Expressiveness and Complexity of the Logic of Public Announcements

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1 The Language

We assume the logic of public announcement without a common knowledge operator.

**Definition 1 (Syntax).** Let \( n \in \mathbb{N} \cup \{\omega\} \) be a number of agents, and let \( \text{PL} \) be a countably infinite set of propositional letters. The formulas of public announcement logic \( (PAL_n) \) are built according to the following syntax rule:

\[
\varphi ::= p \mid \neg p \mid (\varphi \land \psi) \mid K_i \varphi \mid [\varphi] \psi
\]

where \( p \) ranges over \( \text{PL} \) and \( i \) ranges over \( 1, \ldots, n \). Without the public announcement modality \( [\varphi] \psi \), we get the standard modal language \( (ML_n) \).

We will use \( \hat{K}_i \varphi \) as an abbreviation for \( \neg K_i \neg \varphi \) and \( \langle \varphi \rangle \psi \) as an abbreviation for \( \neg [\varphi] \neg \psi \).

**Definition 2 (Semantics).** An epistemic model (or model for short) is a triple \( M = (S, \sim_i, \forall) \) with \( S \) a non-empty set of states, \( \sim^i \) an equivalence relation for \( i < n \), and \( \forall \) a function assigning a subset \( \forall(p) \subseteq S \) with each \( p \in \text{PL} \). Given a model \( M = (S, \sim^i, \forall) \) and an \( s \in S \), we define:

\[
\begin{align*}
M, s \models p & \iff s \in \forall(p) \\
M, s \models \neg \varphi & \iff M, s \nvdash \varphi \\
M, s \models \varphi \land \psi & \iff M, s \models \varphi \text{ and } M, s \models \psi \\
M, s \models K_i \varphi & \iff s \sim_i t \ implies \ M, t \models \varphi, \ for \ all \ t \in S \\
M, s \models [\varphi] \psi & \iff M, s \models \varphi \ implies \ M|\varphi, s \models \psi
\end{align*}
\]

where \( M|\psi := (T, \approx^i, U) \) is defined as follows:

\[
\begin{align*}
T & := \{t \in S \mid M, t \models \psi\} \\
\approx_i & := \sim_i \cap (T \times T) \\
U(p) & := \forall(p) \cap T
\end{align*}
\]

It is easily checked that the semantics of \( \langle \varphi \rangle \psi \) is as follows:

\[
M, s \models \langle \varphi \rangle \psi \ iff \ M, s \models \varphi \text{ and } M|\varphi, s \models \psi.
\]
2 Expressive Power

We show that public announcement logic has exactly the same expressive power as the standard modal language, for any number of agents \( n \in \mathbb{N} \cup \{ \omega \} \). This is done by giving a translation of PAL\(_n\) formulas \( \varphi \) into ML\(_n\) formulas. We need a bit of notation:

- for a finite sequence of PAL\(_n\) formulas, let \( \text{pre}(\sigma) \) denote the set of all its true prefixes including the empty sequence \( \varepsilon \). For \( \nu \in \text{pre}(\sigma) \), we write \( \nu/\sigma \) to denote the leftmost symbol of \( \sigma \) that is not in \( \nu \).

- for a PAL\(_n\) formula \( \varphi \), we use \( |\varphi| \) to denote the length of \( \varphi \), i.e. the number of symbols needed to write down \( \varphi \), including symbols such as "[" and "]". For a non-empty sequence of PAL\(_n\) formulas \( \sigma = \varphi_1 \cdots \varphi_k \), we use \( |\sigma| \) to denote \( |\varphi_1| + \cdots + |\varphi_k| \).

We are now ready to define the translation. More precisely, we will define an ML\(_n\) formula \( \varphi^\sigma \) for every PAL\(_n\) formula \( \varphi \) and every finite sequence of PAL\(_n\) formulas \( \sigma \) by induction on \( |\varphi| + |\sigma| \):

\[
\begin{align*}
p^\sigma & := p \\
(\neg \varphi)^\sigma & := \neg \varphi^\sigma \\
(\varphi \land \psi)^\sigma & := \varphi^\sigma \land \psi^\sigma \\
(K_i \varphi)^\sigma & := K_i( \bigwedge_{\nu \in \text{pre}(\sigma)} (\nu/\sigma)^\nu \rightarrow \varphi^\sigma) \\
([\varphi]\psi)^\sigma & := \varphi^\sigma \rightarrow \psi^\sigma 
\end{align*}
\]

It is readily checked that, in the \( K_i \varphi \) case, \( \sigma = \varepsilon \) implies that the big conjunction collapses to true (as there are no true prefixes of \( \varepsilon \)). For \( \sigma = \varphi_1 \cdots \varphi_k \) a finite sequence of PAL\(_n\) formulas, we write \( M|\sigma \) as an abbreviation for \(((M|\varphi_1)|\varphi_2) \cdots |\varphi_k)\). As a special case, we assume \( M|\varepsilon = M \).

Lemma 3. For all models \( M = (S, \approx^n, V) \), PAL\(_n\) formulas \( \varphi \), finite sequences of PAL\(_n\) formulas \( \sigma \), and states \( s \in M|\sigma \), we have \( M, s \models \varphi^\sigma \) iff \( M|\sigma, s \models \varphi \).

Proof. The proof is by induction on \( |\varphi| + |\sigma| \). The base case is \( |\varphi| + |\sigma| = 1 \). Then \( \varphi = p \in \text{PL} \) and \( \sigma = \varepsilon \). We have \( M|\sigma = M \) and \( \varphi^\sigma = \varphi \) and are done. For the induction step, let \( M|\sigma = (T, \approx^n, U) \) and make a case distinction on the form of \( \varphi \):

- \( \varphi = p \). Trivial by definition of \( p^\sigma \).

- \( \varphi = \neg \psi \). Easy using the definition of \( (\neg \psi)^\sigma \), the semantics, and the induction hypothesis.

- \( \varphi = (\psi \land \theta) \). As previous case.

- \( \varphi = K_i \psi \). Let \( M, s \models (K_i \psi)^\sigma \). We have to show that \( M|\sigma, t \models \psi \) for all \( t \in T \) with \( s \approx_i t \). Hence, let \( t \in T \) with \( s \approx_i t \). Then we have \( M, t \models (\nu/\sigma)^\nu \) for all \( \nu \in \text{pre}(\sigma) \): assume to the contrary of what is to be
shown that $\mathcal{M}, t \not\models (\nu/\sigma)^\varphi$ for some $\nu \in \text{pre}(\sigma)$. By induction hypothesis, we get $\mathcal{M}[\nu], t \not\models (\nu/\sigma)$ and consequently $t \notin T$, which is a contradiction. Thus $\mathcal{M}, t \models (\nu/\sigma)^\varphi$ for all $\nu \in \text{pre}(\sigma)$, implying $\mathcal{M}, t \models \bigwedge_{\nu \in \text{pre}(\sigma)}(\nu/\sigma)^\varphi$. Since $s \approx_i t$, we additionally have $s \sim_i t$. Together with $\varphi$.

Now let $\mathcal{M}, s \models (K_i \psi)^\sigma$, i.e. that $\mathcal{M}, t \models \psi^\sigma$ for all $t \in S$ with (i) $s \sim_i t$ and (ii) $\mathcal{M}, t \models \bigwedge_{\nu \in \text{pre}(\sigma)}(\nu/\sigma)^\varphi$. Hence, let $t \in S$ such that (i) and (ii) are satisfied. We show by induction on the length of $\nu$ that $t$ is a state in $\mathcal{M}[\nu]$ for all $\nu \in \text{pre}(\sigma) \cup \{\sigma\}$.

- $\nu = \varepsilon$. Trivial since $\mathcal{M}|\varepsilon = \mathcal{M}$.

- $\nu = \nu' \cdot \vartheta$. By induction hypothesis, $t$ is a state in $\mathcal{M}[\nu']$. By (ii), we have $\mathcal{M}, t \models (\nu'/\sigma)^\nu'$. By (outer) induction hypothesis, this yields $\mathcal{M}[\nu'], t \models (\nu'/\sigma)$. Thus, $t$ is a state in $(\mathcal{M}[\nu'])(\nu'/\sigma) = \mathcal{M}[\nu]$.

Thus, $s, t \in T$. Hence, (i) yields $s \approx_i t$. Together with $\mathcal{M}[\sigma], s \models K_i \psi$, we get $\mathcal{M}[\sigma], t \models \psi$. By induction hypothesis, we get $\mathcal{M}, t \models \psi^\sigma$ as required.

- $\varphi = [\psi]\vartheta$. Then $\mathcal{M}, s \models ([\psi]\vartheta)^\sigma$ if $\mathcal{M}, s \not\models \psi^\sigma$ or $\mathcal{M}, s \models \psi^\sigma\psi$ if $\mathcal{M}[\sigma], s \not\models \psi$ or $\mathcal{M}[\sigma \cdot \psi], s \models \vartheta$ iff $\mathcal{M}[\sigma], s \not\models \psi$ or $\mathcal{M}[\sigma]|\psi, s \models \vartheta$ iff $\mathcal{M}[\sigma], s \models [\psi]\vartheta$.

The first iff holds by definition of $([\psi]\vartheta)^\sigma$, the second by induction hypothesis, the third since $\mathcal{M}[\sigma \cdot \psi] = (\mathcal{M}[\sigma]|\psi)$, and the fourth by the semantics.

Lemma 3 implies that, for all models $\mathcal{M}$, PAL sentences $\varphi$, and states $s$ of $\mathcal{M}$, we have $\mathcal{M}, s \models \varphi$ if $\mathcal{M}, s \models \varphi^\varepsilon$. Thus we get the following theorem.

**Theorem 4.** PAL$_n$ has the same expressive power as ML$_n$, for all $n \in \mathbb{N} \cup \{\omega\}$.

Observe that, for proving this theorem, we did not use the fact that we have epistemic models. Indeed, the result holds independently of the frame class.

It is readily checked that the above translation produces an exponential blowup in formula length. In the following, we show that such a blowup cannot be avoided if we assume an infinite number of agents.

**Theorem 5.** For $i \geq 0$, define $\varphi_i := \langle \cdots \langle \langle p \rangle_k \hat{K}_i \text{true} \rangle_{i+1} \cdots \hat{K}_i \text{true} \rangle_i$. Then every ML$_n$-formula $\psi$ with $\psi \equiv \varphi_i$ is of length at least $2^i - 1$, for all $i \geq 0$.

**Proof.** For $n \in \mathbb{N}$, let $S(n)$ denote the set of all sequences $i_1 \cdots i_k$, $k \geq 0$, such that $1 \leq i_j \leq k$ for $1 \leq j \leq k$ and $i_j < i_{j+1}$ for $1 \leq j < k$. It is not hard to check that there are exactly $2^n - 1$ such sequences. A careful inspection of the formula $\varphi_i$ and the translation given above shows that, for all $i \geq 0$, we have

$$\varphi_i^\varepsilon \equiv \psi_i := \bigwedge_{i_1 \cdots i_k \in S(i)} \hat{K}_{i_1} \cdots \hat{K}_{i_k} p.$$  \[1\]

1When $i_1 \cdots i_k$ is the empty sequence, the corresponding conjunct collapses to $p$.  

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It is not too difficult to prove that there exists no ML formula \( \vartheta \) such that \( \vartheta \equiv \psi_i \) and \( |\vartheta| < 2^n - 1 \). Before I go into detail, I’d like to think a bit about the finite agent case.

Open problem: I do currently not know whether we can observe the same succinctness effect in the finite agent case. The above class of formulas clearly relies on an unbounded number of agents.

3 PSPACE-Completeness

We proceed in two steps: first, we show that epistemic logic extended with a union operator on agents (as in PDL) is in PSpace. Then we reduce public announcement logic to this extended version of epistemic logic, using a modified version of the reduction presented in the previous section.

3.1 A PSPACE algorithm for ML\(^\cup\) on Epistemic Models

The language ML\(^\cup\)\(_n\) is obtained from the language ML\(_n\) by replacing the modalities \( K_i \varphi \) with modalities \([i_1 \cup \ldots \cup i_k] \varphi\), where \( i_1, \ldots, i_k \) are numbers for agents, i.e., \( i_j < n \) for \( 1 \leq j \leq k \). We will sometimes write \([i] \varphi\) as \( K_i \varphi\). The semantics of the new operator is given as follows:

\[
\mathcal{M}, s \models [i_1 \cup \ldots \cup i_k] \varphi \text{ iff } s \sim_{i_j} t \text{ implies } \mathcal{M}, t \models \varphi, \text{ for all } t \in S \text{ and } j \in \{1, \ldots, k\}.
\]

The idea for obtaining a PSPACE algorithm for epistemic ML\(^\cup\)\(_n\) is simply to devise a standard K-worlds style algorithm.

**Definition 6 (Type).** Let \( \Gamma \) be a set of ML\(^\cup\)\(_n\)-formula. We use \( \text{cl}(\Gamma) \) to denote the smallest superset of \( \text{sub}(\Gamma) \) that satisfies the following properties:

- \( \Gamma \subseteq \text{cl}(\Gamma) \);
- \( \text{cl}(\Gamma) \) is closed under subformulas and single negations;
- if \([i_1, \ldots, i_k] \psi \in \text{cl}(\Gamma)\), then \([i_1] \psi, \ldots, [i_k] \psi \in \text{cl}(\Gamma)\).

A *type* for \( \Gamma \) is a subset \( t \subseteq \text{cl}(\Gamma) \) satisfying the following properties:

1. \( \neg \psi \in t \text{ iff } \psi \notin t \text{ for all } \neg \psi \in \text{cl}(\Gamma)\);
2. \( \psi \land \vartheta \in t \text{ iff } \psi, \vartheta \in t \text{ for all } \psi \land \vartheta \in \text{cl}(\Gamma)\);
3. \([i_1, \ldots, i_k] \psi \in t \text{ iff } [i_1] \psi, \ldots, [i_k] \psi \in t \text{ for all } [i_1, \ldots, i_k] \psi \in \text{cl}(\Gamma)\);
4. \([i] \psi \in t \text{ implies } \psi \in t\).
We now devise the PSPACE algorithm: an ML∪ⁿ-formula ϕ₀ is satisfiable in an epistemic model iff there exists a set Ψ ⊆ cl(ϕ₀) with ϕ₀ ∈ Ψ such that eML∪ⁿ-World(Ψ, cl(ϕ₀), ⊥) returns true:

**define procedure** eML∪ⁿ-World(Δ, Γ, k)

  if Δ is not a type for Γ then
    return false
  for all ¬[i]ϕ ∈ Δ with i ≠ k do
    set Ψ := {¬ϕ} ∪ {ψ, [i]ψ | [i]ψ ∈ Δ} ∪ {¬[i]ψ | ¬[i]ψ ∈ Δ}
  non-deterministically choose a subset Δ′ ⊆ cl(Ψ)
  if eML∪ⁿ-World(Δ′, cl(Ψ), i) = false then
    return false
  return true

It is readily checked that the recursion depth of the algorithm is bounded by the modal depth of the input formula. By Savitch’s Theorem, we thus get the following result:

**Theorem 7.** Satisfiability of ML∪ⁿ formulas in epistemic models is PSPACE-complete, for n ∈ N ∪ {ω}.

### 3.2 Reducing Public Announcement Logic

The idea is to modify the translation given in Section 2 such that it only yields a polynomial blowup in formula length. To do this, we have to trade preservation of equivalence for preservation of satisfiability.

Let ϕ be a PALₙ formula. With sub(ϕ), we denote the set of all subformulas of ϕ, including ϕ. With Σ(ϕ), we denote the set of all pairs (σ, ψ), where ψ ∈ sub(ϕ) and σ is a finite (and possibly empty) sequence of formulas from sub(ϕ). The subset R(ϕ) ⊆ Σ(ϕ) of relevant pairs is defined as follows:

\[
R(a) := \{(ε, a)\} \\
R(¬ϕ) := R(ϕ) \cup \{(ε, ¬ϕ)\} \\
R(ϕ ∧ ψ) := R(ϕ) \cup R(ψ) \cup \{(ε, ϕ ∧ ψ)\} \\
R(K_iϕ) := R(ϕ) \cup \{(ε, K_iϕ)\} \\
R([ϕ]ψ) := R(ϕ) \cup \{(ϕ · σ, ϑ | (σ, ϑ) ∈ R(ψ)) \cup \{(ε, [ϕ]ψ)\}
\]

We use ad(ϕ) to denote the announcement depth of a formula ϕ, i.e. the maximum nesting depth of announcement operators [ψ]ϑ in ϕ, where only nestings via the second argument are counted.

**Lemma 8.** For all PALₙ formulas ϕ, we have the following:

1. |R(ϕ)| ≤ |ϕ|;
2. for all (σ, ψ) ∈ R(ϕ), the length of the sequence σ is bounded by ad(ϕ).
Now let $\varphi_0$ be the PAL$_n$ formula whose satisfiability is to be decided. We introduce a set of fresh propositional letters

$$L_{\varphi_0} := \{p_\varphi^\sigma \mid (\sigma, \varphi) \in R(\varphi_0)\}.$$ 

For every $(\sigma, \varphi) \in R(\varphi_0)$, we define a biimplication $B_\varphi^\sigma$ as follows:

$$B_\varphi^\sigma := p_\varphi^\sigma \leftrightarrow q,$$

$$B_\varphi^\sigma := p_\varphi^\sigma \leftrightarrow \neg p_\varphi^\sigma,$$

$$B^\sigma_{\varphi \land \psi} := p_\varphi^\sigma \land p_\psi^\sigma,$$

$$B^\sigma_{\varphi \lor \psi} := p_\varphi^\sigma \lor p_\psi^\sigma,$$

$$B^\sigma_{\varphi \rightarrow \psi} := p_\varphi^\sigma \rightarrow p_\psi^\sigma,$$

$$B^\sigma_{\varphi \leftrightarrow \psi} := p_\varphi^\sigma \leftrightarrow p_\psi^\sigma.$$ 

We use $md(\varphi)$ to denote the modal depth of the formula $\varphi$. Let $i_1, \ldots, i_k$ be the agents used in $\varphi_0$. Now define

$$\varphi_0^* := p_{\varphi_0}^\epsilon \land \bigwedge_{j \leq md(\varphi_0)} \bigwedge_{(\sigma, \varphi) \in R(\varphi_0)} [i_1 \cup \cdots \cup i_k] B^\sigma_{\varphi},$$ 

where $[\alpha]^{\varphi}$ is an abbreviation for the $j$-fold nesting $[\alpha] \cdots [\alpha][\alpha]$, and $[\alpha]^0\varphi$ is simply $\varphi$. Observe that $|\varphi_0|$ is polynomial in $\varphi_0$: by Point 2 of Lemma 8, $|B^\sigma_{\varphi}|$ is polynomial in $|\varphi_0|$ for each $(\sigma, \varphi) \in R(\varphi_0)$. By Point 1 of Lemma 8 and since $md(\varphi_0)$ is linear in $|\varphi_0|$, we get polynomiality of $|\varphi_0^*|$. 

**Lemma 9.** $\varphi_0$ is satisfiable iff $\varphi_0^*$ is satisfiable.

Before we prove the lemma, we need a new notion. Let $M = (S, \sim, V)$ be a model and $s, t \in S$. Then $d_M(s, t)$ denotes the distance between $s$ and $t$, i.e., the minimal number $k \in \mathbb{N}$ such that there are $s_1, \ldots, s_k \in S$ with $s = s_1$, $s_k = t$, and $s_i \sim_s s_{i+1}$ for some $j \leq n$, for $1 \leq i < k$. Note that $d_M(s, s) = 0$, and that $d_M(s, t)$ may be undefined in the case of disconnected models. Now for the proof of Lemma 9.

**Proof.** “if”. Let $M = (S, \sim, V)$ be a model of $\varphi_0^*$, and let $s_0 \in S$ with $M, s_0 \models \varphi_0^*$. We show that, for all $s \in S$ and all $(\sigma, \varphi) \in R(\varphi_0)$ with $d_M(s_0, s) \leq md(\varphi_0) - md(\varphi)$, we have

$$M, s, s_0 \models \varphi_0^*$$

By Lemma 3, from this we get $M|\sigma, s = s_0 \models \varphi$ iff $M, s \models p_\varphi^\sigma$. Since $M|\epsilon = M$ and $M, s_0 \models p_\varphi^\sigma$, by definition of $\varphi_0^*$, this yields $M, s_0 \models \varphi_0$ as required. The proof is by induction on $|\varphi| + |\sigma|$. For the induction start, we have $\varphi = q$ and $\sigma = \epsilon$. Then $\varphi^\sigma = q$. Since $|d_M(s_0, s)| \leq md(\varphi_0) - md(\varphi)$, $M, s_0 \models \varphi_0^*$ implies $M, s \models B_\varphi^\sigma = p_\varphi^\sigma \leftrightarrow q$ and we are done. For the induction step, we make a case distinction according to the structure of $\varphi$:

- $\varphi = q$. Identical to the induction start.
\begin{itemize}
  \item \( \varphi = \neg \psi \). Then \( \mathcal{M}, s \models (\neg \psi)^{\sigma} \) iff \( \mathcal{M}, s \models \neg \psi^{\sigma} \) iff \( \mathcal{M}, s \not\models \psi^{\sigma} \) iff \( \mathcal{M}, s \not\models p^{\sigma}_\psi \) iff \( \mathcal{M}, s \models \neg p^{\sigma}_\psi \).

  The first “iff” holds by definition of \( (\neg \psi)^{\sigma} \), the second by the semantics, the third by induction hypothesis, the fourth by the semantics, and the last since \(|d_\mathcal{M}(s_0, s)| \leq \text{md}(\varphi_0) - \text{md}(\varphi)\) and \( \mathcal{M}, s_0 \models \varphi_0^{\sigma} \) implies \( \mathcal{M}, s \models B^{\sigma}_\neg \psi = p^{\sigma}_\neg \psi \iff \neg p^{\sigma}_\psi \).

\end{itemize}

\begin{itemize}
  \item \( \varphi = \psi \land \vartheta \). Similar to the previous case.

  \item \( \varphi = K_i \psi \). We have \( \mathcal{M}, s \models (K_i \psi)^{\sigma} \) iff

\[
\mathcal{M}, s \models K_i (\bigwedge_{\nu \in \text{pre}(\sigma)} (\nu/\sigma)^{\nu} \rightarrow \varphi^{\sigma}).
\]

which is the case iff

\[
\text{for all } t \in S, s \sim_i t \text{ implies } \mathcal{M}, t \models (\bigwedge_{\nu \in \text{pre}(\sigma)} (\nu/\sigma)^{\nu} \rightarrow \varphi^{\sigma}).
\]

Since \( \varphi \) contains a modal operator, we have \( d_\mathcal{M}(s_0, s) < \text{md}(\varphi_0) \). Thus, \( d_\mathcal{M}(s_0, t) \leq \text{md}(\varphi_0) \) for all \( t \in S \) with \( s \sim_i t \). By induction hypothesis, for all such \( t \) we thus get (i) \( \mathcal{M}, t \models (\nu/\sigma)^{\nu} \) iff \( \mathcal{M}, t \models p^{\nu}_{\nu/\sigma} \) for all \( \nu \in \text{pre}(\sigma) \) and (ii) \( \mathcal{M}, t \models \varphi^{\sigma} \) iff \( \mathcal{M}, t \models p^{\sigma}_\varphi \). Thus, (i) holds iff

\[
\text{for all } t \in S, s \sim_i t \text{ implies } \mathcal{M}, t \models (\bigwedge_{\nu \in \text{pre}(\sigma)} p^{\nu}_{\nu/\sigma} \rightarrow p^{\sigma}_{\varphi}).
\]

Clearly, \(|d_\mathcal{M}(s_0, s)| \leq \text{md}(\varphi_0) - \text{md}(\varphi)\) and \( \mathcal{M}, s_0 \models \varphi_0^{\sigma} \) implies

\[
\mathcal{M}, s \models p^{\sigma}_{K_i \psi} \iff K_i (\bigwedge_{\nu \in \text{pre}(\sigma)} p^{\nu}_{\nu/\sigma} \rightarrow p^{\sigma}_{\varphi}).
\]

Thus, (i) holds iff \( \mathcal{M}, s \models p^{\sigma}_{K_i \psi} \) and we are done.

\item \( \varphi = [\psi] \vartheta \). We have \( \mathcal{M}, s \models ([\psi] \vartheta)^{\sigma} \) iff \( \mathcal{M}, s \models \varphi^{\sigma} \rightarrow \psi^{\sigma \vartheta} \) iff \( \mathcal{M}, s \models p^{\sigma}_\varphi \rightarrow p^{\sigma \vartheta}_\psi \)

iff \( \mathcal{M}, s \models p^{\sigma \vartheta}_\psi \).

The first “iff” holds by definition of \( ([\psi] \vartheta)^{\sigma} \), the second by the semantics and induction hypothesis, and the third since \(|d_\mathcal{M}(s_0, s)| \leq \text{md}(\varphi_0) - \text{md}(\varphi)\) and \( \mathcal{M}, s_0 \models \varphi_0^{\sigma} \) implies \( \mathcal{M}, s \models p^{\sigma \vartheta}_\psi \) iff \( (p^{\vartheta}_\psi \rightarrow p^{\sigma \vartheta}_\psi) \).

“only if”. Let \( \mathcal{M} = (S, \sim_i, V) \) be a model of \( \varphi_0 \), and let \( s_0 \in S \) with \( \mathcal{M}, s_0 \models \varphi_0 \). Define a model \( \mathcal{M}' \) as \( \mathcal{M} \) by additionally setting, for \( (\sigma, \varphi) \in R(\varphi_0) \),

\[
V(p^{\sigma}_\varphi) := \{ s \in S \mid \mathcal{M}, s \models \varphi^{\sigma} \}.
\]

Using the definition of the translation \( \varphi^{\sigma} \) and the implications \( B^{\sigma}_\varphi \), it is straightforward to show by induction on \(|\varphi| + |\sigma|\) that

\[
\mathcal{M}' \models B^{\sigma}_\varphi \text{ for all } (\sigma, \varphi) \in R(\varphi_0).
\]

Since \( \mathcal{M}, s_0 \models \varphi_0 \), we additionally have \( \mathcal{M}', s_0 \models p^{\sigma \vartheta}_\psi \). Thus, \( \mathcal{M}', s_0 \models \varphi_0^{\sigma} \).

\end{itemize}

**Theorem 10.** \( \text{PAL}_n \) satisfiability is \( \text{PSPACE-complete} \), for \( n \in \mathbb{N} \cup \{ \omega \} \).