Action Evaluation

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1. Partial Evaluation in Action Models

Let $A$ be an action model. Let $\Sigma$ be the set of preconditions occurring in $A$. (assume these are expressions from language $\text{LANG}_0$). Let $\sigma$ range over $\Sigma$. Consider the modal language over $\Sigma$:

$$\phi ::= \text{true} \mid p \mid \neg \phi \mid \phi_1 \land \phi_2 \mid [i]\phi$$

Call this the modal language of $A$. We define verification and falsification of formulas from the language of $A$ in $A$.

**DEFINITION 1.** (Verification and Falsification in Action Models).

- $A \models_s \text{true}$ always
- $A \models_s \neg \phi$ if $A \models_s \phi$
- $A \models_s \neg \phi$ if $A \models_s \phi$
- $A \models_s \phi_1 \land \phi_2$ if $A \models_s \phi_1$ and $A \models_s \phi_2$
- $A \models_s [i]\phi$ if either $\text{pre}_{s'} \models [i]\phi$ or for all $s'$ with $s \stackrel{i}{\rightarrow} s'$ : $A \models_{s'} \phi$
- $A \models_s [i]\phi$ if either $\text{pre}_{s'} \models -[i]\phi$ or for some $s'$ with $s \stackrel{i}{\rightarrow} s'$ : $A \models_{s'} \phi$

Read $A \models_s \phi$ as “$s$ verifies $\phi$ in $A$”, and $A \models_s \phi$ as “$s$ falsifies $\phi$ in $A$”. Note that the clauses for truth and falsity of $p$ and of $[i]\phi$ use the concept of logical consequence for the logic of $\Sigma$. The evaluation uses the strong Kleene scheme (invented by Kleene [?]) to describe the
Therefore, by induction hypothesis, either $A$ or $\neg A$ of verification for immediately from induction hypothesis. For examples where neither $A \models s \phi$ nor $A \npre s \phi$, think of a state $s$ with precondition $p$. Then $A \not\models s q$ and $A \not\pre s q$, for $q$ does not follow from the precondition of $s$ nor is $q$ inconsistent with the precondition of $s$. But we can also have contradictions! Take an action model $A$ with two states $s$ and $s'$ with $s \to s'$ (and no other pairs in $R_i$), with $\pre s = [i]p$ and $\pre s' = \neg p$. Then $A \models s [i]p$ because $[i]p$ equals the precondition of $s$, but also $A \npre s [i]p$, because for the only $i$-accessible state, $s'$, it holds that $A \npre s' p$ (since $p$ is inconsistent with the precondition of $s'$).

For a full treatment we will have to extend strong Kleene evaluation to the full language $\mathcal{L}_{NG0}$, but it is rather obvious how to do this.

**LEMMA 2.** For all action models $A$, all $A$-states $s$, all $\phi$ in the language of $A$: $\pre s \models \phi$ implies $A \models s \phi$, and $\pre s \models \neg \phi$ implies $A \npre s \phi$.

Proof. Induction on the structure of $\phi$. If $\phi$ equals $true$ the statement certainly holds. If $\phi$ equals $p$, then $\pre s \models p$ implies $A \models s p$ by the definition of verification, and $\pre s \models \neg p$ implies $A \npre s p$, again by the definition of verification.

Assume the statement holds for $\phi_1, \phi_2$. We show that it also holds for $\neg \phi_1, \phi_1 \land \phi_2, [i] \phi_1$.

Assume $\pre s \models \neg \phi_1$. We have to show $A \models s \neg \phi_1$. This follows immediately from $A \models s \phi$ (by induction hypothesis) and the definition of verification for $\neg$. Assume $\pre s \models \neg \neg \phi_1$. Then $\pre s \models \phi_1$, and, by induction hypothesis, $A \models s \phi$, and hence $A \models s \neg \phi$.

Assume $\pre s \models \phi_1 \land \phi_2$. Then $\pre s \models \phi_1$ and $\pre s \models \phi_2$, hence by twice the induction hypothesis, $A \models s \phi_1$ and $A \models s \phi_2$. Hence $A \models s \phi_1 \land \phi_2$. Assume $\pre s \models \neg (\phi_1 \land \phi_2)$. Then $\pre s \models \neg \phi_1$ or $\pre s \models \neg \phi_2$. Therefore, by induction hypothesis, either $A \models s \phi_1$ or $A \models s \phi_2$. Hence $A \models s \phi_1 \land \phi_2$. 


Assume $\text{pre}_s \models [i] \phi_1$. Then $A \models_s [i] \phi_1$ by definition of verification for $[i]$. Assume $\text{pre}_s \models \neg [i] \phi_1$. Then $A \not\models_s [i] \phi_1$ by the clause for falsification of $[i]$.

If we know that the formulas are purely propositional, we can turn this around:

**Lemma 3.** For all action models $A$, all $A$-states $s$, all $\phi$ in the purely propositional fragment of the language of $A$:

$$A \models_s \phi \iff \text{pre}_s \models \phi,$$

and

$$A =\models_s \phi \iff \text{pre}_s \models \neg \phi.$$

**Proof.** Induction on the propositional structure of $\phi$. If $\phi$ equals $\text{true}$ the statement certainly holds. If $\phi$ equals $p$, then $A \models_s p$ iff $\text{pre}_s \models p$ by the definition of verification, and $A =\models_s p$ iff $\text{pre}_s \models \neg p$, again by the definition of verification.

Assume the statement holds for $\phi_1, \phi_2$. We show that it also holds for $\neg \phi_1$ and $\phi_1 \land \phi_2$.

$A \models_s \neg \phi_1$ iff (definition of verification) $A =\models_s \phi_1$ iff (induction hypothesis) $\text{pre}_s \models \neg \phi_1$. $A \models_s \phi_1 \land \phi_2$ iff (definition of falsification) $A \models_s \phi_1$ and $A =\models_s \phi_2$ iff $\text{pre}_s \models \phi_1 \land \phi_2$.

$A =\models_s \phi_1 \land \phi_2$ iff (definition of verification) $A =\models_s \phi_1$ or $A =\models_s \phi_2$ iff $\text{pre}_s \models \neg \phi_1$ or $\text{pre}_s \models \neg \phi_2$ iff $\text{pre}_s \models \neg (\phi_1 \land \phi_2)$.

## Smoothness

**Definition 4.** Call a formula $\phi$ a **constraint formula** if $\phi$ is equivalent to a formula built from purely propositional formulas by means of $[i], \land$ and $\lor$ (i.e., a formula in the language

$$\phi ::= \psi \mid \phi_1 \land \phi_2 \mid \phi_1 \lor \phi_2 \mid [i] \phi,$$

where $\psi$ is purely propositional).

So here is a way to single out the smooth action models: smooth are the action models where every constraint formula that follows logically from the precondition of a state is consistent with that state.

**Definition 5.** Action model $A$ is smooth if for all $s \in W_A$ and all constraint formulas $\phi$ in the language of $A$: if $\text{pre}_s \models \phi$ then $A \not\models_s \phi$. 

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Figure 1. Action models for the Moore announcement (left) and the Al Gore announcement (right).

The reason to restrict the consistency requirement to constraint formulas is to allow action models with preconditions that ‘falsify themselves’, such as an announcement of the Moore sentence \( p \land \neg [i]p \). The action model for this (Figure ?? left, with \( \land \) written as \&) has formulas that are falsified at state 0 while they also follow from the precondition of 0: we have \( p \land \neg [i]p \models \neg [i]p \), but also \( s_0 \models \neg [i]p \). Since \( p \) is true at 0, and 0 is the only i-accessible state from 0, \( s_0 \models [i]p \), and therefore \( s_0 \models \neg [i]p \).

This does not contradict smoothness; the precondition \( p \land \neg [i]p \) does not constrain the accessibility relation in the action model itself, for the precondition has no non-trivial modal consequences of the form \( [i]\phi_1 \lor \cdots \lor [i]\phi_n \). A similar thing holds for the Al Gore announcement \( p \land \neg [(a_1 \cup \cdots \cup a_n)^*]p \) in Figure ??, right.

Note that smoothness is a natural extension of consistency:

**Proposition 6.** If \( A \) is smooth then all preconditions in \( A \) are consistent.

**Proof.** Suppose \( A \) is smooth. Let \( s \) be a state in \( A \) and assume \( \text{pre}_s \) is inconsistent. Then \( \text{pre}_s \models \text{false} \) and \( A \models \text{false} \), and since \( \text{false} \) is a constraint formula, it follows that \( A \) is not smooth.

**Proposition 7.** If each precondition of \( A \) is a consistent purely propositional formula, then \( A \) is smooth.

**Proof.** Let \( A \) be an action model with each precondition a consistent purely propositional formula. Let \( s \) be a state in \( A \), and let \( \phi \) be a constraint formula such that \( \text{pre}_s \models \phi \). Then since \( \text{pre}_s \) is purely propositional, \( \phi \) is equivalent to a purely propositional formula. Because \( \text{pre}_s \) is consistent, \( \text{pre}_s \models \phi \) implies \( \text{pre}_s \models \neg \phi \). Lemma ?? gives that \( A \models \phi \).

**Theorem 8.** If \( A \) is a smooth action model, then for each state \( s \) in \( A \) it holds that the formula set

\[
\{ \text{pre}_s \} \cup \{ \langle i \rangle \text{pre}_{s'} \mid s \xrightarrow{i} s' \}
\]

is consistent.
Proof. Let $A$ be a smooth action model. If for some $s$ in the domain of $A$ the formula set $\{\text{pre}_s\} \cup \{(i)\text{pre}_{s'} | s \xrightarrow{i}s'\}$ is inconsistent, this means that the formula $\bigvee\{[i]\neg\text{pre}_{s'} | s \xrightarrow{i}s'\}$ is implied by $\text{pre}_s$. But this is a constraint formula, so smoothness of $A$ gives:

$$A \not\models_s \bigvee\{[i]\neg\text{pre}_{s'} | s \xrightarrow{i}s'\}$$

Contradiction with

$$A \models_s \bigvee\{[i]\neg\text{pre}_{s'} | s \xrightarrow{i}s'\}$$

which we get from $A \models_s [i]\neg\text{pre}_{s'}$ for each $s'$ with $s \xrightarrow{i}s'$.

3. Filtration and Canonical Models

Our goal in Section ?? is to provide a recipe for turning any action model into an equivalent smooth action model. Next, in Section ??, we will prove the converse of Theorem ?? for all smooth action models, i.e., we will extend Theorem ?? from purely propositional action models to all smooth action models.

For both goals we need a technique (called filtration) for constructing models from sets of formulas. The filtration technique in modal logic is used to construct a finite model for a consistent modal formula $\phi$ (see [?]). For ordinary modal logic the construction is based on the set of all sub-formulas of $\phi$, but in PDL we have to be careful in the handling of formulas with complex modalities $\alpha$, so we need so-called Fischer/Ladner closures [?].

**DEFINITION 9.** Let $\Sigma$ be a set of $\text{LANG}_0$ formulas. Then $\text{FL}(\Sigma)$, the Fischer/Ladner closure of $\Sigma$, is the smallest set of formulas $X$ that has $\Sigma \subseteq X$, that is closed under taking sub-formulas, and that satisfies the following constraints:

- if $[\alpha \cup \alpha']\phi \in X$ then $[\alpha]\phi \in X$ and $[\alpha']\phi \in X$,
- if $[\alpha;\alpha']\phi \in X$ then $[\alpha][\alpha']\phi \in X$,
- if $[\alpha^*]\phi \in X$ then $[\alpha][\alpha^*]\phi \in X$.

Note that the definition handles the actual formulas of the language, not their abbreviations. As an example, consider $\Sigma = \{((a \cup b)^*|h)\}$. Then

$$\text{FL}(\Sigma) = \{((a \cup b)^*|h), [(a \cup b)][(a \cup b)^*]|h, [a][(a \cup b)^*]|h, [b][(a \cup b)^*]|h, h\}.$$
DEFINITION 10. (Closure under single negation). For any formula \( \phi \), define \( \sim \phi \), the single negation of \( \phi \), as follows: if \( \phi \) has the form \( \neg \psi \) then \( \sim \phi = \psi \), otherwise \( \sim \phi = \neg \phi \). Then \( \sim \phi \) forms the negation of \( \phi \), while cancelling double negations. A set of formulas \( X \) is closed under single negations if \( \phi \in X \) implies \( \sim \phi \in X \).

DEFINITION 11. (Closure of \( \Sigma \)). For any formula set \( \Sigma \), the closure of \( \Sigma \), notation \( \neg \text{FL}(\Sigma) \) is the smallest set \( X \) which contains \( \text{FL}(\Sigma) \) and is closed under single negations.

As an example, observe that the closure of \( \{[(a \cup b)^* h]\} \) consists of the union of \( \text{FL}(\{[(a \cup b)^* h]\}) \) and the set of all negations of formulas in \( \text{FL}(\{[(a \cup b)^* h]\}) \).

In building epistemic models and action models from sets of formulas \( \Sigma \) we can take worlds (or actions) to be maximal consistent sets of formulas taken from \( \neg \text{FL}(\Sigma) \).

DEFINITION 12. Let \( \Sigma \) be a set of formulas. A set of formulas \( \Gamma \) is an atom over \( \Sigma \) if \( \Gamma \) is a maximal consistent subset of \( \neg \text{FL}(\Sigma) \). Let \( \text{At}(\Sigma) \) be the set of all atoms over \( \Sigma \).

It is easy to show for every consistent formula \( \phi \in \neg \text{FL}(\Sigma) \) there is a \( \Gamma \in \text{At}(\Sigma) \) with \( \phi \in \Gamma \) (see [?]). For any finite formula set \( \Gamma \), let \( \hat{\Gamma} = \bigwedge \Gamma \).

DEFINITION 13. The canonical model \( M_{\Sigma} \) over finite formula set \( \Sigma \) is given by

\[
\begin{align*}
W_{\Sigma} &= \text{At}(\Sigma), \\
V_{\Sigma}(\Gamma) &= \{ p \in \text{Prop} \mid p \in \Gamma \}, \\
R_{\Sigma}(i) &= \{ (\Gamma, \Gamma') \mid \hat{\Gamma} \land (i)\hat{\Gamma}' \text{ is consistent } \}.
\end{align*}
\]

See [?] for a proof that this canonical model ‘works’, in the sense that we can prove the following:

LEMMA 14. (Truth Lemma). For all atoms \( \Gamma \in \text{At}(\Sigma) \) and all \( \phi \in \neg \text{FL}(\Sigma) \) it is the case that \( M_{\Sigma} \models_\Gamma \phi \) iff \( \phi \in \Gamma \).

4. Construction of Smooth Action Models

Let \( \neg \text{FL}(\Sigma) \) be as defined in Section ???. Then define the following operation for turning an action model into an equivalent smooth action model:
DEFINITION 15. Let \( A \) be an action model, and let \( \Sigma \) be the set of preconditions that occur in \( A \). Assume without loss of generality that all preconditions in \( A \) are consistent. (If this is not the case, just restrict \( A \) to the action models with consistent preconditions.) Then \( \text{Sm}(A) \) is the action model \((W, \text{pre}, R)\) given by

\[
W = \{(s, \Gamma) \mid \Gamma \in \text{At}(\Sigma), \text{pre}_s \in \Gamma\},
\]
\[
\text{pre}(s, \Gamma) = \text{pre}(s),
\]
\[
R(i) = \{((s, \Gamma), (s', \Gamma')) \in W^2 \mid s \overset{i}{\rightarrow} s' \text{ and } \hat{\Gamma} \land \langle i \rangle \hat{\Gamma}' \text{ is consistent}\}.
\]

If \( S \) is the set of distinctive points of \( A \), then \( \text{Sm}(S) = \{(s, \Gamma) \in W_{\text{Sm}(A)} \mid s \in S\} \) is the set of distinctive points of \( \text{Sm}(A) \).

Recall that we restrict attention to the modal fragment of \( \text{LANG} \), and assume action models with no complex modalities in their preconditions.

LEMMA 16. Let \( A \) be an action model with precondition set \( \Sigma \). Let \((s, \Gamma)\) be a state in \( \text{Sm}(A) \). Then for all \([i]\phi \in \neg \text{FL}(\Sigma)\):

\[
[i]\phi \in \Gamma \text{ and } (s, \Gamma) \overset{i}{\rightarrow} (s', \Gamma') \text{ implies } \text{pre}_s \land \phi \text{ consistent}.
\]

**Proof.** We assume without loss of generality that all preconditions in \( A \) are consistent.

Assume \([i]\phi \in \Gamma \) and \((s, \Gamma) \overset{i}{\rightarrow} (s', \Gamma')\). Suppose \( \phi \land \text{pre}_s \) inconsistent. Then \( \text{pre}_s \models \neg\phi \), and therefore \( \neg\phi \in \Gamma' \). From \((s, \Gamma) \overset{i}{\rightarrow} (s', \Gamma')\) we have that \( \hat{\Gamma} \land \langle i \rangle \hat{\Gamma}' \) is consistent. Therefore, \( \hat{\Gamma} \land \langle i \rangle \neg\phi \) is also consistent, and by maximality of \( \Gamma \) we get \( \langle i \rangle \neg\phi \in \Gamma \). Contradiction with the fact that \([i]\phi \in \Gamma \).

LEMMA 17. Let \( A \) be an action model with precondition set \( \Sigma \). Let \( \phi \) be a constraint formula in \( \neg \text{FL}(\Sigma) \) and let \((s, \Gamma)\) be a state in \( \text{Sm}(A) \). Then

\[
\text{Sm}(A) \models_{(s, \Gamma)} \phi \text{ implies } \phi \in \Gamma, \text{ and } \text{Sm}(A) \models_{(s, \Gamma)} \neg\phi \text{ implies } \neg\phi \in \Gamma.
\]

**Proof.** Induction on the structure of constraint formula \( \phi \). Let \( \phi \) be purely propositional, and assume \( \text{Sm}(A) \models_{(s, \Gamma)} \phi \). Then by Lemma 16, \( \text{pre}(s, \Gamma) \models \phi \). Since \( \text{pre}(s, \Gamma) = \text{pre}_s \) and \( \text{pre}_s \in \Gamma \), it follows that \( \phi \in \Gamma \). Assume \( \text{Sm}(A) \models_{(s, \Gamma)} \neg\phi \). Then by Lemma 16, \( \text{pre}(s, \Gamma) \models \neg\phi \). Since \( \text{pre}(s, \Gamma) = \text{pre}_s \) and \( \text{pre}_s \in \Gamma \), it follows that \( \neg\phi \in \Gamma \).
Now assume the property holds for constraint formulas $\phi_1, \phi_2$. We show that it also holds for $\phi_1 \land \phi_2, \phi_1 \lor \phi_2, [i] \phi_1$.

Assume $Sm(A) \models (s, \Gamma) \phi_1 \land \phi_2$. Then by the definition of verification, $Sm(A) \models (s, \Gamma) \phi_1$ and $Sm(A) \models (s, \Gamma) \phi_2$. By the induction hypothesis, $\phi_1 \in \Gamma$ and $\phi_2 \in \Gamma$. It follows that $\phi_1 \land \phi_2 \in \Gamma$.

Assume $Sm(A) \models (s, \Gamma) \phi_1 \lor \phi_2$. Then by the definition of falsification, $Sm(A) \models (s, \Gamma) \phi_1$ or $Sm(A) \models (s, \Gamma) \phi_2$. By the induction hypothesis, $\sim \phi_1 \in \Gamma$ or $\sim \phi_2 \in \Gamma$. It follows that $\sim (\phi_1 \lor \phi_2) \in \Gamma$.

Assume $Sm(A) \models (s, \Gamma) \phi_1 \lor \phi_2$. Then by the definition of falsification, $Sm(A) \models (s, \Gamma) \phi_1$ and $Sm(A) \models (s, \Gamma) \phi_2$. By the induction hypothesis, $\sim \phi_1 \in \Gamma$ and $\sim \phi_2 \in \Gamma$. It follows that $\sim (\phi_1 \lor \phi_2) \in \Gamma$.

Assume $Sm(A) \models (s, \Gamma) [i] \phi_1$. Then by the definition of verification, either $\text{pre}(s, \Gamma) \models [i] \phi_1$ or for all $(s', \Gamma')$ with $(s, \Gamma) \not\rightarrow (s', \Gamma')$ it holds that $Sm(A) \models (s', \Gamma') \phi_1$. In the first case, $\text{pre}(s) \models [i] \phi_1$, and therefore $[i] \phi_1 \in \Gamma$. In the second case, the induction hypothesis yields $\phi_1 \in \Gamma'$ for all $(s', \Gamma')$ with $(s, \Gamma) \not\rightarrow (s', \Gamma')$. It follows that all $\Gamma'$ with $\hat{\Gamma} \land (i) \hat{\Gamma'}$ consistent satisfy $\phi_1 \in \Gamma'$. Therefore, $[i] \phi_1$ is consistent with $\Gamma$. It follows by maximality of $\Gamma$ that $[i] \phi \in \Gamma$.

Assume $Sm(A) \models (s, \Gamma) \not[i] \phi_1$. Then by the definition of falsification, either $\text{pre}(s, \Gamma) \models \not[i] \phi_1$ or there is a $(s', \Gamma')$ with $(s, \Gamma) \not\rightarrow (s', \Gamma')$ and $Sm(A) \models (s', \Gamma') \phi_1$. In the first case, $\text{pre}(s) \models \not[i] \phi_1$, and therefore $\not[i] \phi \in \Gamma$. In the second case, the induction hypothesis yields $\sim \phi_1 \in \Gamma'$. $(s, \Gamma) \not\rightarrow (s', \Gamma')$ implies $\hat{\Gamma} \land (i) \hat{\Gamma'}$ is consistent. Therefore, $(i) \sim \phi$ is consistent with $\Gamma$, i.e., $\sim [i] \phi$ is consistent with $\Gamma$. It follows by maximality of $\Gamma$ that $\sim [i] \phi \in \Gamma$. \hfill\Box

We show that $Sm(A)$ is smooth.

THEOREM 18. For each action model $A$ it holds that $Sm(A)$ is smooth.

Proof. Let $A$ be any action model with precondition set $\Sigma$. Again, assume without loss of generality that each precondition in $A$ is consistent. Let $\phi$ be a constraint formula in the language of $A$, and let $(s, \Gamma)$ be a state of $Sm(A)$ with $Sm(A) \models (s, \Gamma) \phi$. We show with induction on the structure of $\phi$ that $Sm(A) \not\models (s, \Gamma) \phi$.

We assume without loss of generality that $\phi \in \neg FL(\Sigma)$.

Basis. If $\phi$ is purely propositional then $Sm(A) \models (s, \Gamma) \phi$ implies $\text{pre}(s, \Gamma) \models \neg \phi$, by Lemma ??, and therefore $\text{pre}(s, \Gamma) \not\models \phi$, by consistency of the preconditions, and again by Lemma ??, $Sm(A) \not\models (s, \Gamma) \phi$. 

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Induction step. Suppose the property holds for constraint formulas \( \phi_1 \) and \( \phi_2 \). Let \( \phi = \phi_1 \land \phi_2 \). Then by the verification definition, it follows from

\[
Sm(A) \models_{(s, \Gamma)} \phi_1 \land \phi_2
\]

that

\[
Sm(A) \models_{(s, \Gamma)} \phi_1 \text{ and } Sm(A) \models_{(s, \Gamma)} \phi_2.
\]

By the induction hypothesis, \( Sm(A) \not\models_{(s, \Gamma)} \phi_1 \) and \( Sm(A) \not\models_{(s, \Gamma)} \phi_2 \).

By the definition of falsification, \( Sm(A) \not\models_{(s, \Gamma)} \phi_1 \land \phi_2 \).

Let \( \phi = \phi_1 \lor \phi_2 \). Then by the verification definition, it follows from

\[
Sm(A) \models_{(s, \Gamma)} \phi_1 \lor \phi_2
\]

that

\[
Sm(A) \models_{(s, \Gamma)} \phi_1 \text{ or } Sm(A) \models_{(s, \Gamma)} \phi_2.
\]

By the induction hypothesis, \( Sm(A) \not\models_{(s, \Gamma)} \phi_1 \) or \( Sm(A) \not\models_{(s, \Gamma)} \phi_2 \).

Let \( \phi = [i] \phi_1 \). Then by the verification definition, it follows from

\[
Sm(A) \models_{(s, \Gamma)} [i] \phi_1
\]

that either \( pre(s, \Gamma) \models [i] \phi_1 \) or for all \( (s', \Gamma') \) with \( (s, \Gamma) \rightarrow (s', \Gamma') \) it holds that \( Sm(A) \models_{(s', \Gamma')} \phi_1 \).

In the first case, \( pre(s, \Gamma) \models [i] \phi_1 \) yields \([i] \phi_1 \in \Gamma \), so by maximality of Gamma, \( \sim [i] \phi_1 \notin \Gamma \). From this, by Lemma ?? and contraposition: \( Sm(A) \not\models_{(s, \Gamma)} \phi \).

In the second case, we can assume \( pre(s, \Gamma) \not\models [i] \phi_1 \). The induction hypothesis gives that for all \( (s', \Gamma') \) with \( (s, \Gamma) \rightarrow (s', \Gamma') \) it holds that \( Sm(A) \not\models_{(s', \Gamma')} \phi_1 \). By the definition of falsification and the fact that \( pre(s, \Gamma) \not\models [i] \phi_1 \), it follows that \( Sm(A) \not\models_{(s, \Gamma)} [i] \phi_1 \).

Question: can we extend this to the case of PDL formulas, where there is no notion of modal degree?

**THEOREM 19.** For every distinctive action model \((A, S)\) it holds that \((A, S) \equiv (Sm(A), Sm(S))\).

**Proof.** Let \( M \) be an arbitrary epistemic model. The relation

\[
C \subseteq W_{M \otimes A} \times W_{M \otimes Sm(A)}
\]

given by

\[
(w, s)C(w', (s', \Gamma)) : = w = w' \land s = s' \land M \models_w \Gamma
\]
is a bisimulation that connects $S$ to $Sm(S)$. Thus,

$$(A, S) \equiv (Sm(A), Sm(S)).$$

5.

**Lemma 20.** $(A, S)$ is smooth iff $(A, S) \equiv (Sm(A), Sm(S))$.

**Proof.** Let $(A, S)$ be smooth. Let $\Sigma$ be the preconditions in $A$. For any state $s$ of $A$, let $\Phi_s$ be the set

$$\Phi_s = \{ \phi \in \neg FL(\Sigma) \mid \phi \text{ is a constraint formula and } \text{pre}_s \models \phi \}.$$

We show that the relation

$$C \subseteq W_A \times W_{Sm(A)}$$

given by

$$sC(s', \Gamma) \text{ iff } s = s' \text{ and } \Phi_s \subseteq \Gamma$$

is a bisimulation.

First observe that it follows from Lemma 19 that $\{\text{pre}_s\} \cup \Phi_s$ is consistent.

Assume $sC(s, \Gamma)$. Then invariance certainly holds, as $\text{pre}_s = \text{pre}(s, \Gamma)$.

For the zig condition, suppose $s \xrightarrow{i} s'$. Then since $A$ is smooth, the set

$$X = \{ \phi \in \neg FL(\Sigma) \mid \phi \text{ is a constraint formula and } \text{pre}_{s'} \models \phi \}$$

is consistent. Let $\Gamma'$ be a maximally consistent extension of $X$. We show that $\hat{\Gamma} \land \langle i \rangle \hat{\Gamma}'$ is consistent. Suppose not.

**Lemma 21.** Let $A$ be a smooth action model, and let $M_\Sigma$ the canonical model constructed from its set of preconditions $\Sigma$. Then for all $s, s' \in W_A$ and for all $\Gamma \in M_\Sigma$:

if $s \xrightarrow{i} s'$ and $\text{pre}_s \in \Gamma$ then $\exists \Gamma' \in W_\Sigma : \Gamma \xrightarrow{i} \Gamma'$ and $\text{pre}_{s'} \in \Gamma'$.

**Proof.** Let $A$ be smooth, let $s, s' \in W_A$ with $s \xrightarrow{i} s'$, and let $\Gamma \in M_\Sigma$.

Suppose $\hat{\Gamma} \land \langle i \rangle \text{pre}_{s'}$ is inconsistent. Then $\langle i \rangle \text{pre}_{s'} \notin \Gamma$. Since $\Gamma$ is an atom over $\Sigma$ and $\text{pre}_{s'} \in \Sigma$, we get from this that $[i] \neg \text{pre}_{s'} \in \Gamma$.

Now either $\text{pre}_s \land \langle i \rangle \text{pre}_{s'}$ is inconsistent, and contradiction with the smoothness of $A$, or $\langle i \rangle \text{pre}_{s'}$ is inconsistent with $\Gamma - \{\text{pre}_s\}$, i.e., $[i] \neg \text{pre}_{s'} \in \Gamma - \{\text{pre}_s\}$.

not local.
Assume there are \(s, s' \in W_A\) and there is a \(\Gamma \in W_\Sigma\) with for all \(\Gamma' \in W_\Sigma\): \(\Gamma \xrightarrow{\cdot} \Gamma'\) implies \(\text{pre}_{s'} \notin \Gamma'\).

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Next, can we show that for smooth action models there is a simple (structural) definition of action emulation that matches action model equivalence?

6.

Kleene evaluation in smooth action models might be a powerful tool for investigating update effect. We can investigate what happens to formulas that evaluate to true in a smooth update model. Will they all be made true by the update, in the sense that if \(M \models_w \text{pre}_s\) and \(A \models_s \phi\) then \(M \otimes A \models_{(w,a)} \phi\)?