$ in the following way:

$$
\begin{aligned}
\|C\|_{\infty,\mathbb{K}'} &= \max_{\phi'_1,\dots,\phi'_N\in\mathcal{S}(l_\infty^n)} \left|\sum_I C[I]\,\phi'_1(i_1)\cdots\phi'_N(i_N)\right| \\
&= \max_{\phi'_1,\dots,\phi'_N\in\mathcal{S}(l_\infty^n)} \left|\sum_I A[I]\,\phi_1(i_1)\phi'_1(i_1)\cdots\phi_N(i_N)\phi'_N(i_N)\right| \\
&= \max_{\phi''_1,\dots,\phi''_N\in\mathcal{S}(l_\infty^n)} \left|\sum_I C[I]\,\phi''_1(i_1)\cdots\phi''_N(i_N)\right| \\
&= \|A\|_{\infty,\mathbb{K}'}
\end{aligned}
$$

Combining Eqs. (22) and (23) gives the result in the case where all $f_x(i_x)$ have the tensor structure we described earlier. If not, since Φ is linear, writing their Schmidt decomposition will result in a weighted sum of expressions involving only unit vectors of this form. The weighted sum can be bounded by its maximum component, for which we can apply the reasoning above. □

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