Action Emulation

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Abstract

A key notion of equivalence for modal and epistemic logic is bisimulation. However, to capture the update effects of action models in epistemic update logic, this notion turns out to be too strong. We propose necessary and sufficient conditions for having the same update effect, in the cases of action models with propositional preconditions and action models with modal precondions. Next, the notion of an action emulation is proposed as a notion of equivalence more appropriate for action models than bisimulation. It is proved that every bisimulation is an action emulation, but not vice versa, and that in the context of action models with propositional or modal preconditions, action emulation provides a full characterisation of update effect.

1 Introduction

Actions with epistemic effects, such as informing someone that something is the case, are quite similar to situations with epistemic aspects, such as models of the states of knowledge of groups of agents. Knowledge of agents is encoded in epistemic models, with transition relations $\xrightarrow{i}$ modelling the epistemic state of each agent i, and valuations over a set of proposition letters modelling factual states of affairs.

Definition 1 *Let a set of propositional variables P and a finite set of agents* Ag *be given. An epistemic model is a triple $M = (W, V, R)$ where W is a set of worlds, $V : W \to \mathcal{P}(P)$ assigns a valuation to each world $w \in W$, and $R : \text{Ag} \to \mathcal{P}(W^2)$ assigns an accessibility relation $\xrightarrow{i}$ to each agent $i \in$* Ag.

A pair $\mathbf{M} = (M, U)$ with $U \subseteq W$ is a multiple pointed epistemic model, indicating that the actual world is among U.

[4] proposes to model epistemic actions as epistemic models, with valuations replaced by preconditions. (See also: [1, 2, 3, 5, 6, 7, 8, 9, 10].)

Definition 2 (Action models for a given language $\mathcal{L}$) *Let a finite set of agents* Ag *and an epistemic language $\mathcal{L}$ be given. An action model for $\mathcal{L}$ is a triple $A = (W, pre, R)$ where W is a set of action states, $pre : W \to \mathcal{L}$ assigns a precondition to each action state, and $R : \text{Ag} \to \mathcal{P}(W^2)$ assigns an accessibility relation $\xrightarrow{i}$ to each agent $i \in \text{Ag}$.*

A pair $\mathbf{A} = (A, S)$ with $S \subseteq W$ is a multiple-pointed action model, indicating that the actual action that takes place is a member of S.

The epistemic language $\mathcal{LANG}$ is defined as follows.

Definition 3 ($\mathcal{LANG}$) *Assume p ranges over set of basic propositions P, i ranges over the set of agents* Ag *and B ranges over subsets of* Ag*. The formulas of $\mathcal{LANG}$ are given by:*

$$\varphi \ ::= \ \top \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box_i\varphi \mid E_B\varphi \mid C_B\varphi \mid \ [A, S]\varphi$$

where (A, S) is a multiple pointed finite $\mathcal{LANG}$ (action) model.

We employ the usual abbreviations. In particular, $\varphi_1 \vee \varphi_2$ is shorthand for $\neg(\neg\varphi_1 \wedge \neg\varphi_2)$, $\varphi_1 \to \varphi_2$ for $\neg(\varphi_1 \wedge \neg\varphi_2)$, $\Diamond_i\varphi$ for $\neg\Box_i\neg\varphi$, $\langle A, S\rangle\varphi$ for $\neg[A, S]\neg\varphi$.

The reason to employ multiple pointed models for updating is that it allows us to handle choice. Suppose we want to model the action of testing whether φ followed by a public announcement of the result. This involves *choice*: if the outcome of the test is affirmative, then do this, else do that. Choice is modelled in a straightforward way in multiple pointed action models. Once we allow multiple pointed action models, it is reasonable to also take our epistemic models to be multiple pointed, with the multiple points constraining the whereabouts of the actual world.

Let MOD be the class of multiple pointed epistemic models and ACT the class of multiple pointed finite $\mathcal{LANG}$ models. Then $\mathcal{LANG}$-update is an operation of the following type:

$$\otimes : \text{MOD} \times \text{ACT} \to \text{MOD}.$$

The operation $\otimes$ and the truth definition for $\mathcal{LANG}$ are defined by mutual recursion, as follows.

Definition 4 (Update, Truth) *Given a multiple pointed epistemic model (M, U) and an action model (A, S), we define the update $M \otimes A$ as (W', V', R') and the update*

$$(M, U) \otimes (A, S)$$

as

$$((W', V', R'), \{(u, s) \mid u \in U, s \in S, (u, s) \in W'\}),$$

where

$$\begin{aligned} W' &:= \{(w,s) \mid w \in W_M, s \in W_A, M \models_w pre_A(s)\}, \\ V'(w,s) &:= V_M(w), \\ (w,s) \xrightarrow{i} (w',s') \in R' &:\equiv w \xrightarrow{i} w' \in R_M \text{ and } s \xrightarrow{i} s' \in R_A, \end{aligned}$$

and where the truth definition is given by:

$$\begin{aligned} M &\models_w \top && \text{always} \\ M &\models_w p &:\equiv\;& p \in V_M(w) \\ M &\models_w \neg\varphi &:\equiv\;& \text{not } M \models_w \varphi \\ M &\models_w \varphi_1 \wedge \varphi_2 &:\equiv\;& M \models_w \varphi_1 \text{ and } M \models_w \varphi_2 \\ M &\models_w \Box_i\varphi &:\equiv\;& \text{for all } w' \text{ with } w \xrightarrow{i} w'\ M \models_{w'} \varphi \\ M &\models_w E_B\varphi &:\equiv\;& \text{for all } w' \text{ with } w \xrightarrow{B} w'\ M \models_{w'} \varphi \\ M &\models_w C_B\varphi &:\equiv\;& \text{for all } w' \text{ with } w \xrightarrow{B^*} w'\ M \models_{w'} \varphi \\ M &\models_w [A,S]\varphi &:\equiv\;& \text{for all } s \in S, \\ &&& \text{if } M \models_w pre_s \text{ then } M \otimes A \models_{(w,s)} \varphi. \end{aligned}$$

In this definition $\xrightarrow{B}$ is the relation $\bigcup_{i \in B} \xrightarrow{i}$, and $\xrightarrow{B^*}$ its reflexive transitive closure.

This paper addresses the question of the appropriate notion of equivalence for action models. It may seem that generalizing bisimulations to action models in the obvious way to 'precondition preserving bisimulation', as is proposed in [2], is the way to go.

Definition 5 *Let A, B be $\mathcal{L}$ action models. Let $\equiv_{\mathcal{L}}$ be the appropriate equivalence notion for $\mathcal{L}$. Then relation $C \subseteq W_A \times W_B$ is a $\mathcal{L}$ bisimulation if whenever mCn the following hold:*

- $pre_m \equiv_{\mathcal{L}} pre_n$,
- *for all $a \in \mathrm{Ag}$, all states m' with $m \xrightarrow{a} m'$ there is a state n' with $n \xrightarrow{a} n'$ and $m'Cn'$.*
- *same requirement vice versa.*

A pointed bisimulation between (A,S) and (B,T) is a bisimulation that connects each $s \in S$ to some $t \in T$, and vice versa. If there is a pointed bisimulation between (A,S) and (B,T), this is expressed as $(A,S) \underline{\leftrightarrow} (B,T)$.

Note that pointed bisimulations are defined in terms of a lift of a bisimulation at the state level to one at the state-set level.

Given two epistemic models $\mathbf{M} = (M, U)$ and $\mathbf{N} = (N, V)$, we say that $\mathbf{M} \underline{\leftrightarrow} \mathbf{N}$ if every actual world $u \in U$ of $\mathbf{M}$ is connected by a bisimulation to some actual world $v \in V$ of $\mathbf{N}$, and vice versa. The following two theorems summarize what can be said in terms of bisimulations.

Theorem 6 (Preservation of epistemic bisimulation; Baltag, Moss, Solecki) *The action update operation $\otimes$ preserves ordinary bisimulation on epistemic models: if $\mathbf{M} \underline{\leftrightarrow} \mathbf{N}$ then $\mathbf{M} \otimes \mathbf{A} \underline{\leftrightarrow} \mathbf{N} \otimes \mathbf{A}$.*

Of course, we can also look at the action models modulo $\mathcal{LANG}$ bisimulation:

Theorem 7 (Preservation of action bisimulation) *The action update operation preserves action bisimulation:*

$$\textit{if } \mathbf{A} \underline{\leftrightarrow} \mathbf{B} \textit{ then } \mathbf{M} \otimes \mathbf{A} \underline{\leftrightarrow} \mathbf{M} \otimes \mathbf{B}.$$

Proof. We have to show that for every (u, s_i) among the actual worlds of $\mathbf{M} \otimes \mathbf{A}$ there is a (v, t_j) among the actual worlds of $\mathbf{M} \otimes \mathbf{B}$ with $(u, s_i) \underline{\leftrightarrow} (v, t_j)$, and vice versa. This follows immediately from the existence of the bisimulation $\underline{\leftrightarrow}$ between $\mathbf{A}$ and $\mathbf{B}$, for the relation on $\mathbf{M} \otimes \mathbf{A} \times \mathbf{M} \otimes \mathbf{B}$ defined by means of

$$(u, s)C(v, t) \text{ iff } u = v \text{ and } s \underline{\leftrightarrow} t$$

is a bisimulation. □

2 Same Update Effect

Thinking of the finite multiple pointed action models $\mathbf{A}$ as 'action programs', the basic semantic notion of equivalence between such programs is that of having the same update effect:

Definition 8 (Same update effect) $\mathbf{A} \equiv_{ACT} \mathbf{B}$ *iff*

$$\forall \mathbf{M} : \ \mathbf{M} \otimes \mathbf{A} \underline{\leftrightarrow} \mathbf{M} \otimes \mathbf{B}.$$

From the update bisimulation preservation theorem it follows that:

Theorem 9 *$\mathbf{A} \underline{\leftrightarrow} \mathbf{B}$ implies $\mathbf{A} \equiv_{ACT} \mathbf{B}$.*

Can we turn this around? No we cannot. Here is a simple counterexample. Let

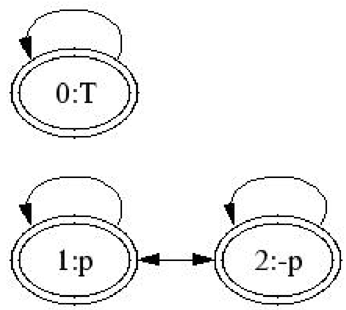
$$\mathbf{A} = ((\{0\}, 0 \mapsto \bot, \emptyset), \{0\}),$$

and let

$$\mathbf{0} = ((\emptyset, \emptyset, \emptyset), \emptyset).$$

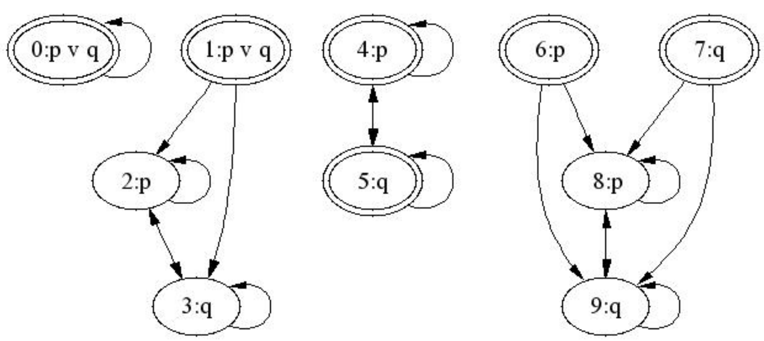
Then $\mathbf{A} \equiv_{\text{ACT}} \mathbf{0}$, but $\mathbf{A}$ and $\mathbf{0}$ are not bisimilar. Removing the inconsistent states (the states with a precondition equivalent to $\bot$) from an action model does not affect its update potential, so we might as well assume that action models contain only consistent states. This would reduce $\mathbf{A}$ to $\mathbf{0}$. However, Figure 1 provides another example of non-bisimilar action models with consistent states and with the same update potentials.

Figure 1: Non-bisimilar actions with the same update effects



Clearly, state 0 in Figure 1 is not bisimilar to 1, for these states have different preconditions. Also 0 is not bisimilar to 2, for the same reason. Still the two action models have the same update effects: they both act as the identity update.

Figure 2: More non-bisimilar actions with the same update effects



For another example, consider Figure 2. Each of the action models in this figure has the effect of selecting the accessibility paths with $p \vee q$ holding at every node along the paths.

Examples like these suggest that the notion of $\mathcal{LANG}$ bisimilarity is too strong to capture the 'essence' of our update actions.

In the next two section, we characterize the notion of having the same update effect in terms of a kind of canonical updates. On the basis of that, we will define a structural relation on action models directly, called action emulation, and show that this notion exactly captures the update effects of action models.

3 Action Emulation

We now proceed to give a structural condition for equivalence of action models. The relation of action emulation between action models, to be defined below, can be viewed as a suitably weakened of bisimulation, adapted to the case where valuations are replaced by preconditions.

Instead of insisting that the preconditions are the same, we just require that the preconditions are compatible.

Instead of insisting on a precise match in the zig and zag clauses, we merely require that that an approppriate choice from a list of possible matches can be made. The idea behind this is that to match a pair (w, s) in $M \otimes \mathbf{A}$, we need a pair (w, t) in $M \otimes \mathbf{B}$. For (w, t) to exist, the precondition of t should be satisfied by w. Requiring that s and t have the same precondition would be too strong. Instead we require that there is a choice between finitely many t_i the preconditions of which are jointly implied by that of s.

These considerations are reflected in the following definition.

Definition 10 (Action Emulation) *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are actions with sets of action states* W_A, W_B, *respectively, then a relation* $R \subseteq W_A \times W_B$ *is an action emulation if whenever* sRt *the following hold:*

Preconditions $pre(s) \wedge pre(t)$ *is consistent.*

Zig *If* $s \xrightarrow{a} s'$ *then there are* $t_1, \ldots, t_n$ *with*

$$t \xrightarrow{a} t_1, \ldots, t \xrightarrow{a} t_n, pre(s') \models pre(t_1) \vee \cdots \vee pre(t_n) \text{ and } s'Rt_1, \ldots, s'Rt_n.$$

Zag *If* $t \xrightarrow{a} t'$ *then there are* $s_1, \ldots, s_n$ *with*

$$s \xrightarrow{a} s_1, \ldots, s \xrightarrow{a} s_n, pre(t') \models pre(s_1) \vee \cdots \vee pre(s_n) \text{ and } s_1Rt', \ldots, s_nRt'.$$

Observe that the examples of actions with the same update effects all satisfy this structural requirement.

Use $\mathbf{A} \leftrightarrows \mathbf{B}$ to indicate that an action emulation exists between $\mathbf{A}$ and $\mathbf{B}$. Observe that it is obvious that action emulation is a weakening of bisimulation, in the following sense.

Theorem 11 *If* $\mathbf{A} \underline{\leftrightarrow} \mathbf{B}$ *then* $\mathbf{A} \leftrightarrows \mathbf{B}$.

Proof. Check that every action bisimulation is an action emulation. □

The proof that the existence of an action emulation between **A** and **B** guarantees that **A** and **B** have the same update effect is also straigtforward:

Theorem 12 *If* $A \leftrightarrows B$ *then* $\mathbf{A} \equiv_{ACT} \mathbf{B}$.

Proof. Let M be an arbitrary epistemic model. Assume $A \leftrightarrows B$ and let Z be an action emulation between A and B. Define R by means of: $(w, s)R(v, t) :\equiv w = v \wedge sZt$. We show that R is a bisimulation.

Preconditions From $(w, s)R(v, t)$ we get that $w = v$ and $M \models_w \text{pre}(s)$, and $M \models_w \text{pre}(t)$. This shows that $\text{pre}(s) \wedge \text{pre}(t)$ is consistent.

Zig Let $(w, s) \xrightarrow{a} (w', s')$. Then $w \xrightarrow{a} w'$, $s \xrightarrow{a} s'$, and $M \models_{w'} \text{pre}(s')$. From $(w, s)R(v, t)$ we have that sZt. By sZt, there are $t_1, \ldots, t_n$ with $\text{pre}(s') \models \text{pre}(t_1) \vee \cdots \vee \text{pre}(t_n)$ and $s'Zt_1, \ldots, s'Zt_n$. Since $M \models_{w'} \text{pre}(s')$ it follows from $\text{pre}(s') \models \text{pre}(t_1) \vee \cdots \vee \text{pre}(t_n)$ that there is some t_i with $M \models_{w'} \text{pre}(t_i)$. Thus $(w', s')R(w', t_i)$.

Zag Same reasoning vice versa.

□

The theorem shows that action emulation is a sufficient condition for having the same update effect. To see whether it is also necessary, we will make a case separation. as follows.

Call an action model *propositional* if all preconditions that occur in it are purely propositional formulas. Call an action model *modal* if all preconditions that occur in it are multimodal formulas. In the next two sections we will look at the update effects of propositional and modal action models, and show that in both cases having the same update effect implies the existence of an action emulation.

4 Update Effects of Propositional Action Models

In this section we will show that in the case of actions with propositional preconditions, having the same update effect can be characterized in terms of the update effects in some special case.

Let Q be a set of proposition letters. Then a valuation over Q is a subset of Q. Use $\mathbf{v} \models \varphi$ to express that valuation $\mathbf{v}$ models a propositional formula φ.

The epistemic model VAL_Q is the model (W, V, R) where $W = \mathcal{P}(Q)$, V is the identity function, and R is the universal relation on W for every agent i. Thus, worlds are valuations, and the valuation at each world is that world itself.

If $\mathbf{A} = (A, S), \mathbf{B} = (B, T)$, Q is the set of all proposition letters occurring in $\mathbf{A}, \mathbf{B}$, and

$$U = \{\mathbf{v} \subseteq Q \mid \exists x \in S \cup T \text{ with } \mathbf{v} \models \text{pre}_x\}.$$

Then we abbreviate $(\text{VAL}_Q, U) \otimes \mathbf{A}$ as $\mathbf{A}^\circ$ and $(\text{VAL}_Q, U) \otimes \mathbf{B}$ as $\mathbf{B}^\circ$. Call $\mathbf{A}^\circ$ the *expansion* of action model $\mathbf{A}$.

Clearly we have:

Theorem 13 *Let $\mathbf{A}, \mathbf{B}$ have propositional preconditions. Then $\mathbf{A} \equiv_{ACT} \mathbf{B}$ implies $\mathbf{A}^\circ \underline{\leftrightarrow} \mathbf{B}^\circ$.*

Proof. What holds for an arbitrary epistemic model $\mathbf{M}$ certainly holds for (VAL_Q, U). □

Next, we prove the implication from bisimulation of expanded models to having the same update effect:

Theorem 14 *Let $\mathbf{A}, \mathbf{B}$ have propositional preconditions. Then $\mathbf{A}^\circ \underline{\leftrightarrow} \mathbf{B}^\circ$ implies $\mathbf{A} \equiv_{ACT} \mathbf{B}$.*

Proof. Assume $(\text{VAL}_Q, U) \otimes (A, S) \underline{\leftrightarrow} (\text{VAL}_Q, U) \otimes (B, T)$, with Q the set of all proposition letters occurring in $\mathbf{A}, \mathbf{B}$, and

$$U = \{\mathbf{v} \subseteq Q \mid \exists x \in S \cup T \text{ with } \mathbf{v} \models \text{pre}_x\}.$$

Let $\mathbf{M} = (M, V)$ be an arbitrary epistemic model. We have to show that $\mathbf{M} \otimes \mathbf{A} \underline{\leftrightarrow} \mathbf{M} \otimes \mathbf{B}$.

Define a relation $C \subseteq W_{\mathbf{M} \otimes \mathbf{A}} \times W_{\mathbf{M} \otimes \mathbf{B}}$ by means of

$$(w, s) C (v, t) \text{ iff } w = v \text{ and } (V(w), s) \underline{\leftrightarrow} (V(w), t),$$

where $\underline{\leftrightarrow}$ is the bisimulation linking $(\text{VAL}_Q, U) \otimes \mathbf{A}$ to $(\text{VAL}_Q, U) \otimes \mathbf{B}$.

We show that C is a bisimulation. Assume $(w, s) C (v, t)$. Then $w = v$ and $(\mathbf{v}, s) \underline{\leftrightarrow} (\mathbf{v}, t)$, where $\mathbf{v}$ is the valuation of w. We check the three bisimulation conditions:

Invariance Immediate from the fact that the valuation of (w, s) equals the valuation of w equals the valuation of (w, t).

Zig Let $(w, s) \xrightarrow{i} (w', s')$. Then $w \xrightarrow{i} w'$ and $s \xrightarrow{i} s'$. Let $\mathbf{v}'$ be the valuation of w'. Then it holds in (VAL_Q, U) that $(\mathbf{v}, s) \xrightarrow{i} (\mathbf{v}', s')$. By the zig condition for $(\mathbf{v}, s) \underline{\leftrightarrow} (\mathbf{v}, t)$, it follows from this that there is a t' with $(\mathbf{v}, t) \xrightarrow{i} (\mathbf{v}', t')$ and $(\mathbf{v}', s') \underline{\leftrightarrow} (\mathbf{v}', t')$. So $(w', s') C (w', t')$, as desired.

Zag Similar.

$\square$

Combining these, we get:

Theorem 15 *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *have propositional preconditions.*
Then $\mathbf{A}^\circ \underline{\leftrightarrow} \mathbf{B}^\circ$ *iff* $\mathbf{A} \equiv_{ACT} \mathbf{B}$*.*

Proof. Immediate from theorems 13 and 14. $\square$

In this section we have shown that, in the context of propositional action models, having the same update effect is equivalent to bisimilarity of expanded action models.

5 Update Effects of Modal Action Models

We now turn to the case where the preconditions are multimodal formulas, i.e., where they belong to the language defined by:

$$\varphi \ ::= \ \top \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box_i \varphi$$

Extend the definition of action model expansions, as follows. Let $\mathbf{A}, \mathbf{B}$ be action models with modal preconditions Π. Let Q be the set of all proposition letters occurring in Π. Let MCONS_Π be the set of all maximal consistent subsets taken from $\neg\text{Sub}\Pi$, where Sub denotes taking subformulas and $\neg$ denotes closure under single negations. Let EXP_Π be the triple (W, V, R) where $W = \text{MCONS}_\Pi$, V is the function that assigns to every maximal consistent subset Γ of MCONS_Π the Q-valuation $\Gamma \cap Q$, and R assigns to each agent i the relation $\xrightarrow{i}$ given by:

$$\Gamma \xrightarrow{i} \Gamma' \text{ iff } \forall \varphi \in \Gamma', \Box_i \neg\varphi \notin \Gamma.$$

Thus, the accessibilities now take the modal constraints imposed by the preconditions into acccount.

If Π is a set of preconditions including those of $\mathbf{A}, \mathbf{B}$, then abbreviate $\text{EXP}_\Pi \otimes \mathbf{A}$ as $\mathbf{A}^\odot$ and $\text{EXP}_\Pi \otimes \mathbf{B}$ as $\mathbf{B}^\odot$. Then, obviously:

Theorem 16 *Let* $\mathbf{A}, \mathbf{B}$ *have modal preconditions.*
Then $\mathbf{A} \equiv_{ACT} \mathbf{B}$ *implies* $\mathbf{A}^\odot \underline{\leftrightarrow} \mathbf{B}^\odot$*.*

Proof. What holds for an arbitrary epistemic model $\mathbf{M}$ certainly holds for EXP_Π. $\square$

Next, here is a proof of the fact that for actions with modal preconditions, bisimilarity of expanded models implies having the same update effect.

Theorem 17 *Let* $\mathbf{A}, \mathbf{B}$ *have modal preconditions.*
Then $\mathbf{A}^{\odot} \underline{\leftrightarrow} \mathbf{B}^{\odot}$ *implies* $\mathbf{A} \equiv_{ACT} \mathbf{B}$.

Proof. Assume $(\text{EXP}_{\Pi}, U) \otimes (A, S) \underline{\leftrightarrow} (\text{EXP}_{\Pi}, U) \otimes (B, T)$, with Π the set of preconditions occurring in $\mathbf{A}, \mathbf{B}$, and

$$U = \{\Gamma \in \text{MCONS}_{\Pi} \mid \exists x \in S \cup T \text{ with } \text{pre}_x \in \Gamma\}.$$

Let $\mathbf{M} = (M, V)$ be an arbitrary epistemic model. We have to show that $\mathbf{M} \otimes \mathbf{A} \underline{\leftrightarrow} \mathbf{M} \otimes \mathbf{B}$.

Let Π^M_w be the set $\{\varphi \in \neg\text{Sub}\Pi \mid M \models_w \varphi\}$. Note that $\Pi^M_w \in \text{MCONS}_{\Pi}$.

Define a relation $C \subseteq W_{\mathbf{M}\otimes\mathbf{A}} \times W_{\mathbf{M}\otimes\mathbf{B}}$ by means of

$$(w, s)C(v, t) \quad \text{iff} \quad w = v \text{ and } (\Pi^M_w, s) \underline{\leftrightarrow} (\Pi^M_w, t).$$

where $\underline{\leftrightarrow}$ is the bisimulation linking $\mathbf{A}^{\odot}$ to $\mathbf{B}^{\odot}$.

We show that C is a bisimulation. Assume $(w, s)C(v, t)$. Then $w = v$ and $(\Pi^M_w, s) \underline{\leftrightarrow} (\Pi^M_w, t)$. We check the three bisimulation conditions:

Invariance Immediate from the fact that the valuation of (w, s) equals the valuation of w equals the valuation of (w, t).

Zig Let $(w, s) \xrightarrow{i} (w', s')$. Then $w \xrightarrow{i} w'$, $s \xrightarrow{i} s'$, and $M \models_{w'} \text{pre}_{s'}$. It follows from $M \models_{w'} \text{pre}_{s'}$ that $\text{pre}_{s'} \in \Pi^M_{w'}$.

Let $\varphi \in \Pi^M_{w'}$. Assume, for a contradiction, that $\Box_i\neg\varphi \in \Pi^M_w$. Then, because Π^M_w is maximally consistent, $\Diamond_i\varphi \notin \Pi^M_w$, and contradiction with the fact that $M \models_w \Diamond_i\varphi$. It follows that $\Box_i\neg\varphi \notin \Pi^M_w$. Thus, $\Pi^M_w \xrightarrow{i} \Pi^M_{w'}$.

From $\text{pre}_{s'} \in \Pi^M_{w'}$ we get that $(\Pi^M_{w'}, s')$ is among the states of $\mathbf{A}^{\odot}$, and from $s \xrightarrow{i} s'$ and $\Pi^M_w \xrightarrow{i} \Pi^M_{w'}$ it follows that $(\Pi^M_w, s) \xrightarrow{i} (\Pi^M_{w'}, s')$.

Since $(\Pi^M_w, s) \underline{\leftrightarrow} (\Pi^M_w, t)$, it follows from $(\Pi^M_w, s) \xrightarrow{i} (\Pi^M_{w'}, s')$ that there is a t' with $(\Pi^M_w, t) \xrightarrow{i} (\Pi^M_{w'}, t')$ and $(\Pi^M_{w'}, s') \underline{\leftrightarrow} (\Pi^M_{w'}, t')$. Therefore $(w', s')C(w', t')$, as desired.

Zag Similar.

□

Combining the above, we have:

Theorem 18 *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *have modal preconditions.*
Then $\mathbf{A}^{\odot} \underline{\leftrightarrow} \mathbf{B}^{\odot}$ *iff* $\mathbf{A} \equiv_{ACT} \mathbf{B}$.

Proof. Immediate from theorems 16 and 17. □

6 Action Emulation

Let $\mathbf{A}$ and $\mathbf{B}$ be action models with propositional or modal preconditions. Let Π be the preconditions occurring in $\mathbf{A}, \mathbf{B}$. Let MCONS_Π be the set of all maximal consistent subsets taken from $\neg\text{Sub}\Pi$, where Sub denotes taking subformulas and $\neg$ denotes closure under single negations. Since $\neg\text{Sub}\Pi$ is finite, these maximal consistent subsets are finite as well. For $x \in W_\mathbf{A} \cup W_\mathbf{B}$, let $G(x)$ be the set

$$\{\Gamma \mid \Gamma \in \text{MCONS}_\Pi, \text{pre}_x \in \Gamma\}.$$

Thus, $G(x)$ is the set of those maximal consistent subsets taken from $\neg\text{Sub}\Pi$ that contain the precondition of x.

Definition 19 (Indexed Bisimulation) *Let $\mathbf{A}$ and $\mathbf{B}$ be action models. A relation (Γ) on $W_\mathbf{A} \times W_\mathbf{B}$ is an indexed bisimulation if whenever $w(\Gamma)v$ the following hold:*

Index condition $\Gamma \models pre_w$, $\Gamma \models pre_v$,

Zig *if $w \xrightarrow{i} w'$ and $\Gamma' \in G(w')$, then there is a $v \in W_\mathbf{B}$ with $v \xrightarrow{i} v'$ and $w'(\Gamma')v'$.*

Zag *if $v \xrightarrow{i} v'$ and $\Gamma' \in G(v')$, then there is a $w \in W_\mathbf{A}$ with $w \xrightarrow{i} w'$ and $w'(\Gamma')v'$.*

We now define action emulation in terms of indexed bisimulation, as follows:

Definition 20 (Action Emulation) *Let $\mathbf{A}$ and $\mathbf{B}$ be action models. Then a relation*

$$\mathbf{R} \subseteq \mathcal{P}(W_\mathbf{A}) \times \mathcal{P}(W_\mathbf{B})$$

is an action emulation if whenever $S\ \mathbf{R}\ T$ the following hold:

- *if $s \in S$, $\Gamma \in G(s)$, then there is a $t \in T$ with $s(\Gamma)t$.*
- *if $t \in T$, $\Gamma \in G(t)$, then there is an $s \in S$ with $s(\Gamma)t$.*

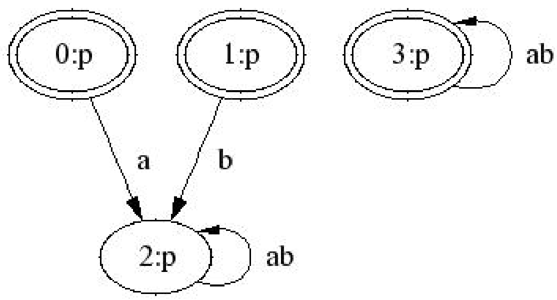
Use $(\mathbf{S}, S) \leftrightarrows (\mathbf{T}, T)$ if there is an action emulation that connects S and T. It is easy to check that in the example of Figure 1, $\{0\} \leftrightarrows \{1, 2\}$, and in the example of Figure 2, $\{0\} \leftrightarrows \{1\} \leftrightarrows \{4, 5\} \leftrightarrows \{6, 7\}$. A non-example is given in Figure 3.

Clearly, every bisimulation on action models corresponds to an action emulation:

Theorem 21 *Let $\mathbf{A}, \mathbf{B}$ be action models. If $\mathbf{A} \underline{\leftrightarrow} \mathbf{B}$ then $\mathbf{A} \leftrightarrows \mathbf{B}$.*

Proof. Let R be a bisimulation between $\mathbf{A}$ and $\mathbf{B}$, at the level of states. Assume sRt. We will show that for any $\Gamma \in G(s) \cup G(t)$, R is a Γ indexed bisimulation between s and t. Let $\Gamma \in G(s) \cup G(t)$.

Figure 3: No action emulation between $\{0, 1\}$ and $\{3\}$.



Index condition Assume $\Gamma \in G(s)$. Then by definition, $\Gamma \models \text{pre}_s$ and because of the invariance condition for s and t, $\text{pre}_s \equiv \text{pre}_t$. It follows that $\Gamma \models \text{pre}_t$. Similarly for the case where $\Gamma \in G(t)$.

Zig Let $s \xrightarrow{i} s'$. Then by the fact that R is a bisimulation with sRt, there is a t' with $t \xrightarrow{i} t'$ and $s'Rt'$. By assumption, R is a Γ' indexed bisimulation between s' and t', for any $\Gamma' \in G(s') \cup G(t')$.

Zag Same reasoning vice versa.

Clearly, the lift of R to the state set level is an action emulation. □

As we already saw, action models may emulate without being bisimilar. We will now prove that emulating action models have the same update effects:

Theorem 22 *If* $\mathbf{A} \leftrightarrows \mathbf{B}$*, then* $\mathbf{A} \equiv_{ACT} \mathbf{B}$.

Proof. Let $\mathbf{R}$ be a relation witnessing $\mathbf{A} \leftrightarrows \mathbf{B}$. We wish to show for arbitrary $\mathbf{M}$ that $\mathbf{M} \otimes \mathbf{A} \underline{\leftrightarrow} \mathbf{M} \otimes \mathbf{B}$.

Let $\mathbf{M}$ be arbitrary. Define $\mathbf{C}$ on $\mathcal{P}(W_{\mathbf{M}\otimes\mathbf{A}}) \times \mathcal{P}(W_{\mathbf{M}\otimes\mathbf{B}})$ by means of

$$(X \otimes S)\ \mathbf{C}\ (Y \otimes T) \text{ iff } X = Y \text{ and } S \leftrightarrows T$$

Note that $X \otimes S$ is the set of pairs $\{(x, s) \mid x \in X, s \in S, M \models_x \text{pre}_s\}$, and $Y \otimes T$ is the set of pairs $\{(y, t) \mid y \in Y, t \in T, M \models_y \text{pre}_t\}$,

Assume $(X \otimes S)\ \mathbf{C}\ (Y \otimes T)$ and let $(x, s) \in X \otimes S$. We have to show that there is a $(y, t) \in Y \otimes T$ with $(x, s) \underline{\leftrightarrow} (y, t)$.

Define a relation C on $W_{\mathbf{M}\otimes\mathbf{A}} \times W_{\mathbf{M}\otimes\mathbf{B}}$ by means of:

$$(x, s)C(y, t) :\equiv x = y \text{ and } s(\Pi^M_x)t.$$

From the fact that $(X \otimes S)$ $\mathbf{C}$ $(Y \otimes T)$ and $\Pi_x^M \in G(x)$ it follows that there is a t with $s(\Pi_x^M)t$. This shows that there is a t with $(x,s)C(x,t)$. We still have to show that C is a bisimulation.

Suppose $(x,s)C(y,t)$. We show that C is a bisimulation.

Invariance (x,s) and (y,t) have the same valuation.

Zig Let $(x,s) \xrightarrow{i} (x',s')$. Then $x \xrightarrow{i} x'$, $s \xrightarrow{i} s'$, and $M \models_{x'} \text{pre}_{s'}$. By the fact that $s(\Pi_x^M)t$, it follows that there is a t' with $t \xrightarrow{i} t'$ and $s'(\Pi_{x'}^M)t'$. From this, by the index condition, $\Pi_{x'}^M \models \text{pre}_{t'}$, i.e., $M \models_{x'} \text{pre}_{t'}$. It follows that $(x',t') \in W_{\mathcal{M}\otimes\mathbf{B}}$, and thus, $(x',t') \in W_{\mathbf{M}\otimes\mathbf{B}}$. Thus, we have a t' with $(x,t) \xrightarrow{i} (x',t')$ and $s'(\Pi_{x'}^M)t'$, i.e., $(x',s')C(x',t')$.

Zag Same reasoning vice versa.

□

Theorem 22 shows that action emulation is a sufficient condition for having the same update effect.

7 Further Issues

Action emulation is defined at the state set level, in terms of a set-theoretic lift of indexed bisimulation. This family tie with standard bisimulation generates a number of behavioural similarities to bisimulation. E.g., as with bisimulations, there always is a largest action emulation.

Theorem 23 *If $\mathbf{R}_j$ ($j \in J$) are action emulations between $\mathbf{A}$ and $\mathbf{B}$, then $\bigcup_{j\in J} \mathbf{R}_j$ is an action emulation between $\mathbf{A}$ and $\mathbf{B}$.*

Proof. Let $\mathbf{R}_j$ ($j \in J$) be action emulations between $\mathbf{A}$ and $\mathbf{B}$, with corresponding indexed bisimulations (j, Γ). Suppose $X \bigcup_{j\in J} \mathbf{R}_j Y$. Let $x \in X$ and $\Gamma \in G(x)$. Then there is a $j \in J$ and a $y \in Y$ with $x(j,\Gamma)y$.

**** proof has to be completed ****

□

It follows from this that the union of all action emulations connecting $\mathbf{A}$ and $\mathbf{B}$ is an action emulation, i.e., there always is a largest action emulation connecting $\mathbf{A}$ and $\mathbf{B}$.

We end with two open questions:

Question 24 *What is the (modal) language characterization of action emulation (compare the characterization theorems for bisimulation)?*

Question 25 *What is the complexity of determining whether two action models emulate? Is this more complex than bisimulation, or is it also polynomial, like the decision problem for bisimilarity? In particular, can something like a partition refinement algorithm in the style of [11] be made to work for this?*

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