Quantifier Decomposition

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Abstract

Functions of type $\langle n \rangle$ are characteristic functions on n-ary relations. In Beyond the Frege Boundary [6], Keenan established their importance for natural language semantics, by showing that natural language has many examples of irreducible type $\langle n \rangle$ functions, where he called a function of type $\langle n \rangle$ reducible if it can be represented as a composition of functions of type $\langle 1 \rangle$.

We will give a normal form theorem for functions of type $\langle n \rangle$, and use this to show that natural language has many examples of irreducible type $\langle n \rangle$ functions in a much stronger sense, where we take a function to be reducible if it can be represented as a composition of functions of lower types.

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- Alternative analysis: look at the complex expression

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  and interpret this as a function that takes a relation as its argument (a denotation of a transitive verb, such as cheated, accused, defended) and produces a truth value.

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Quantification patterns that only allow the alternative analysis are

beyond the Frege boundary.

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- Keenan [5, 6, 7, 8]
- Van Benthem [2]
- Ben-Shalom [1]
- Dekker [3]
- Van Eijck [4]

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- Two detectives interviewed a total of twenty witnesses.
- The boys gave the same presents to the same girlfriends on the same occasions.

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- A type $\langle 2\rangle$ quantifier is a function of type $(e \longrightarrow e \longrightarrow t) \longrightarrow t$. Type $\langle 2\rangle$ quantifiers are properties of binary relations between individuals,
- A type $\langle n\rangle$ quantifier is a function of type

$$(\underbrace{e \longrightarrow \cdots \longrightarrow e}_{n \text{ times}} \longrightarrow t) \longrightarrow t.$$

Type $\langle n\rangle$ quantifiers are properties of n-ary relations between individuals.

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$$\textbf{INJ} \; xy \cdot R(x,y) \;\; :\Leftrightarrow \;\; \forall x \forall y (x \neq y \Rightarrow \lambda u \cdot R(x,u) \neq \lambda v . R(y,v)).$$

The set injectivity quantifier captures the meaning of Different students gave different answers.

Quantifiers as Relation Reducers

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- A type $\langle n \rangle$ quantifier corresponds to a function that maps $m+n$-ary relations to m-ary relations.
- If $F :: \langle n \rangle$ and R is an $m+n$-ary relation, then

  $$\begin{aligned} F(R) \ = \ & \{\langle d_1, \ldots, d_m \rangle \mid \\ & \qquad F\{\langle d_{m+1}, \ldots, d_{m+n} \rangle \mid \\ & \qquad\qquad \langle d_1, \ldots, d_m, d_{m+1}, \ldots, d_{m+n} \rangle \in R\} = 1\}. \end{aligned}$$

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How do we find out whether F is reducible? The fact that we cannot easily find f, g with $F = f \circ g$ proves nothing . . .

Characterizing Irreducibility

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The sentence asserts that there is a one-to-one correspondence between students and sets of questions they asked.

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Let $\mathbf{0}$ be the type $\langle 1\rangle$ function that is false for any argument. Then, by the above, $F(A \times B) = \mathbf{0} \circ \mathbf{0}(A \times B)$ for any product relation $A \times B$.

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But obviously, F is different from the composition $\mathbf{0} \circ \mathbf{0}$, for F is true of $\{\langle s_1, q_1 \rangle, \langle s_2, q_2 \rangle\}$, and $\mathbf{0} \circ \mathbf{0}$ is not. Thus, by Keenan's theorem, F is not reducible.

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Fact 1 (Keenan) *Let* f *be a positive function of type* $\langle 1 \rangle$ *and let* $P, Q \subseteq E$. *Then:*

$$f(P \times Q) = \begin{cases} P & \textit{if } f(Q) = 1 \\ \emptyset & \textit{otherwise} \end{cases}$$

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Corollary: for positive f, g : $g \circ f(P \times Q) = 1$ iff $g(P) = 1 \wedge f(Q) = 1$.

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Proof: If $F = G$ then their behaviour on products is the same.

For the other direction, first assume F, G both positive, and suppose F, G have the same behaviour on products. Because of reducibility there are positive f_1, f_2, g_1, g_2 with $F = f_1 \circ f_2$ and $G = g_1 \circ g_2$. Then by fact 1, $f_1 = g_1$ and $f_2 = g_2$. Thus $F = f_1 \circ f_2 = g_1 \circ g_2 = G$.

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Now assume F, G negative. Then, because of reducibility there are f_1, f_2, g_1, g_2, with f_1, g_1 negative, f_2, g_2 positive, $F = f_1 \circ f_2$ and $G = g_1 \circ g_2$. Clearly, if $f_2 = g_2$, then by Fact 1, $f_1 = g_1$, and $F = f_1 \circ f_2 = g_1 \circ g_2 = G$.

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On the other hand, if $\exists Q : 0 = f_2(Q) \neq g_2(Q) = 1$, then for any P, $f_1 \circ f_2(P \times Q) = f_1(\emptyset) = 1 = g_1 \circ g_2(P \times Q)$. It follows that $f_1 = g_1 = \mathbf{1}$, and $F = G = \lambda R.\top$. ■

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Definition 3 (Reduct) *The reduct $F^\bullet$ of a positive type $\langle n \rangle$ function F is defined as $F^\bullet = f_1 \circ \cdots \circ f_n$ with f_i given by:*

$$\begin{aligned} f_i(\emptyset) &:= 0 \\ f_i(Q \neq \emptyset) = 1 &:\Leftrightarrow \exists Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_n \subseteq E, \\ &\quad\; F(Q_1 \times \cdots \times Q_{i-1} \times Q \times Q_{i+1} \times \cdots \times Q_n) = 1 \end{aligned}$$

Characterizing the Reducible Functions

Keenan does give an indirect characterisation of the reducible functions. Dekker [3] proposes an improvement, in terms of a notion of 'invariance for sets in products'. We will propose a direct criterion.

Definition 3 (Reduct) *The reduct* $F^{\bullet}$ *of a positive type* $\langle n \rangle$ *function* F *is defined as* $F^{\bullet} = f_1 \circ \cdots \circ f_n$ *with* f_i *given by:*

$$\begin{aligned} f_i(\emptyset) &:= 0 \\ f_i(Q \neq \emptyset) = 1 &:\Leftrightarrow \exists Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_n \subseteq E, \\ &\quad\; F(Q_1 \times \cdots \times Q_{i-1} \times Q \times Q_{i+1} \times \cdots \times Q_n) = 1 \end{aligned}$$

Theorem 4 *For all positive type* $\langle n \rangle$ *functions* F*:*

$F = F^{\bullet}$ *iff* F *is reducible.*

Example of reducibility

$$F = \mathbf{E}_A \mathbf{A}_B xy \cdot R(x, y) \ \ :\Leftrightarrow \ \ \exists x \in A \ \forall y \in B \ R(x, y)$$

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Then:

$$\begin{aligned} F^\bullet &= f_1 \circ f_2 \text{ with} \\ & f_1(P \neq \emptyset) = 1 \text{ iff } \exists Q \subseteq E \text{ with } F(P \times Q) = 1 \\ & \text{ iff } P \cap A \neq \emptyset \wedge \exists Q \; B \subseteq Q \text{ iff } P \cap A \neq \emptyset \\ & f_2(Q \neq \emptyset) = 1 \text{ iff } \exists P \subseteq E \text{ with } F(P \times Q) = 1 \\ & \text{ iff } B \subseteq Q \wedge \exists P \; P \cap A \neq \emptyset. \end{aligned}$$

Clearly, $F = F^\bullet$, so F is reducible.

Example of irreducibility

$$F = \textsf{\textbf{Symm}}\ xy \cdot R(x,y) \ \ :\Leftrightarrow\ \ \forall x \forall y (R(x,y) \Rightarrow R(y,x)))$$

This is a negative quantifier, so consider its negation:

$$(\neg F) = \textsf{\textbf{NotSymm}}\ xy \cdot R(x,y) \ \ :\Leftrightarrow\ \ \exists x \exists y (R(x,y) \wedge \neg R(y,x)))$$

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Then:

$$\begin{aligned}(\neg F)^\bullet \;\; = \;\; & f_1 \circ f_2 \text{ with} \\ & f_1(P \neq \emptyset) = 1 \text{ if } \exists Q \subseteq E \text{ with } (\neg F)(P \times Q) \text{ iff always} \\ & \text{ so } f_1 = \mathbf{1}. \\ & f_2(Q \neq \emptyset) = 1 \text{ if } \exists P \subseteq E \text{ with } (\neg F)(P \times Q) \text{ iff always} \\ & \text{ so } f_2 = \mathbf{1}.\end{aligned}$$

So $(\neg F)^\bullet = \mathbf{1} \circ \mathbf{1} \neq \neg F$, so $\neg F$ is irreducible, and so is F.

Reduction at the Far Side...

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Definition 5 (m-n-reduct) *The* (m, n)*-reduct of a positive* $\langle m+n \rangle$ *function* $\mathbf{F}$ *is the composition* $F \circ G$*, with* F *of type* $\langle m \rangle$ *and* G *of type* $\langle n \rangle$*, with* F *defined by* $F(\emptyset) = 0$ *and*

$$F(R \neq \emptyset) = 1 :\Leftrightarrow \exists S \subseteq E^n : \ \mathbf{F}(R \times S) = 1$$

and G *by* $G(\emptyset) = 0$ *and*

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Theorem 6 *A positive type* $\langle m+n\rangle$ *function* $\mathbf{F}$ *is equal to its own* (m,n)*-reduct iff* $\mathbf{F}$ *is* (m,n)*-reducible.*

Normal Forms for Characteristic Functions on n-ary Relations

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- Fact: if a function $F :: \langle 3 \rangle$ is both $(1, 2)$ reducible and $(2, 1)$ reducible, then it is fully reducible.

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- This fact can be generalized to a Diamond Property for reduction.

Theorem 7 (Diamond Property) *If* $\mathbf{F} = F \circ G = K \circ M$ *with* $F :: \langle m \rangle$ *and* $G :: \langle n \rangle$, $m < m'$, $K :: \langle m' \rangle$, $M :: \langle m + n - m' \rangle$, *all of* $\mathbf{F}, F, G, K, M$ *positive, then there is a positive function* $H :: \langle m' - m \rangle$ *such that* $\mathbf{F} = F \circ H \circ M$.

$$\begin{tikzcd}
\mathbf{F} \arrow[r] \arrow[d] & F \circ G \arrow[d] \\
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\end{tikzcd}$$

Theorem 8 (Normal Form) *Every positive* $\mathbf{F} :: \langle n \rangle$ *is uniquely representable as*

$$\mathbf{F} = F_1 \circ \cdots \circ F_k,$$

with F_i *positive and irreducible for all* $i : 1 \leq i \leq k$. *Moreover, there exists an algorithm for finding this normal form NF*$(\mathbf{F})$.

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Normal form:

$$\begin{gathered}(\lambda R.((\text{dom}(R) \cap \text{HERMIT}) \neq \emptyset \land \forall x\ xRx) \\ \circ \quad \lambda Q.\text{PLEASURE} \subseteq Q)(\text{FORBID}).\end{gathered}$$

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It holds for every $R \subset E^2$ and $Q \subseteq E$ that $R \times Q$ is in the quantifier relation, for $R \times Q$ expresses that every (p, s) pair in R is related to every c in Q, so if (p_1, s) and (p_2, s) both in R then p_1 and p_2 charge s with the same crimes, namely all crimes in Q.

This shows the following fact about natural language:

Fact 9 *Natural languages can express type* $\langle 3 \rangle$ *quantifiers that cannot be reduced to any composition of lesser types.*

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Fact 10 *For all reasonable* n*, natural languages present examples of type* $\langle n \rangle$ *quantificational expressions that cannot be reduced to any composition of quantifiers of lesser degree.*

General Programme of Polyadic Quantification

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- interpret a sentence by applying its NP list interpretation to its VP interpretation.
- interpret scope reordering as a parallel operation on lists of quantifiers (all in reduced form) and VP interpretations.

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