On the Shannon capacity of sums and products of graphs

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Abstract Let $\Theta(G)$ denote the Shannon capacity of a graph G. We give an elementary proof of the equivalence, for any graphs G and H, of the inequalities $\Theta(G \sqcup H) > \Theta(G) + \Theta(H)$ and $\Theta(G \boxtimes H) > \Theta(G)\Theta(H)$. This was shown independently by Wigderson and Zuiddam [2022] using Kadison-Dubois duality and the Axiom of choice.

Keywords: graph, stable set number, Shannon capacity

1. Introduction

Let G be a graph. (All graphs in this paper are undirected and simple.) A stable set in G is a set of pairwise nonadjacent vertices. The stable set number $\alpha(G)$ is the maximum cardinality of a stable set in G.

The sum G + H of graphs G and H is the disjoint union of G and H. Trivially,

(1)
$$\alpha(G+H) = \alpha(G) + \alpha(H).$$

The strong product GH of G and H is the graph with vertex set $V(G) \times V(H)$ where distinct (u, v) and (u', v') in $V(G) \times V(H)$ are adjacent if and only if (i) u and u' are equal or adjacent in G and (ii) v and v' are equal or adjacent in H.

Since sum and strong product are associative, commutative, and distributive (up to isomorphism), this makes the set of graphs to a commutative semiring, with unit the one-vertex graph K_1 . Sum and strong product are often denoted by $G \sqcup H$ and $G \boxtimes H$, but the semiring notation G + H and GH is more efficient here.

As the cartesian product of stable sets in G and H is a stable set in GH we have

(2)
$$\alpha(GH) \ge \alpha(G)\alpha(H),$$

but strict inequality may occur, even if G = H (for instance for $G = H = C_5$, the five-cycle). This made Shannon [1956] define what is now called the *Shannon capacity* $\Theta(G)$ of a graph G:

(3)
$$\Theta(G) := \sup_{k \in \mathbb{N}} \alpha(G^k)^{1/k} = \lim_{k \to \infty} \alpha(G^k)^{1/k}.$$

The second equality in (3) follows from (2) and Fekete's lemma [1923]. (In fact, Shannon introduced $\log \Theta(G)$ as the 'zero-error capacity' of the 'channel' G.)

Inequality (2) implies

(4)
$$\Theta(GH) \ge \Theta(G)\Theta(H).$$

Haemers [1979] (disproving a conjecture of Shannon [1956]) gave examples of graphs G, H with strict inequality in (4). In fact, Haemers showed that the 'Schläfli graph' G satisfies $\Theta(G)\Theta(\overline{G}) < |V(G)| \le \alpha(G\overline{G}) \le \Theta(G\overline{G})$. Here \overline{G} is the graph complementary to G.

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On the other hand, for each graph G and $n \in \mathbb{N}$:

(5)
$$\Theta(G^n) = \Theta(G)^n,$$

as follows directly from definition (3).

The value of $\Theta(C_5)$ was for a long time an open question, until Lovász [1979] introduced the upper bound $\vartheta(G)$ on $\Theta(G)$ yielding $\Theta(C_5) = \sqrt{5}$. Since, as Lovász proved, $\vartheta(GH) = \vartheta(G)\vartheta(H)$ for all G, H, the Haemers examples imply that $\Theta(G) < \vartheta(G)$ may occur.

As for the sum, Shannon showed that for all graphs G and H one has

(6)
$$\Theta(G+H) \ge \Theta(G) + \Theta(H).$$

(For completeness, we give a proof in Section 2 below.) Shannon conjectured that for all G, H equality holds in (6). This was disproved by Alon [1998], by displaying graphs G and H with $\Theta(G + H) > \Theta(G) + \Theta(H)$. In fact, strict inequality holds for any G and H that satisfy $\Theta(GH) > \Theta(G)\Theta(H)$, as follows (using (5) and (6)) from

(7)
$$\Theta(G+H)^{2} = \Theta((G+H)^{2}) = \Theta(G^{2}+2GH+H^{2}) \ge \Theta(G^{2})+2\Theta(GH)+\Theta(H^{2}) = \Theta(G)^{2}+2\Theta(GH)+\Theta(H)^{2} > \Theta(G)^{2}+2\Theta(G)\Theta(H)+\Theta(H)^{2} = (\Theta(G)+\Theta(H))^{2}.$$

So Haemers' counterexamples G, H for products also work for sums.

In this paper we give an elementary proof of the fact that for all G, H:

(8)
$$\Theta(GH) > \Theta(G)\Theta(H) \iff \Theta(G+H) > \Theta(G) + \Theta(H)$$

(see Section 3). This was proved (independently) by Wigderson and Zuiddam [2022], using Strassen's theory of asymptotic spectra (based on Kadison-Dubois duality) and the Axiom of choice.

More strongly, consider any $n \in \mathbb{N}$ and graphs G_1, \ldots, G_n . Then for any polynomial $p \in \mathbb{N}[x_1, \ldots, x_n]$ one has

(9)
$$\Theta(p(G_1,\ldots,G_n)) \ge p(\Theta(G_1),\ldots,\Theta(G_n)).$$

(This follows from (6) and (4).) Now if equality holds in (9) for one polynomial p in which each of the variables x_1, \ldots, x_n occurs, then equality holds in (9) for all polynomials p. For this result of Wigderson and Zuiddam [2022] we also give an elementary proof in Section 4.

2. Shannon's inequality

For self-containedness of this paper, we give a proof of Shannon's inequality:

Theorem 1 (Shannon [1956]). $\Theta(G + H) \ge \Theta(G) + \Theta(H)$.

Proof. For all $n, t \ge 1$, using (1) and (2):

(10)
$$\alpha((G+H)^{n}) = \alpha(\sum_{k=0}^{n} {n \choose k} G^{k} H^{n-k}) = \sum_{k=0}^{n} {n \choose k} \alpha(G^{k} H^{n-k}) \ge \sum_{k=0}^{n} {n \choose k} \alpha(G^{k}) \alpha(H^{n-k}) \ge \sum_{k=0}^{n} {n \choose k} \alpha(G^{t})^{\lfloor k/t \rfloor} \alpha(H^{t})^{\lfloor (n-k)/t \rfloor} \ge$$

$$\sum_{k=0}^{n} {n \choose k} \alpha(G^{t})^{k/t} \alpha(G^{t})^{-1} \alpha(H^{t})^{(n-k)/t} \alpha(H^{t})^{-1} = (\alpha(G^{t})^{1/t} + \alpha(H^{t})^{1/t})^{n} \alpha(G^{t})^{-1} \alpha(H^{t})^{-1}.$$

So for each $t \ge 1$:

(11)
$$\Theta(G+H) = \sup_{n \in \mathbb{N}} \alpha((G+H)^n)^{1/n} \ge \sup_{n \in \mathbb{N}} (\alpha(G^t)^{1/t} + \alpha(H^t)^{1/t}) \alpha(G^t)^{-1/n} \alpha(H^t)^{-1/n} = \alpha(G^t)^{1/t} + \alpha(H^t)^{1/t}.$$

So letting $t \to \infty$ gives the theorem.

(Note that this proof also applies if α is replaced by any superadditive and supermultiplicative graph function.)

3. Equivalence of $\Theta(GH) > \Theta(G)\Theta(H)$ and $\Theta(G+H) > \Theta(G) + \Theta(H)$

Theorem 2. $\Theta(GH) > \Theta(G)\Theta(H)$ if and only if $\Theta(G+H) > \Theta(G) + \Theta(H)$.

Proof. Necessity follows from (7). To see sufficiency, assume $\Theta(GH) \leq \Theta(G)\Theta(H)$. Then for all $i, j \in \mathbb{N}$, using (4) and (5):

(12)
$$\Theta(G^{i}H^{j})\Theta(G)^{j}\Theta(H)^{i} = \Theta(G^{i}H^{j})\Theta(G^{j})\Theta(H^{i}) \le \Theta((GH)^{i+j}) = \Theta(GH)^{i+j} \le \Theta(G)^{i+j}\Theta(H)^{i+j}.$$

So $\Theta(G^i H^j) \leq \Theta(G)^i \Theta(H)^j$. Hence for each *n*, using (1):

(13)
$$\alpha((G+H)^n) = \alpha(\sum_{k=0}^n \binom{n}{k} G^k H^{n-k}) = \sum_{k=0}^n \binom{n}{k} \alpha(G^k H^{n-k}) \leq \sum_{k=0}^n \binom{n}{k} \Theta(G^k H^{n-k}) \leq \sum_{k=0}^n \binom{n}{k} \Theta(G)^k \Theta(H)^{n-k} = (\Theta(G) + \Theta(H))^n.$$

Taking *n*-th roots and letting $n \to \infty$ gives $\Theta(G + H) \le \Theta(G) + \Theta(H)$.

4. Extension to polynomials

We also give an elementary proof of the following extension of Theorem 2, that was shown by Wigderson and Zuiddam [2022] using Kadison-Dubois duality and the Axiom of choice.

For given graphs G_1, \ldots, G_n , define

(14)
$$\mathcal{P} = \{ p \in \mathbb{N}[x_1, \dots, x_n] \mid \Theta(p(G_1, \dots, G_n)) = p(\Theta(G_1), \dots, \Theta(G_n)) \}.$$

Theorem 3. Let G_1, \ldots, G_n be graphs with at least one vertex. Then $\mathcal{P} = \mathbb{N}[x_1, \ldots, x_n]$ if and only if \mathcal{P} contains a polynomial in which all variables x_1, \ldots, x_n occur.

Proof. Necessity being trivial, we prove sufficiency. Let $\underline{G} := (G_1, \ldots, G_n)$ and $\Theta(\underline{G}) := (\Theta(G_1), \ldots, \Theta(G_n))$. So $p(\Theta(\underline{G})) \leq \Theta(p(\underline{G}))$ for any polynomial $p \in \mathbb{N}[x_1, \ldots, x_n]$.

We first show that for $p, q \in \mathbb{N}[x_1, \ldots, x_n]$:

(15) if
$$p + q \in \mathcal{P}$$
, then $p \in \mathcal{P}$.

Indeed,

(16)
$$\Theta((p+q)(\underline{G})) = (p+q)(\Theta(\underline{G})) = p(\Theta(\underline{G})) + q(\Theta(\underline{G})) \le \Theta(p(\underline{G})) + \Theta(q(\underline{G})) \le \Theta(p(\underline{G}) + q(\underline{G})) = \Theta((p+q)(\underline{G})).$$

Hence we have equality throughout, implying $\Theta(p(\underline{G})) = p(\Theta(\underline{G}))$. This proves (15). Similarly,

(17) if
$$pq \in \mathcal{P}$$
 and $q \neq 0$, then $p \in \mathcal{P}$.

Indeed,

$$\begin{array}{ll} (18) \qquad \qquad \Theta((pq)(\underline{G})) = (pq)(\Theta(\underline{G})) = p(\Theta(\underline{G}))q(\Theta(\underline{G})) \leq \Theta(p(\underline{G}))\Theta(q(\underline{G})) \leq \\ \qquad \qquad \Theta(p(\underline{G})q(\underline{G})) = \Theta((pq)(\underline{G})). \end{array}$$

Hence we have equality throughout, implying $\Theta(p(\underline{G})) = p(\Theta(\underline{G}))$. This proves (17). Moreover, for $p \in \mathbb{N}[x_1, \ldots, x_n]$ and $k \in \mathbb{N}$,

(19) if
$$p \in \mathcal{P}$$
 then $p^k \in \mathcal{P}$.

Indeed, if $p \in \mathcal{P}$, then

(20)
$$\Theta(p^k(\underline{G})) = \Theta(p(\underline{G})^k) = (\Theta(p(\underline{G})))^k = (p(\Theta(\underline{G})))^k = (p^k(\Theta(\underline{G}))),$$

proving (19).

Now let $p \in \mathcal{P}$ with each x_1, \ldots, x_n occurring in p. Then for some $k \in \mathbb{N}$, p^k contains as term a monomial q in which each variable occurs at least once. As $p^k \in \mathcal{P}$ by (19), we know by (15) that $q \in \mathcal{P}$. Now for each monomial μ in $\mathbb{N}[x_1, \ldots, x_n]$ there exists a large enough N such that μ is a divisor of q^N . So by (14), each monomial belongs to \mathcal{P} .

Now consider any polynomial $r = q_1 + \cdots + q_t$ in $\mathbb{N}[x_1, \ldots, x_n]$, where each q_i is a monomial. Then for each $i_1, \ldots, i_t \in \mathbb{N}$, $\mu := \prod_{j=1}^t q_j^{i_j}$ is a monomial, implying

(21)
$$\Theta(\prod_{j=1}^{t} q_j(\underline{G})^{i_j}) = \Theta(\mu(\underline{G})) = \mu(\Theta(\underline{G})) = \prod_{j=1}^{t} q_j(\Theta(\underline{G}))^{i_j}.$$

This implies, for each $k \in \mathbb{N}$, using the additivity ((1)) of the function α :

$$(22) \qquad \alpha(r(\underline{G})^{k}) = \alpha((\sum_{j=1}^{t} q_{j}(\underline{G}))^{k}) = \alpha(\sum_{\substack{i_{1},\dots,i_{t} \in \mathbb{N} \\ i_{1}+\dots+i_{t}=k}} {\binom{k}{i_{1},\dots,i_{t}}} \prod_{j=1}^{t} q_{j}(\underline{G})^{i_{j}}) = \sum_{\substack{i_{1},\dots,i_{t} \in \mathbb{N} \\ i_{1}+\dots+i_{t}=k}} {\binom{k}{i_{1},\dots,i_{t}}} \alpha(\prod_{j=1}^{t} q_{j}(\underline{G})^{i_{j}}) \leq \sum_{\substack{i_{1},\dots,i_{n} \in \mathbb{N} \\ i_{1}+\dots+i_{t}=k}} {\binom{k}{i_{1},\dots,i_{t}}} \Theta(\prod_{j=1}^{t} q_{j}(\underline{G})^{i_{j}}) =$$

$$\sum_{\substack{i_1,\dots,i_n\in\mathbb{N}\\i_1+\dots+i_t=k}} {k \choose {i_1,\dots,i_t}} \prod_{j=1}^t q_j(\Theta(\underline{G}))^{i_j} = \left(\sum_{j=1}^t q_j(\Theta(\underline{G}))\right)^k = (r(\Theta(\underline{G})))^k.$$

Taking k-th roots and letting $k \to \infty$ we obtain $\Theta(r(\underline{G})) \leq r(\Theta(\underline{G}))$. So $r \in \mathcal{P}$.

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