Epistemic Probability Logic Simplified

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ABSTRACT

We propose a logic for reasoning about (multi-agent) epistemic probability models, and for epistemic probabilistic model checking. Epistemic probability models are multi-agent Kripke models that assign to each agent an equivalence relation on worlds, together with function from worlds to positive rationals (a lottery). The difference with the usual approach is that probability is linked to knowledge rather than belief, and that knowledge is equated with certainty.

Contributions of this paper are a semantics with a single lottery, an adaptation of the Hennessy-Milner Theorem, a completeness result for epistemic probability logic, and information about its model checking complexity. In particular, we state the PSPACE-completeness of the model checking in the dynamic version with action models of this framework.

One of the purposes of the logic is model checking for epistemic probability logic. A prototype model checker for the logic exists.

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Contributions of this paper are a semantics with a single lottery, an adaptation of the Hennessy-Milner Theorem, a completeness result for epistemic probability logic, and information about its model checking complexity. In particular, we state the PSPACE-completeness of the model checking in the dynamic version with action models of this framework.

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1. PROBABILITY AS A FUNCTION OF DEGREE OF INFORMATION

A classical view of probability theory is that probability measures degree of information. Here is a characteristic quote from [16]:

"Dans les choses qui ne sont que vraisemblables, la différence des données que chaque homme a sur elles, est une des causes principales de la diversité des opinions que l’on voit régner sur les mêmes objets. (Laplace)"

This paper presents a logic of probability and knowledge where the two are related as follows:

Agent $a$ knows $\phi$ iff the probability $a$ assigns to $\phi$ equals 1.

Let $P_a\phi$ be the probability that agent $a$ assigns to $\phi$.

Certainty implies Truth

$(P_a\phi = 1) \rightarrow \phi$.

Positive Introspection into Certainty

$(P_a\phi = 1) \rightarrow P_b(P_a\phi = 1) = 1$.

Negative Introspection into Certainty

$(P_a\phi < 1) \rightarrow P_b(P_a\phi < 1) = 1$.

Our proposal has obvious relations to earlier proposals on combining knowledge and probability [8,14,3,5,4,11], and many more. A key difference is that these proposals do not equate knowledge with certainty.

Although, in real applications, knowledge and certainty are strongly related. We deal with such examples in section 5 when we present lotteries. This notably simplify the framework of epistemic probability logic we present in section 3. In particular, we will present models with a single lottery and in section 4 we prove that semantics with a single lottery and with several lotteries are the same. We then prove the Hennessy-Milner Theorem for epistemic probability logic in section 5. In section 6 we give an axiomatization for our epistemic probabilistic logic based on [8] and we prove that $S5$ axioms can be retrieved. In section 7 we deal with the model checking procedure that runs in polynomial time. In section 8 we see how to add action models of Dynamic epistemic logic and we provide a PSPACE-completeness proof for the model checking problem with a dynamic operator in the language.

2. LOTTERIES

A $W$-lottery $l$ is a function from a (finite) set of worlds $W$ to the set of positive (non-zero) rationals, i.e., $l : W \rightarrow Q^+$. If we have a lottery $l : W \rightarrow Q^+$ and a block $B \subseteq W$ in a partition of $W$, then this determines a probability distribution $P$ on $B$, by means of (we assume that $B \neq \emptyset$):

$P(w) = \frac{\sum \{l(w') \mid w' \in B\}}{\sum \{l(w) \mid w \in B\}}$

Example 1 Say there are two urns, $U$ and $V$. $U$ contains one black marble and two white marbles, $V$ contains one black marble and one white marble.

This is common knowledge among $a$, $b$ and $c$. Now $a$ selects one of the urns, without revealing which one to $b$, $c$. Then $b$ picks a marble from it, without revealing the marble to $a$, $c$.

Representation (accessibility represented as existence of a path; solid lines for $a$, dashed for $b$, dotted for $c$):

$0 : (U, \bullet) \xrightarrow{a} 2 : (V, \bullet)$

$\{0 : \frac{1}{6}, 1 : \frac{1}{3}, 2 : \frac{1}{4}, 3 : \frac{1}{4}\}$

$1 : (U, \bullet) \xrightarrow{a} 3 : (V, \bullet)$

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Another Representation, in terms of two worlds (a and c confuse the worlds, b does not) and two lotteries (b and c confuse the lotteries, a does not):

\[
\begin{align*}
0 : \mathbb{C} & \quad l_0 : \{0 : \frac{1}{2}, 1 : \frac{1}{2}\} \\
1 : \mathbb{C} & \quad l_1 : \{0 : \frac{1}{2}, 1 : \frac{1}{2}\}
\end{align*}
\]

Example 2 (Lotteries over lotteries) This example is from \cite{12}. There are five urns with the following compositions: 2 urns with 2 white balls and 1 black ball, 3 black balls each, 2 urns with 1 white and 4 black balls each, and one urn with 4 white balls and 1 black ball.

\[
\begin{align*}
\text{\ding{172}} \text{\ding{173}} \text{\ding{174}} \text{\ding{175}} \text{\ding{176}} & \quad \text{\ding{172}} \text{\ding{173}} \text{\ding{174}} \text{\ding{175}} \text{\ding{176}}
\end{align*}
\]

A ball is chosen from one of the urns taken at random. It turns out to be white. What is the probability (after the experiment) that the ball was taken from the last urn?

Representation in terms of a lottery over lotteries:

\[
\begin{align*}
0 : \mathbb{C} & \quad l_0 : \{0 : \frac{1}{2}, 1 : \frac{1}{2}\} \\
1 : \mathbb{C} & \quad l_1 : \{0 : \frac{1}{2}, 1 : \frac{1}{2}\}
\end{align*}
\]

Another Representation, in terms of single lotteries:

\[
\begin{align*}
0 : \mathbb{C} & \quad l_0 : \{0 : \frac{1}{2}, 1 : \frac{1}{2}\} \\
1 : \mathbb{C} & \quad l_1 : \{0 : \frac{1}{2}, 1 : \frac{1}{2}\}
\end{align*}
\]

Finally, a representation in terms of one lottery:

\[
\begin{align*}
(\mathbb{C}, u_0) & : \frac{3}{5} \quad (\mathbb{C}, u_0) : \frac{2}{5} \quad (\mathbb{C}, u_1) : \frac{3}{5} \quad (\mathbb{C}, u_1) : \frac{2}{5} \\
(\mathbb{C}, u_2) & : \frac{4}{5} \quad (\mathbb{C}, u_2) : \frac{1}{5} \\
(\mathbb{C}, u_3) & : \frac{4}{5} \quad (\mathbb{C}, u_3) : \frac{1}{5} \quad (\mathbb{C}, u_4) : \frac{4}{5}
\end{align*}
\]

Example 3 (Coin Tossing) Suppose Alice is tossing a coin while Bob is watching. Both know that the coin can either be fair or biased (say, with bias \(\frac{1}{2}\) towards heads). Bob does not know which coin Alice is using, but Alice knows. Representation (solid lines for Alice, dashed lines for Bob):

\[
\begin{align*}
0 : H & \quad l_0 : \{0 : \frac{1}{2}, 1 : \frac{1}{2}\} \\
1 : T & \quad l_1 : \{0 : \frac{1}{2}, 1 : \frac{1}{2}\}
\end{align*}
\]

Again, this can be represented by a single lottery:

\[
\begin{align*}
H, F, \frac{1}{2} & \quad H, B, \frac{1}{2} \\
\langle \rangle & \quad \langle \rangle \\
T, F, \frac{1}{2} & \quad T, B, \frac{1}{2}
\end{align*}
\]

3. **EPISTEMIC PROBABILISTIC LOGIC**

In this section, we recall standard epistemic logic. Then we present our epistemic lottery models (with the variant with a single lottery even for the multi-agent case). We then present the language of our version of epistemic probabilistic logic and its semantics and finally we present how to embed standard epistemic logic in our framework.

3.1 **Standard epistemic logic**

Let us start with the standard definition of epistemic Kripke models and then we give the standard multi-agent epistemic logic.

**Definition 1** A standard epistemic model \(M\) for a set \(P\) of propositions and a set \(A\) of agents is a tuple \((W, V, R, L)\) where

- \(W\) is a non-empty set of worlds,
- \(V\) is a valuation function that assigns to every \(w \in W\) a subset of \(P\).
- \(R\) is a function that assigns to every agent \(a \in A\) an equivalence relation \(R_a\) on \(W\).

The language \(\mathcal{L}_0\) of multi-agent epistemic logic is defined as follows.

**Definition 2** Let \(p\) range over a countable set \(P\) of basic propositions and \(r\) ranges over a finite set of agents \(A\).

\[
\varphi ::= T \mid p \mid \neg \varphi \mid \varphi \land \varphi \mid K_a \varphi
\]

\(K_a \varphi\) is read as ‘agent \(a\) knows that \(\varphi\) is true’ and the semantics is given by: \((W, R, V), w \models K_a \varphi\) iff for all \(u \in R_a(w)\), \((W, R, V), w \models \varphi\).

3.2 **Epistemic lottery models**

To turn a standard epistemic model into an epistemic probability model, we assign to each agent an equivalence relation over a list of lotteries. This represents subjective probabilities.

**Definition 3** An epistemic lottery model \(M\) is a tuple \((W, V, R, L)\) where \(W, V, R\) are as in Definition \(\text{[1]}\) and \(L\) is a function that assigns to every agent \(a \in A\) a \(W\)-lottery.

We say that an epistemic lottery model is normalized if \(L_a\) restricted to \(E\) is a probability measure for all agents \(a\) and for all \(R_a\)-equivalence classes \(E\).

Now, we define an epistemic lottery model where the lotteries are the same for each agent, that is \(L_a = L_b\) for all agents \(a, b\). We will write \(L\) instead of \(L_a\) for a given agent \(a\). Models where there is a single lottery seem easier to manipulate. Formally:

**Definition 4** An epistemic single lottery model \(M\) is a tuple \((W, V, R, L)\) where \(W, V, R\) are as in Definition \(\text{[1]}\) and \(L\) is a \(W\)-lottery.
3.3 Epistemic probability logic language

The language $\mathcal{L}$ of multi-agent epistemic probability logic is defined as follows.

**Definition 5** Let $p$ range over $\mathbb{P}$, $a$ over $\mathbb{A}$, $q$ over $\mathbb{Q}$. Then $\mathcal{L}$ is given by:

$$
\phi ::= \top | p | \neg \phi | \phi \land \phi | t_a \geq 0 | t_a = 0
$$

Some useful abbreviations:
- $\bot, \varphi_1 \lor \varphi_2, \varphi_1 \rightarrow \varphi_2, \varphi_1 \leftrightarrow \varphi_2$.
- $t \geq t'$ for $t + (-1)t' \geq 0$.
- $t < t'$ for $-t \geq t'$.
- $t \geq t'$ for $t' \geq t$.
- $t \leq t'$ for $t' \leq t$.
- $t \neq t'$ for $-t = -t'$.
- $P_a(\varphi_1 \land \varphi_2) = q$ for $q \cdot P_a(\varphi_1) = P_a(\varphi_1 \land \varphi_2)$.

$t_a$ generates expressions linear expressions dealing with subjective probabilities of agent $a$. A formula of the form $t_a = 0$ or $t_a > 0$ is called an $a$-probability formula.

Given these, we have:
- $P_a q = q$ expresses that the probability of $\varphi$ according to $a$ equals $q$.
- $P_a(\varphi_1 \land \varphi_2) = q$ expresses that according to $a$, the probability of $\varphi_2$, conditional on $\varphi_1$, equals $q$.

The truth definition for $\mathcal{L}$ are given below.

**Definition 6** Let $\mathcal{M} = (W, V, R, L)$ be an epistemic lottery model and let $w \in W$.

$$
\mathcal{M}, w \models \top \quad \text{always}
$$

$$
\mathcal{M}, w \models p \quad \text{iff} \quad p \in V(w)
$$

$$
\mathcal{M}, w \models \phi \quad \text{iff} \quad \text{it is not the case that} \mathcal{M}, w \not\models \varphi
$$

$$
\mathcal{M}, w \models \varphi_1 \land \varphi_2 \quad \text{iff} \quad \mathcal{M}, w \models \varphi_1 \quad \text{and} \quad \mathcal{M}, w \models \varphi_2
$$

$$
\mathcal{M}, w \models t_a = 0 \quad \text{iff} \quad t_a[w] = 0
$$

$$
[\phi]_w := q
$$

$$
[q \cdot P_a \phi]_w := q \cdot P_a,w(\phi)
$$

$$
[t_a + t'_a]_w := [t_a]_w + [t'_a]_w
$$

$$
P_{a,w}(\varphi) = \frac{\sum_{u \in L_a(u) \mid w \in R_{a,u} \text{ and } u \models \varphi}}{\sum_{u \in L_a(u) \mid w \in R_{a,u}}}
$$

Remark that $L_a(u) > 0$ for all $u \in W$ so that there is no division by zero. The interpretation of formulas in epistemic single lottery model is similar except that we use directly $L$ instead of $L_a$ for a given agent $a$.

3.4 Relating Knowledge to Certainty

We use $K_a(\varphi)$ as an abbreviation for $P_a(\varphi) = 1$. This interprets knowledge as certainty and makes $K_a$ behave as an $S5$-operator.

**Example 4** (Uncertainty about $q$-bias)

$$
0 : pq \quad 2 : \overline{pq}
$$

$$
1 : pq \quad 3 : \overline{pq}
$$

$$
l_0 = \{0 : \frac{1}{4}, 1 : \frac{1}{4}, 2 : \frac{1}{4}, 3 : \frac{3}{4} \}
$$

$$
l_1 = \{0 : \frac{1}{2}, 1 : \frac{1}{2}, 2 : \frac{1}{2}, 3 : \frac{3}{4} \}
$$

Figure 1: Example of an epistemic probability model $\mathcal{M}$

As above, this representation can be viewed as shorthand for a model with a single lottery, with worlds

$$\{(0, l_0), (1, l_0), (2, l_0), (3, l_0), (0, l_1), (1, l_1), (2, l_1), (3, l_1)\}.$$

In the model of this example, at world $(0, l_0)$, the probability that $a$ (represented by solid lines) assigns to $p$ is 1, so $K_a p$ is true at $(0, l_0)$. $K_a q$ is false at $(0, l_0)$, for the probability that $a$ assigns to $q$ is less than 1. In fact, we have:

**Proposition 1** Let $\phi$ be a formula of standard epistemic logic. Then $\phi$ is satisfiable in a standard epistemic model iff $tr(\phi)$ is satisfiable in an epistemic lottery model where $tr$ is defined by $tr(K_a \phi) = P_a tr(\phi) = 1$.

**Proof.** We transform a standard epistemic model into an epistemic lottery model by adding a fake single lottery that assigns 1 to all worlds. Conversely, we transform a lottery model into a standard epistemic model by dropping the lotteries. Those transformations preserves the satisfiability via the $tr$.

**Remark 1** Notice that if we define a belief operator $B_a \phi$ by $P_a(\phi) > \alpha$ for some $\alpha \in (\frac{1}{2}, 1)$, the formula $B_a p \land B_a q \land \neg B_a(p \land q)$ is satisfiable. That is, $B_a$ behaves as a non-normal operator and not as a $KD45$ operator. This provides a way out of the so-called lottery paradox [15].

4. EQUIVALENCE OF THE SEMANTICS

In this section, we prove that the semantics given in terms of epistemic lottery models (definition 3) and the semantics given in terms of epistemic single lottery models (definition 4) are equivalent. Epistemic single lottery are easier to manipulate because we only attribute one single value to a world and a value for each agent.

**Proposition 2** Given an epistemic lottery model $\mathcal{M} = (W, V, R, L)$, given a world $w$, there exists an epistemic single lottery model $\mathcal{M}' = (W', V', R', L')$ such that for all formulas $\phi$, $\mathcal{M}, w \models \phi$ iff $\mathcal{M}', w \models \phi$.

Before starting the proof, let us consider the following example. We start with the model $\mathcal{M}$ depicted in Figure 1. In order to get a model with a single lottery, we unravel the model $\mathcal{M}$ and we obtain the infinite epistemic lottery model of Figure 2. The proof formalizes this transformation.

**Proof.** The construction goes as follows. The set of worlds $W'$ is the set of all sequences of the form:

$$w_0 a_0 w_1 a_1 \ldots w_{n-1} a_{n-1} w_n$$

such that, $n \geq 0$, $w_0 = w$, $w_i$ are worlds, $a_i$ are agents, $(w_i, w_{i+1}) \in R_{a_i}$ and $a_i \neq a_{i+1}$.

For any sequence $\overline{s}$, we write $\text{end}(\overline{s})$ for the last world in the sequence that is, $\text{end}(w_0 a_0 w_1 a_1 \ldots w_{n-1} a_{n-1} w_n) = w_n$.

The valuation $V'$ is defined by $V'(\overline{s}) = V(\text{end}(\overline{s}))$.

The relation $R'_{a_i}$ is defined as follows:
implies that these are normalized.

We prove by induction on \( M \) of the Hennessy-Milner Theorem. We say that an epistemic lottery model is image-finite if for all worlds \( w \) in \( M \), and for all agents \( a \), \( R_a(w) \) is finite.

Proposition 4 Let \( M, M' \) be two image-finite models and \( w \) and \( w' \) be respectively two worlds of \( M \) and \( M' \). \( M, w \) and \( M', w' \) are bisimilar if, and only if, \( M, w, M', w' \) satisfy the same formulas.

Proof. We show the right to left direction and for that, we prove that the relation \( \equiv \) of modal equivalence on the two models is itself a bisimulation.

The condition 1 is immediate: if \( w \) and \( w' \) satisfy the same formulas, they satisfy the same atomic propositions. Assume that \( w \equiv w' \) and let \( E \) be a subset of \( R_a(w) \). We will prove condition 2 by arriving at a contradiction by assuming that there is no \( E' \subseteq R'_a(w') \) such that

- for all \( u' \in E' \), there exists \( u \in E \) such that \( u \equiv u' \);
- and \( L_A(E) \leq L_A(E') \).

That is, we assume that for all subsets \( E' \subseteq R'_a(w) \) such that \( L_A(E) \leq L_A(E') \), there exists \( u' \in E' \) such that for all \( u \in E \) we have \( u \not\equiv u' \).

Let \( S' = \{ E'_1, \ldots, E'_n \} \) be an enumeration of subsets \( E' \subseteq R'_a(w) \) such that \( L_A(E) \leq L_A(E') \). For all \( i \in \{ 1, \ldots, n \} \), there exists \( u' \in E'_i \) and a (finite) collection of formulas \( \{ \psi_{u,u'} \}_{u \in E} \) such that for all \( u \in E, M, u = \psi_{u,u} \) and \( M', u' \not\equiv \psi_{u,u} \).

Let \( \varphi = \bigwedge_{i=1,n} \bigvee_{u \in E} \psi_{u,u} \). On the one hand, we have that for all \( u \in E, M, u = \varphi \). Thus, if we pose \( \alpha = L_A(E) \), we have \( M, w \models P_a(\varphi) \geq \alpha \).

On the other hand, for all \( i \in \{ 1, \ldots, n \} \), there exists a world \( u' \in E'_i \) such that \( M', u' = \bigwedge_{u \in E} \psi_{u,u} \). That is, \( M', u' = \bigwedge_{u \in E} \psi_{u,u} \). In particular, the set \( \{ u' \in R_a(w') \} \) is not in \( S' \) and is therefore of probability strictly lower than \( \alpha \). So, \( M, w' \not\models P_a(\varphi) \geq \alpha \).

So \( w \) and \( w' \) do not satisfy the same formulas and there is a contradiction hence condition 2 holds. Condition 3 is symmetrical and may be checked in a similar way.

6. AXIOMATIZATION

Figure 3 shows a complete axiomatization of epistemic probabilistic logic. We show in subsection 6.1 the principles of standard epistemic logic S5 (where \( K_a \varphi \) is replaced by \( P_a \varphi = 1 \)) are derivable from the axiomatization. In subsection 6.2 we adapt the proof of completeness of [8] to our simplified logic.

6.1 Principles of S5 are derivable

Principles of S5 are the following:

- the necessitation rule: if \( \vdash_{S5} \varphi \) then \( \vdash_{S5} K_a \varphi \);
- the K-principle: \( K_a \varphi \land K_a (\varphi \rightarrow \psi) \rightarrow K_a \varphi \);
- the T-axiom or truth axiom: \( K_a \varphi \rightarrow \varphi \);
The necessitation rule for certainty is derivable:

\[ \vdash P_a \varphi = 1 \rightarrow P_a (P_a \varphi = 1) = 1 \quad (4) \]
\[ \vdash P_a \varphi > 0 \rightarrow P_a (P_a \varphi > 0) = 1 \quad (5) \]

They correspond respectively to axiom 4 (positive introspection) and axiom 5 (negative introspection) in standard epistemic logic S5.

### 6.2 Soundness and Completeness

**Theorem 7** The EPL calculus is sound.

**Proof.** All axioms are valid in all EPL models. All rules preserve validity. □

The completeness works as follows. We prove that:

**Proposition 8** If \( \varphi \) is consistent, then \( \varphi \) is satisfiable.

We adapt the proof from [8]. First we construct a canonical epistemic probability model. Contrary to the proof in [8], the epistemic relations are inferred from probabilities.

Let \( SF(\varphi) \) be the set of all subformulas of \( \varphi \) augmented with the negations of subformulas. Let us define the canonical model \( M = (W, V, R, \bot) \). \( W \) is the set of all maximal consistent subsets of \( SF(\varphi) \). \( W \) is not empty because \( \varphi \) is supposed to be consistent.

Valuations are defined as follows: \( V(w) = P \cap w \).

Let \( sat(w) = \{ \psi \mid w \vdash \psi \} \), that is, \( sat(w) \) is the set of formulas that are provable from \( w \). Relations are defined as follows: \( wR_u w \) iff \( sat(w) \cap sat(u) \) contain the same \( a \)-probability formulas.

Now it remains to define \( \bot \). Let us consider an agent \( a \) and an equivalence class \( R_u(w) \) in the canonical model \( M \). All worlds \( u \) of \( R_u(w) \) contain the same \( a \)-probability formulas. In the sequel, we are transforming all the \( a \)-probability formulas in a system of linear inequalities that is consistent.

For all \( w \in W \), we write \( \varphi_u \) the conjunction of all formulas in \( u \).

We have:

- \( \vdash \varphi_u \rightarrow \neg \varphi_u \) if \( u \neq w \) by CPL

Given any formulas \( \psi \) of \( SF(\varphi) \), we have

- \( \vdash \psi \leftrightarrow \bigwedge_{w\in W \mid \psi \in u} \varphi_u \) by CPL

Let \( \psi \) be any formula of \( SF(\varphi) \). By axioms \( \text{ProbaRule} \) and \( \text{ProbaAdditivity} \) and ..., we have:

- \( \vdash P_a(\psi) = \sum_{w\in W \mid \psi \in u} P_a(\varphi_u) \).

Thus, if we take any \( a \)-probability formula \( \psi \), and we replace any term \( P_a(\chi) \) by \( \sum_{w\in W \mid \psi \in u} P_a(\varphi_u) \), we obtain

\[ \sum_{w\in W} c_w P_a(\varphi_u) \geq b \]

where \( c_w, b \in \mathbb{Q} \). Now, when we evaluate the value of \( P_a(\varphi_u) \) in \( w \), we should obtain non-zero if, and only if, \( u \in R_a(w) \). Let us prove it.

- If \( u \in R_a(w) \), we have:
  1. \( \vdash \varphi_u \rightarrow P_a(\varphi_u) > 0 \) by \( \text{ProbaGeq0} \)
  2. \( P_a(\varphi_u) > 0 \) in \( sat(u) \);
  3. \( P_a(\varphi_u) > 0 \) in \( sat(w) \) because \( u \in R_a(w) \).

There, \( P_a(\varphi_u) > 0 \) should be also true in \( w \).
• If \( u \not\in \mathcal{R}_n(w) \), \( a \) and \( w \) differ by at least one \( a \)-probability formula \( \psi \in \mathcal{SF}(\varphi) \) such that \( \psi \in w \) and \( \neg \psi \not\in u \) without loss of generality. We have:

1. \( \vdash \varphi_u \rightarrow \psi \) by \([\text{CPL}]\).
2. \( \vdash \psi \rightarrow \neg \varphi_u \) by \([\text{CPL}]\).
3. \( \vdash \psi \rightarrow \text{P}_a(\psi) = 1 \) by \([\text{CPL}]\) and \([\text{Modus Ponens}]\).
4. \( \vdash \varphi_u \rightarrow \text{P}_a(\psi_u) = 1 \) by \([\text{CPL}]\) and \([\text{Modus Ponens}]\).
5. \( \vdash \varphi_u \rightarrow \text{P}_a(-\varphi_u) = 1 \) by \( \mathbb{Z}_2 \) and 4.
6. \( \vdash \varphi_u \rightarrow \text{P}_a(\varphi_u) = 0 \).

Therefore \( \text{P}_a(\varphi_u) = 0 \) should be true in \( w \).

Thus, \( \psi \) should be equivalent to

\[ \sum_{u \in \mathcal{R}_n(w)} c_u \text{P}_a(\varphi_u) \geq b \]

where \( c_u, b \in \mathbb{Q} \). This yields to the following linear inequations system made up inequations such as \( \sum_{u \in \mathcal{R}_n(w)} c_u x_u \geq b \) when \( \psi \in w \) or \( \sum_{u \in \mathcal{R}_n(w)} c_u x_u < b \) when \( \psi \not\in w \). We add also \( \sum_{u \in \mathcal{R}_n(w)} x_u = 1 \) and \( x_u > 0 \) for all \( u \in \mathcal{R}_n(w) \). The set \( \mathcal{s}at(w) \) is consistent so the above system is re-writing of some inequations that are in \( \mathcal{s}at(w) \) is also consistent and therefore satisfiable [9] [Theorem 2.2]. Let \( (x_u)s \in \mathcal{R}_n(w) \) be a solution. We define \( \Delta_n(u) = x_u * \).

Lemma 9 (truth lemma) For all formulas \( \psi \in \mathcal{SF}(\varphi) \), we have \( M, w \models \psi \iff \psi \in w \).

PROOF. By induction on \( \psi \).

7. MODEL CHECKING

Here is the algorithm for model checking where the input model \( M \) is assumed to be a normalized epistemic lottery model.

**Function** \( \text{mc}(M = (W, V, R, L), \varphi) \)

if \( T[\varphi] \) is defined then

\[ \text{return } T[\varphi]; \]

endIf

match \( \varphi \) do

\[ \text{case } T; \]

\[ T[\varphi] := W; \]

\[ \text{return } T[\varphi]; \]

\[ \text{case } \neg \psi; \]

\[ T[\varphi] := \{ w \in W \mid p \in V(w) \}; \]

\[ \text{return } T[\varphi]; \]

\[ \text{case } \psi_1 \land \psi_2; \]

\[ T[\varphi] := \text{mc}(M, \psi_1) \cap \text{mc}(M, \psi_2); \]

\[ \text{return } T[\varphi]; \]

\[ \text{case } t_u \geq q; \]

\[ T[\varphi] := \{ w \in W \mid \text{get}(M, t_u, w, i) \geq q \}; \]

\[ \text{return } T[\varphi]; \]

endCase

endMatch

endFunction

**Function** \( \text{get}(M, t, w, i) \)

match \( t / d \) do

\[ \text{case } q; \]

\[ \text{return } q; \]

\[ \text{case } q \cdot \text{P}_a(\varphi); \]

\[ \Sigma := \text{mc}(M, \varphi); \]

\[ v := \sum_{u \in \mathcal{X}u \in \mathcal{R}_n(u)} \Delta_n(u) \]

\[ \text{return } q \times v; \]

\[ \text{case } t_1 + t_2; \]

\[ \text{return } \text{get}(M, t_1, w) + \text{get}(M, t_2, w); \]

endCase

endMatch

endFunction

**Theorem 10** A call to \( \text{mc}(M, \varphi) \) returns the set \( \{ w \in W \mid M, w \models \varphi \} \).

PROOF. By induction on \( \varphi \).

**Theorem 11** A call to \( \text{mc}(M, \varphi) \) requires \( O(|\varphi|^2 \times |W|^3) \) elementary operations.

PROOF. A call to \( \text{mc}(M, \varphi) \) calls \( \text{mc}(M, \psi) \) where \( \psi \) is a subformula of \( \varphi \). As the algorithm \( \text{mc}(M, \psi) \) is based on memoization: for a given \( \psi \), the call \( \text{mc}(M, \psi) \) is called at most once. So it is sufficient to compute an upper bound of the number of elementary operations performed in one call \( \text{mc}(M, \psi) \). Then we multiply this upper bound by an upper bound of the number of calls, that is the number of subformulas of \( \varphi \) which is \( O(|\varphi|) \).

8. UPDATES

8.1 Example

Consider the following story. An urn contains a single marble, either white or black. Mr A and Mrs B know this, and they also know that both possibilities are equally likely. Next, Mr A looks in the urn, while Mrs B is watching. Mr A puts another marble in the urn, a white one, and Mrs B sees this. The urn now contains two marbles. Next, Mrs B draws one of the two marbles from the urn. It turns out to be white. What is the probability, according to Mr A, that the other marble is also white? What is the probability, according to Mrs B, that the other marble is also white? (This is a multi-agent variation on a puzzle by Lewis Carroll, see [10].)

Call the first white marble \( p \) and the second one \( q \). We start with a situation where there is nothing in the urn, and both agents know this. Update this with the action of tossing a fair coin and making \( p \) true in case the coin shows heads. It is assumed that the two agents \( a \) and \( b \) see that the action happens, but do not see what the outcome is. The action model for this (solid arrows for \( a \), dashed arrows for \( b \)):

\[ p := \text{T}; \quad p := \text{T}; \]

The update of the initial model looks like this:

\[ 0 : p = \frac{1}{2} \cdot 1 : p = \frac{1}{2} \]

The action where \( a \) takes a look, while \( b \) sees this but does not observe what \( a \) sees:

\[ p : \frac{1}{2} \quad \text{---} \quad 1 : p : \frac{1}{2} \]

The situation after \( a \) has taken a look:

\[ 0 : p = \frac{1}{2} \cdot 1 : p = \frac{1}{2} \]
The action of putting another white marble (represented by $q$) in the urn:

\[ 0 : q := \top \]

The result of updating with this:

\[ pq : \frac{1}{2} \]

Extracting a white marble from the box is represented as either the act of removing $p$ or the act of removing $q$, with neither nor both seeing the difference. The act of removing $p$ (making $p$ false) has as precondition that $p$ is true, the act of removing $q$ has as precondition that $q$ is true.

\[ p, p := \bot : \frac{1}{2} \]
\[ q, q := \bot : \frac{1}{2} \]

The result of updating with this is the following model:

\[ w : q : \frac{1}{3} \]
\[ u : p : \frac{1}{3} \]
\[ v : : \frac{1}{3} \]

We see that in $w$ it holds that $P_0(p \lor q) = 1$, $P_0(p \lor \ell) = \frac{1}{2}$. Same values in $u$, while in $v$ it holds that $P_0(p \lor q) = 0$, $P_0(p \lor \ell) = \frac{2}{3}$.

### 8.2 Definitions

Formally, an action model $E$ for epistemic probability logic is the result of replacing the valuation function in an epistemic lottery model by a pair of functions PRE and POST that assign to every world (or event) a precondition and a postcondition, where the precondition $\varphi$ is a formula of the epistemic probability language, and the postcondition is a finite set of bindings $p := \varphi$, with $p$ in the set of basic proposition letters of the epistemic probability language, and $\varphi$ a formula of the language.

Update is defined as the product construction of $E$, with the extra proviso that $L_{\varphi}(w, e) = L_{\varphi}(w) \times L_{\varphi}(e)$, where $L_{\varphi}$ is a lottery model.

If the initial epistemic lottery model is normalized and the update model is normalized, then the product is also an epistemic lottery model.

We consider a probabilistic version of the language extended with a dynamic epistemic operators logic simplified (DEPL). The truth conditions are defined as follows:

- $M, w \models [E, \psi]$ iff $M, w \models PRE(e)$ implies $M \otimes E, (w, e) \models \psi$.

8.3 Model checking with updates

We study now the model checking in DEPL. We adapt the model checking procedure written in subsection [7]. Now, array $T$ is replaced by $T_M$ where $M$ is the current epistemic lottery model.

The implemented version works as follows:

\[
T_M(\text{PRE}(e)) = \text{mc}(M, \text{PRE}(e));
\]
\[
T_M([E, \psi]) := \begin{cases} w \in W & w \not\in T_M(\text{PRE}(e)) \{ (w, e) \in \text{mc}(M \times E, \psi) \}; \\
\text{return } T_M([E, \psi]);
\end{cases}
\]

This leads to an algorithm which is running in exponential time and that uses an exponential amount of memory. We may write an algorithm that only use a polynomial amount of memory in the size of the initial model and the size of the formula, that is inspired by the algorithm provided in [1]: we browse the product models on the fly. Thus our model checking in the dynamic case is in PSPACE. Nevertheless, the PSPACE-hardness bound for DEL without probability, with $\bigcup$ operator (and where preconditions are all $\top$) shown in [1] does not provide a lower bound because we can not reduce the DEL without probability on models without constraints to the model checking problem in DEPL. Nevertheless the idea of the proof of [1] can be adapted and it provides the following lemma:

**Lemma 12** The model checking problem when the initial models and action models are $S5$-models, when we have the $\cup$-operator in the language and when there are at least two agents is PSPACE-hard. The result holds even if all preconditions of the action models are propositional formulas.

**Proof.** Without loss of generality, we only consider in this proof quantified Boolean formulas of the form $\forall_0 p_1 \exists_0 p_2 \forall_0 p_3 \ldots \forall_0 p_{k-1} \exists_0 p_k \psi(p_1, \ldots, p_k)$, where $\psi(p_1, \ldots, p_k)$ is a Boolean formula over the atomic propositions $p_1, \ldots, p_k$.

The quantified Boolean formula satisfiability problem takes as an input a natural number $k$ and a quantified Boolean formula $\varphi \equiv \forall_0 p_1 \exists_0 p_2 \forall_0 p_3 \ldots \forall_0 p_{k-1} \exists_0 p_k \psi(p_1, \ldots, p_k)$. It returns yes if $\varphi$ is true in quantified Boolean logic.

Let $\varphi$ be such a quantified Boolean formula. We define a pointed epistemic model $M, \varphi'$, $2k$ pointed event models $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{2k}$, a pointed event model $\mathcal{E}_0, \psi'$, and an epistemic formula $\psi'$ that are computable in polynomial time in the size of $\varphi$ such that:

- $\varphi$ is satisfiable in quantified Boolean logic
- $\psi'$ is the formula $\psi'$

where

- $M$ is depicted below:
- $\mathcal{E}_0, \mathcal{E}_i$ is the action model made up of a single event $\psi_i$ with preconditions $\top$.
- $\psi'$ is the formula $\psi'$ where all $p_i$ occurrences are substituted by $(K_0 \ell_0)^0 \ell_0, K_0, K_0, K_0, \ell_0$.

$p_i$ is true is interpreted as the existence of a branch that stop at $\ell_i$ world. Making the product with $\mathcal{E}_i, \psi_i$ will add such a branch in the model whereas making the product with $\mathcal{E}_i, \psi_i$ will leave the epistemic model as it is. The universal and existential choices of values for the $p_i$'s are simulated by the dynamic epistemic operators.

**Proposition 13** Model checking for DEPL with at least two agents and with the $\cup$-operator is PSPACE-complete.

**Proof.** Membership in PSPACE comes from the remark above. We polynomially reduce the model checking of $S5$-models that is PSPACE-hard (Lemma [12] in the model checking of DEPL). To do so, we add to the models 'artificial' lotteries and we recursively replace all subformulas $K_0 \varphi$ by $P_a(\varphi) = 1$. Thus, we obtain the lower-bound.
9. CONNECTIONS, FURTHER WORK
The assumption that agents have a common prior, widely used in epistemic game theory, is not built into our concept of an epistemic probability model. If we want to impose common prior conditions, say for proposition $p$, then a natural way to express this would be by means of:

$$\bigwedge_{a,b \in A} P_a p = P_b p.$$ 

Currently, this is not in our language, but if we allow such expressions, then this formula rules out models like the following:

$$(0 : p : a : \frac{1}{2}, b : \frac{2}{3}) \, 1 : p : a : \frac{1}{2}, b : \frac{1}{3}.$$

This model describes a situation where $a$ and $b$ ‘agree to disagree’ on the probability of $p$. If they are both willing to take bets on the truth of $p$, they are not rational, for then they make themselves vulnerable to a pair of bets that forms a Dutch book [2]. In finite models with a single lottery Dutch books cannot occur.

**Question 1** Can we strengthen the language to allow for a axiom that forces lotteries to be single?

If we want to allow lotteries with unknowns in our models, then the language should be extended with expressions $B_p$ meaning: the (unknown) probability of $p$, and lotteries should allow for factors $B_p$. To handle cases where it is given that no probability distribution for an event exists, we can allow lotteries with unknown factors. A $W$-lottery with unknowns $X \subseteq P$ (or: a $W$-lottery functional over $X$) is a function from $(0..1)^X$ to $W$-lotteries, where $(0..1)$ is the open unit interval $\subseteq Q^+$. Thus, the type of a $W$-lottery with unknowns $X$ is:

$$(X \rightarrow (0..1)) \rightarrow W \rightarrow Q^+.$$ 

Let $B$ be a function that assigns probabilities to the members of $X$, i.e., $B : X \rightarrow (0..1)$. Let $l$ be a normalized $W$-lottery (i.e., a lottery with values summing up to 1 over $W$), and let $V$ be a valuation for $W$. Then $L_{l,V,B}$ is the $W$-lottery given by:

$$L_{l,V,B}(w) = l(w) \times \prod\{B(p) \mid p \in Q, p \in V(w)\} \times \prod\{1 - B(p) \mid p \in P, p \notin V(w)\}.$$ 

Then for all $w \in W$, $L_{l,V,B}(w) \in (0..1) \subseteq Q$, so $L_{l,V,B}$ is a $W$-lottery. The function $B \mapsto L_{l,V,B}$ is a lottery functional.

**Example 5 (Von Neumann’s Trick)** How to obtain fair results from a coin with unknown bias [17]:

Toss the coin twice. If the results match, start over and forget both results. If the results differ, use the first result.

Here is the explanation. Represent the coin as a lottery functional for the set $\{h\}$. Let $B$ assign a probability to $h$. That is, $B_h = b$ is the coin bias. Then the probabilities of the four possible outcomes of Von Neumann’s procedure are represented by the following lottery:

$$\{hh : b^2, ht : b - b^2, th : b - b^2, tt : (1 - b)^2\}.$$ 

This shows that the cases $ht$ and $th$ are equally likely, so interpreting the first as $h$ and the second as $t$ gives indeed a model of a fair coin.

**Example 6 (Model representing a coin with unknown bias)**

$$p : B_p \quad \quad \quad \quad \quad \quad \quad p : 1 - B_p$$

Model checking and model update for epistemic probability logic is implemented in [7]. This allows to solve urn problems in a multi-agent setting by means of epistemic model checking. This extension generates lots of further logical questions. Also, it can serve as a solid basis for the design and analysis of probabilistic protocol languages for epistemic probability updating. Hooking up to more sophisticated model checkers like NuSMV (nuasmv.fbk.eu) is future work. Finally, we would like to further explore the obvious connections with Bayesian learning.

10. REFERENCES