

## Pencil and Paper Exercises Week 4

1. Prove that the result of updating an S5 epistemic model with an S5 action model is an S5 model.
2. In the following exercises we look at the muddy children problem from an update perspective. Here is the story again, now in a more detailed version. Three children are standing in a circle. Each child is facing the others. The children *have their eyes closed*. Assume that it is common knowledge among the children that they all have their eyes closed. Also assume that in fact children *a* and *b* have mud on their foreheads, but child *c* is clean. Let  $p_1$  express that *a* is muddy,  $p_2$  that *b* is muddy, and  $p_3$  that *c* is muddy. Give the epistemic model for this initial situation.
3. The children open their eyes. Two of them see mud on the forehead of one of the others. The third one (*c*) sees two muddy faces. Define an appropriate action model to capture what goes on in this scenario, specifically in the action of the children opening their eyes.
4. Give the epistemic model that results from updating the epistemic model from Exercise 2 with the action model for Exercise 3.
5. Father says: “*At least one of you is muddy*”. Give the corresponding action model.
6. Give the epistemic model that results from updating the model from Exercise 4 with the action model from Exercise 5.
7. Children *a*, *b* and *c* make public announcements to the effect that they do not know whether they are muddy or not. Give a single action model to capture this.
8. Give the epistemic model that results from updating the epistemic model from Exercise 6 with the action model from Exercise 7.
9. Now children *a* and *b* publicly announce that they know whether they are muddy or not. Give an appropriate action model for this.
10. Give the epistemic model that results from updating the epistemic model from Exercise 8 with the action model from Exercise 9.
11. Let a set of agents  $B$  and a set of basic propositions  $P$  be given. Assume that in the following definition of a multimodal language with public announcements,  $b$  ranges over  $B$  and  $p$  ranges over  $P$ .

$$\phi ::= \perp \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid [b]\phi \mid [\phi_1!]\phi_2$$

$[\phi_1!]\phi_2$  expresses that  $\phi_2$  holds after successful public announcement of  $\phi_1$ . The formal truth definition of  $[\phi_1!]\phi_2$  runs as follows:

$$\mathbf{M}, w \models [\phi_1!]\phi_2 \text{ iff } (\mathbf{M}, w \models \phi_1 \text{ implies } \mathbf{M} \upharpoonright \phi_1, w \models \phi_2).$$

Give appropriate definitions of the following abbreviations:

- $\top$ ,
- $\phi_1 \vee \phi_2$ ,
- $\phi_1 \Rightarrow \phi_2$ ,
- $\phi_1 \Leftrightarrow \phi_2$ ,
- $\langle b \rangle \phi$ ,
- $\langle \phi_1! \rangle \phi_2$ .

12. Prove the following equivalences:

$$\begin{array}{ll}
 [\phi!] \perp & \text{iff } \neg \phi, \\
 [\phi!] p & \text{iff } \phi \Rightarrow p, \\
 [\phi!] \neg \psi & \text{iff } \phi \Rightarrow \neg [\phi!] \psi, \\
 [\phi!] (\psi_1 \wedge \psi_2) & \text{iff } [\phi!] \psi_1 \wedge [\phi!] \psi_2, \\
 [\phi!] [b] \psi & \text{iff } \phi \Rightarrow [b] [\phi!] \psi
 \end{array}$$

13. Use the equivalences from the previous exercise to prove that the multimodal language with public announcements from Exercise 11 can be reduced to the multimodal language without public announcements. In other words: for every formula from the multimodal language with public announcements there is an equivalent formula in the multimodal language without public announcements.