Expressivity

**Definition 1** Let the model \( M(n) = (W, R, V) \) be defined by

- \( W = \{ x \mid 0 \leq x \leq n \} \)
- \( R = \{ (x, (x-1)) \mid 1 \leq x \leq n \} \)
- \( V(p) = W \)

These models are simply lines of worlds. The idea is that Spoiler cannot distinguish two of these models if the line is long enough. The only hope that Spoiler has is to force one of the current worlds to an endpoint and the other not to be an endpoint. In that case Spoiler can make a \( \Box \)-move in the world that is not an endpoint and Duplicator is stuck. This will not succeed if the lines are long enough. Note that a \( C \)-move does not help Spoiler. Also a \( [\varphi] \)-move will not help Spoiler. Such a move will shorten the lines, but that will cost as much rounds as it shortens them, so Spoiler still loses if they are long enough. The following Lemma captures this idea.

**Lemma 1** For all \( m, n \), and all \( x \leq m \) and \( y \leq n \) Duplicator has a winning for the public announcement game for \((M(m), x) \) and \((M(n), y) \) with at most \( \min(x, y) \) rounds.

The proof is in the appendix. Also consider the following set of models.

**Definition 2** Let the model \( M(m, n) = (W, R, V) \) where \( 0 < n \leq m \) be defined by

- \( W = \{ x \mid 0 \leq x \leq m \} \cup \{ x' \mid n \leq x \leq m \} \cup \{ \omega \} \)
- \( R = \{ (x, (x-1)) \mid 1 \leq x \leq m \} \cup \{ (x', (x-1)) \mid n+1 \leq x \leq m \} \cup \{ (x, \omega) \mid x \in W \setminus \{ \omega \} \} \)
- \( V(p) = W \setminus \{ \omega \} \)

The picture below represents \( M(2, 1) \).

Let us call these models lemniscates. The idea is that Spoiler cannot distinguish the top line from the bottom line of these models if they are long enough. Note that apart from \( \omega \) this model consists of two lines. So if Spoiler plays \( \Box \)-moves on these lines, Duplicator’s strategy is the same as for the line models described above. If he moves to \( \omega \), Duplicator also moves to \( \omega \), and surely Duplicator
cannot lose the subsequent game in that case. In these models a $C$-move is very bad for Spoiler, since all worlds are connected by the reflexive transitive closure of $R$. A $[\varphi]$-move will either yield two lines which are too long, or it will be a smaller lemniscate model, which will still be too large, since the $[\varphi]$-move reduces the number of available moves. The Lemma below captures this idea.

In the Lemma below we refer to $x^*$ as a variable ranging over $x$ and $x'$ and if $x'$ does not exist it refers to $x$.

**Lemma 2** For all $m$, $n$ and all $x^* \leq m$ and $y^* \leq n$ Duplicator has a winning strategy for the public announcement game for $(M(m, n), x^*)$ and $(M(m, n), y^*)$ with at most $\min(x, y) - m$ rounds.

The proof is in the appendix. Lastly consider the following class of models.

**Definition 3** Let the model $M(m, n) = (W, R, V)$ where $0 < n \leq m$ be defined by

- $W = \{x \mid n \leq x \leq m\} \cup \{x' \mid 0 \leq x \leq m\} \cup \{\alpha, \omega\}$
- $R = \{(x, (x-1)) \mid n + 1 \leq x \leq m\} \cup \{(x', (x-1)') \mid 1 \leq x \leq m\} \cup \{(w, \omega) \mid w \in W \setminus \{\alpha, \omega\}\} \cup \{(m', \alpha)\}$
- $V(p) = W \setminus \{\omega\}$

The picture below represents $M(2, 2)$.

Let us call these models lemniscates with an appendage. The idea is again that Duplicator cannot distinguish the top line from the bottom line of these models when they are long enough. Apart from $\alpha$ the model is just like a lemniscate. So the only new option for Spoiler is to force one of the current worlds to $\alpha$, and the other to another world. Then Spoiler chooses a $\Box$-move and takes a step from the non-$\alpha$ world and Duplicator is stuck at $\alpha$. However if the model is large enough $\alpha$ is too far away. Again a $C$-move does not help Spoiler, because it can be matched exactly by Duplicator. Reducing the model with a $[\varphi]$-move will yield either a lemniscate (with or without an appendage) or two lines, for which Spoiler does not have a winning strategy. The idea leads to the following Lemma.

**Lemma 3** For all $m$, $n$ and all $x^* \leq m$ and $y^* \leq n$ Duplicator has a winning strategy for the public announcement game for $(M(m, n), x^*)$ and $(M(m, n), y^*)$ with at most $\min(x, y) - m$ rounds.
The proof is in the appendix. This Lemma leads to the following theorem.

**Theorem 1** RCL is more expressive than PALC.

**Proof** Lemma 3 implies that the top worlds cannot be distinguished from bottom worlds in lemniscates with appendages in PALC. However, the formula $C(p, \neg \Box p)$ is true in $M(m, m)(m)$, but false in $M(m, m)(m')$. Therefore RCL can distinguish these models and therefore is more expressive.

### Appendix: proofs

**Lemma 1** For all $m$, $n$, and all $x \leq m$ and $y \leq n$ Duplicator has a winning strategy for the public announcement game for $(M(m), x)$ and $(M(n), y)$ with at most $\min(x, y)$ rounds.

**Proof** If $x = y$, the proof is trivial. We proceed by induction on the number of rounds. Suppose the number of rounds is 0. Then $x$ and $y$ only have to agree on propositional variables. They must agree, since $p$ is true everywhere.

Suppose that the number of rounds is $k + 1$ (i.e. $\min(x, y) = k + 1$). Duplicator’s strategy is the following. If Spoiler chooses to play a $\Box$-move, he moves to $x - 1$ (or to $y - 1$). Duplicator responds by choosing $y - 1$ (or $x - 1$). Duplicator has a winning strategy for the resulting subgame by the induction hypothesis.

Suppose Spoiler chooses to play a $C$-move. If Spoiler chooses a $z \leq \min(x, y)$, then Duplicator chooses $z$. Otherwise, Duplicator does not move at all. Duplicator has a winning strategy for the resulting subgame by the induction hypothesis.

Suppose Spoiler chooses to play a $[\varphi]$-move. Spoiler chooses a number of rounds $r$ and some $S$ and $S'$. Observe that for all $z < \min(x, y)$ it must be the case that $z \in S$ if $z \in S'$. Otherwise, Duplicator has a winning strategy by the induction hypothesis by choosing $z$ and $z$. Moreover for all $z \geq \min(x, y)$ it must be the case that $z \in S$ and $z \in S'$. Otherwise, Duplicator has a winning strategy by the induction hypothesis for $\max(x, y)$ and $z$. In Stage 2 the resulting subgame will be for two models bisimilar to models to which the induction hypothesis applies. The number of rounds will be $(k + 1) - r$, and the lines will be at least $\min(x, y) - r$ long (and $\min(x, y) = k + 1$).

**Lemma 2** For all $m$, $n$ and all $x^* \leq m$ and $y^* \leq n$ Duplicator has a winning strategy for the public announcement game for $(M(m, n), x^*)$ and $(M(m, n), y^*)$ with at most $\min(x, y) - m$ rounds.

**Proof** By induction on the number of rounds. Suppose the number of rounds is 0. Then $x^*$ and $y^*$ only have to agree on propositional variables. They must agree, since $p$ is true in all $z^*$.

Suppose that the number of rounds is $k + 1$. Duplicator’s winning strategy is the following. If Spoiler chooses to play a $\Box$-move, he moves to $(x - 1)^*$ (or to $(y - 1)^*$), or to $\omega$. In the last case Duplicator responds by also choosing $\omega$, and has a winning strategy for the resulting subgame. Otherwise Duplicator moves to the $(y - 1)^*$ (or $(x - 1)^*$). Duplicator has a winning strategy for the resulting subgame by the induction hypothesis.

Suppose Spoiler chooses to play a $C$-move. If Spoiler moves to $w$, then Duplicator moves to the same $w$, and has a winning strategy for the resulting subgame.
Suppose Spoiler chooses to play a $[\varphi]$-move. Spoiler chooses a number of rounds $r$ and some $S$. Since there is only one model, Spoiler only chooses one subset of $W$. Moreover for all $z^* \geq \min(x, y) - m - r$ it must be the case that $z^* \in S$. Otherwise, Duplicator has a winning strategy by the induction hypothesis for $\max(x, y)$ and $z^*$. In Stage 2 the result will be two models bisimilar to models to which the induction hypothesis applies, or to which Lemma 1 applies.

**Lemma 3** For all $m$, $n$ and all $x^* \leq m$ and $y^* \leq n$ Duplicator has a winning strategy for the public announcement game for $(M(m, n), x^*)$ and $(M(m, n), y^*)$ with at most $\min(x, y) - m$ rounds.

**Proof** The proof is analogous to the proof of Lemma 2.