# On the Shannon capacity of sums and products of graphs 

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#### Abstract

Let $\Theta(G)$ denote the Shannon capacity of a graph $G$. We give an elementary proof of the equivalence, for any graphs $G$ and $H$, of the inequalities $\Theta(G \sqcup H)>\Theta(G)+\Theta(H)$ and $\Theta(G \boxtimes H)>\Theta(G) \Theta(H)$. This was shown independently by Wigderson and Zuiddam [2022] using Kadison-Dubois duality and the Axiom of choice.


Keywords: graph, stable set number, Shannon capacity

## 1. Introduction

Let $G$ be a graph. (All graphs in this paper are undirected and simple.) A stable set in $G$ is a set of pairwise nonadjacent vertices. The stable set number $\alpha(G)$ is the maximum cardinality of a stable set in $G$.

The sum $G+H$ of graphs $G$ and $H$ is the disjoint union of $G$ and $H$. Trivially,

$$
\begin{equation*}
\alpha(G+H)=\alpha(G)+\alpha(H) \tag{1}
\end{equation*}
$$

The strong product $G H$ of $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ where distinct $(u, v)$ and $\left(u^{\prime}, v^{\prime}\right)$ in $V(G) \times V(H)$ are adjacent if and only if (i) $u$ and $u^{\prime}$ are equal or adjacent in $G$ and (ii) $v$ and $v^{\prime}$ are equal or adjacent in $H$.

Since sum and strong product are associative, commutative, and distributive (up to isomorphism), this makes the set of graphs to a commutative semiring, with unit the onevertex graph $K_{1}$. Sum and strong product are often denoted by $G \sqcup H$ and $G \boxtimes H$, but the semiring notation $G+H$ and $G H$ is more efficient here.

As the cartesian product of stable sets in $G$ and $H$ is a stable set in $G H$ we have

$$
\begin{equation*}
\alpha(G H) \geq \alpha(G) \alpha(H) \tag{2}
\end{equation*}
$$

but strict inequality may occur, even if $G=H$ (for instance for $G=H=C_{5}$, the five-cycle). This made Shannon [1956] define what is now called the Shannon capacity $\Theta(G)$ of a graph G:

$$
\begin{equation*}
\Theta(G):=\sup _{k \in \mathbb{N}} \alpha\left(G^{k}\right)^{1 / k}=\lim _{k \rightarrow \infty} \alpha\left(G^{k}\right)^{1 / k} \tag{3}
\end{equation*}
$$

The second equality in (3) follows from (2) and Fekete's lemma [1923]. (In fact, Shannon introduced $\log \Theta(G)$ as the 'zero-error capacity' of the 'channel' $G$.)

Inequality (2) implies

$$
\begin{equation*}
\Theta(G H) \geq \Theta(G) \Theta(H) \tag{4}
\end{equation*}
$$

Haemers [1979] (disproving a conjecture of Shannon [1956]) gave examples of graphs $G, H$ with strict inequality in (4). In fact, Haemers showed that the 'Schläfli graph' $G$ satisfies $\Theta(G) \Theta(\bar{G})<|V(G)| \leq \alpha(G \bar{G}) \leq \Theta(G \bar{G})$. Here $\bar{G}$ is the graph complementary to $G$.

[^0]On the other hand, for each graph $G$ and $n \in \mathbb{N}$ :

$$
\begin{equation*}
\Theta\left(G^{n}\right)=\Theta(G)^{n} \tag{5}
\end{equation*}
$$

as follows directly from definition (3).
The value of $\Theta\left(C_{5}\right)$ was for a long time an open question, until Lovász [1979] introduced the upper bound $\vartheta(G)$ on $\Theta(G)$ yielding $\Theta\left(C_{5}\right)=\sqrt{5}$. Since, as Lovász proved, $\vartheta(G H)=$ $\vartheta(G) \vartheta(H)$ for all $G, H$, the Haemers examples imply that $\Theta(G)<\vartheta(G)$ may occur.

As for the sum, Shannon showed that for all graphs $G$ and $H$ one has

$$
\begin{equation*}
\Theta(G+H) \geq \Theta(G)+\Theta(H) \tag{6}
\end{equation*}
$$

(For completeness, we give a proof in Section 2 below.) Shannon conjectured that for all $G, H$ equality holds in (6). This was disproved by Alon [1998], by displaying graphs $G$ and $H$ with $\Theta(G+H)>\Theta(G)+\Theta(H)$. In fact, strict inequality holds for any $G$ and $H$ that satisfy $\Theta(G H)>\Theta(G) \Theta(H)$, as follows (using (5) and (6)) from

$$
\begin{align*}
& \Theta(G+H)^{2}=\Theta\left((G+H)^{2}\right)=\Theta\left(G^{2}+2 G H+H^{2}\right) \geq \Theta\left(G^{2}\right)+2 \Theta(G H)+\Theta\left(H^{2}\right)=  \tag{7}\\
& \Theta(G)^{2}+2 \Theta(G H)+\Theta(H)^{2}>\Theta(G)^{2}+2 \Theta(G) \Theta(H)+\Theta(H)^{2}=(\Theta(G)+\Theta(H))^{2}
\end{align*}
$$

So Haemers' counterexamples $G, H$ for products also work for sums.
In this paper we give an elementary proof of the fact that for all $G, H$ :

$$
\begin{equation*}
\Theta(G H)>\Theta(G) \Theta(H) \quad \Longleftrightarrow \quad \Theta(G+H)>\Theta(G)+\Theta(H) \tag{8}
\end{equation*}
$$

(see Section 3). This was proved (independently) by Wigderson and Zuiddam [2022], using Strassen's theory of asymptotic spectra (based on Kadison-Dubois duality) and the Axiom of choice.

More strongly, consider any $n \in \mathbb{N}$ and graphs $G_{1}, \ldots, G_{n}$. Then for any polynomial $p \in \mathbb{N}\left[x_{1}, \ldots, x_{n}\right]$ one has

$$
\begin{equation*}
\Theta\left(p\left(G_{1}, \ldots, G_{n}\right)\right) \geq p\left(\Theta\left(G_{1}\right), \ldots, \Theta\left(G_{n}\right)\right) \tag{9}
\end{equation*}
$$

(This follows from (6) and (4).) Now if equality holds in (9) for one polynomial $p$ in which each of the variables $x_{1}, \ldots, x_{n}$ occurs, then equality holds in (9) for all polynomials $p$. For this result of Wigderson and Zuiddam [2022] we also give an elementary proof in Section 4.

## 2. Shannon's inequality

For self-containedness of this paper, we give a proof of Shannon's inequality:
Theorem 1 (Shannon [1956]). $\Theta(G+H) \geq \Theta(G)+\Theta(H)$.
Proof. For all $n, t \geq 1$, using (1) and (2):

$$
\begin{align*}
& \alpha\left((G+H)^{n}\right)=\alpha\left(\sum_{k=0}^{n}\binom{n}{k} G^{k} H^{n-k}\right)=\sum_{k=0}^{n}\binom{n}{k} \alpha\left(G^{k} H^{n-k}\right) \geq  \tag{10}\\
& \sum_{k=0}^{n}\binom{n}{k} \alpha\left(G^{k}\right) \alpha\left(H^{n-k}\right) \geq \sum_{k=0}^{n}\binom{n}{k} \alpha\left(G^{t}\right)^{\lfloor k / t\rfloor} \alpha\left(H^{t}\right)^{\lfloor(n-k) / t\rfloor} \geq
\end{align*}
$$

$$
\begin{aligned}
& \sum_{k=0}^{n}\binom{n}{k} \alpha\left(G^{t}\right)^{k / t} \alpha\left(G^{t}\right)^{-1} \alpha\left(H^{t}\right)^{(n-k) / t} \alpha\left(H^{t}\right)^{-1}= \\
& \left(\alpha\left(G^{t}\right)^{1 / t}+\alpha\left(H^{t}\right)^{1 / t}\right)^{n} \alpha\left(G^{t}\right)^{-1} \alpha\left(H^{t}\right)^{-1}
\end{aligned}
$$

So for each $t \geq 1$ :

$$
\begin{align*}
& \Theta(G+H)=\sup _{n \in \mathbb{N}} \alpha\left((G+H)^{n}\right)^{1 / n} \geq  \tag{11}\\
& \sup _{n \in \mathbb{N}}\left(\alpha\left(G^{t}\right)^{1 / t}+\alpha\left(H^{t}\right)^{1 / t}\right) \alpha\left(G^{t}\right)^{-1 / n} \alpha\left(H^{t}\right)^{-1 / n}=\alpha\left(G^{t}\right)^{1 / t}+\alpha\left(H^{t}\right)^{1 / t}
\end{align*}
$$

So letting $t \rightarrow \infty$ gives the theorem.
(Note that this proof also applies if $\alpha$ is replaced by any superadditive and supermultiplicative graph function.)

## 3. Equivalence of $\Theta(\boldsymbol{G H})>\Theta(\boldsymbol{H}) \Theta(\boldsymbol{H})$ and $\Theta(\boldsymbol{G}+\boldsymbol{H})>$ $\Theta(G)+\Theta(H)$

Theorem 2. $\Theta(G H)>\Theta(G) \Theta(H)$ if and only if $\Theta(G+H)>\Theta(G)+\Theta(H)$.
Proof. Necessity follows from (7). To see sufficiency, assume $\Theta(G H) \leq \Theta(G) \Theta(H)$. Then for all $i, j \in \mathbb{N}$, using (4) and (5):

$$
\begin{align*}
& \Theta\left(G^{i} H^{j}\right) \Theta(G)^{j} \Theta(H)^{i}=\Theta\left(G^{i} H^{j}\right) \Theta\left(G^{j}\right) \Theta\left(H^{i}\right) \leq \Theta\left((G H)^{i+j}\right)=\Theta(G H)^{i+j} \leq  \tag{12}\\
& \Theta(G)^{i+j} \Theta(H)^{i+j}
\end{align*}
$$

So $\Theta\left(G^{i} H^{j}\right) \leq \Theta(G)^{i} \Theta(H)^{j}$. Hence for each $n$, using (1):

$$
\begin{align*}
& \alpha\left((G+H)^{n}\right)=\alpha\left(\sum_{k=0}^{n}\binom{n}{k} G^{k} H^{n-k}\right)=\sum_{k=0}^{n}\binom{n}{k} \alpha\left(G^{k} H^{n-k}\right) \leq  \tag{13}\\
& \sum_{k=0}^{n}\binom{n}{k} \Theta\left(G^{k} H^{n-k}\right) \leq \sum_{k=0}^{n}\binom{n}{k} \Theta(G)^{k} \Theta(H)^{n-k}=(\Theta(G)+\Theta(H))^{n}
\end{align*}
$$

Taking $n$-th roots and letting $n \rightarrow \infty$ gives $\Theta(G+H) \leq \Theta(G)+\Theta(H)$.

## 4. Extension to polynomials

We also give an elementary proof of the following extension of Theorem 2, that was shown by Wigderson and Zuiddam [2022] using Kadison-Dubois duality and the Axiom of choice.

For given graphs $G_{1}, \ldots, G_{n}$, define

$$
\begin{equation*}
\mathcal{P}=\left\{p \in \mathbb{N}\left[x_{1}, \ldots, x_{n}\right] \mid \Theta\left(p\left(G_{1}, \ldots, G_{n}\right)\right)=p\left(\Theta\left(G_{1}\right), \ldots, \Theta\left(G_{n}\right)\right)\right\} \tag{14}
\end{equation*}
$$

Theorem 3. Let $G_{1}, \ldots, G_{n}$ be graphs with at least one vertex. Then $\mathcal{P}=\mathbb{N}\left[x_{1}, \ldots, x_{n}\right]$ if and only if $\mathcal{P}$ contains a polynomial in which all variables $x_{1}, \ldots, x_{n}$ occur.

Proof. Necessity being trivial, we prove sufficiency. Let $\underline{G}:=\left(G_{1}, \ldots, G_{n}\right)$ and $\Theta(\underline{G}):=$ $\left(\Theta\left(G_{1}\right), \ldots, \Theta\left(G_{n}\right)\right)$. So $p(\Theta(\underline{G})) \leq \Theta(p(\underline{G}))$ for any polynomial $p \in \mathbb{N}\left[x_{1}, \ldots, x_{n}\right]$.

We first show that for $p, q \in \mathbb{N}\left[x_{1}, \ldots, x_{n}\right]$ :

$$
\begin{equation*}
\text { if } p+q \in \mathcal{P} \text {, then } p \in \mathcal{P} \text {. } \tag{15}
\end{equation*}
$$

Indeed,

$$
\begin{align*}
& \Theta((p+q)(\underline{G}))=(p+q)(\Theta(\underline{G}))=p(\Theta(\underline{G}))+q(\Theta(\underline{G})) \leq \Theta(p(\underline{G}))+\Theta(q(\underline{G})) \leq  \tag{16}\\
& \Theta(p(\underline{G})+q(\underline{G}))=\Theta((p+q)(\underline{G})) .
\end{align*}
$$

Hence we have equality throughout, implying $\Theta(p(\underline{G}))=p(\Theta(\underline{G}))$. This proves (15).
Similarly,

$$
\begin{equation*}
\text { if } p q \in \mathcal{P} \text { and } q \neq 0 \text {, then } p \in \mathcal{P} . \tag{17}
\end{equation*}
$$

Indeed,

$$
\begin{align*}
& \Theta((p q)(\underline{G}))=(p q)(\Theta(\underline{G}))=p(\Theta(\underline{G})) q(\Theta(\underline{G})) \leq \Theta(p(\underline{G})) \Theta(q(\underline{G})) \leq  \tag{18}\\
& \Theta(p(\underline{G}) q(\underline{G}))=\Theta((p q)(\underline{G})) .
\end{align*}
$$

Hence we have equality throughout, implying $\Theta(p(\underline{G}))=p(\Theta(\underline{G}))$. This proves (17).
Moreover, for $p \in \mathbb{N}\left[x_{1}, \ldots, x_{n}\right]$ and $k \in \mathbb{N}$,

$$
\begin{equation*}
\text { if } p \in \mathcal{P} \text { then } p^{k} \in \mathcal{P} \text {. } \tag{19}
\end{equation*}
$$

Indeed, if $p \in \mathcal{P}$, then

$$
\begin{equation*}
\Theta\left(p^{k}(\underline{G})\right)=\Theta\left(p(\underline{G})^{k}\right)=(\Theta(p(\underline{G})))^{k}=(p(\Theta(\underline{G})))^{k}=\left(p^{k}(\Theta(\underline{G}))\right), \tag{20}
\end{equation*}
$$

proving (19).
Now let $p \in \mathcal{P}$ with each $x_{1}, \ldots, x_{n}$ occurring in $p$. Then for some $k \in \mathbb{N}, p^{k}$ contains as term a monomial $q$ in which each variable occurs at least once. As $p^{k} \in \mathcal{P}$ by (19), we know by (15) that $q \in \mathcal{P}$. Now for each monomial $\mu$ in $\mathbb{N}\left[x_{1}, \ldots, x_{n}\right]$ there exists a large enough $N$ such that $\mu$ is a divisor of $q^{N}$. So by (14), each monomial belongs to $\mathcal{P}$.

Now consider any polynomial $r=q_{1}+\cdots+q_{t}$ in $\mathbb{N}\left[x_{1}, \ldots, x_{n}\right]$, where each $q_{i}$ is a monomial. Then for each $i_{1}, \ldots, i_{t} \in \mathbb{N}, \mu:=\prod_{j=1}^{t} q_{j}^{i_{j}}$ is a monomial, implying

$$
\begin{equation*}
\Theta\left(\prod_{j=1}^{t} q_{j}(\underline{G})^{i_{j}}\right)=\Theta(\mu(\underline{G}))=\mu(\Theta(\underline{G}))=\prod_{j=1}^{t} q_{j}(\Theta(\underline{G}))^{i_{j}} . \tag{21}
\end{equation*}
$$

This implies, for each $k \in \mathbb{N}$, using the additivity ((1)) of the function $\alpha$ :

$$
\begin{align*}
& \alpha\left(r(\underline{G})^{k}\right)=\alpha\left(\left(\sum_{j=1}^{t} q_{j}(\underline{G})\right)^{k}\right)=\alpha\left(\sum_{\substack{i_{1}, \ldots, i_{t} \in \mathbb{N} \\
i_{1}+\ldots+i_{t}=k}}\binom{k}{i_{1}, \ldots, i_{t}} \prod_{j=1}^{t} q_{j}(\underline{G})^{i_{j}}\right)=  \tag{22}\\
& \sum_{\substack{i_{1}, \ldots, i_{t} \in \mathbb{N} \\
i_{1}+\ldots+i_{t}=k}}\binom{k}{i_{1}, \ldots, i_{t}} \alpha\left(\prod_{j=1}^{t} q_{j}(\underline{G})^{i_{j}}\right) \leq \sum_{\substack{i_{1}, \ldots, i_{n} \in \mathbb{N} \\
i_{1}+\ldots+i_{t}=k}}\binom{k}{i_{1}, \ldots, i_{t}} \Theta\left(\prod_{j=1}^{t} q_{j}(\underline{G})^{i_{j}}\right)=
\end{align*}
$$

$$
\sum_{\substack{i_{1}, \ldots, i_{n} \in \mathbb{N} \\ i_{1}+\cdots+i_{t}=k}}\binom{k}{i_{1}, \ldots, i_{t}} \prod_{j=1}^{t} q_{j}(\Theta(\underline{G}))^{i_{j}}=\left(\sum_{j=1}^{t} q_{j}(\Theta(\underline{G}))\right)^{k}=(r(\Theta(\underline{G})))^{k} .
$$

Taking $k$-th roots and letting $k \rightarrow \infty$ we obtain $\Theta(r(\underline{G})) \leq r(\Theta(\underline{G}))$. So $r \in \mathcal{P}$.

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## References

[1998] N. Alon, The Shannon capacity of a union, Combinatorica 18 (1998) 301-310.
[1923] M. Fekete, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, Mathematische Zeitschrift 17 (1923) 228-249.
[1979] W. Haemers, On some problems of Lovász concerning the Shannon capacity of a graph, IEEE Transactions on Information Theory IT-25 (1979) 231-232.
[1979] L. Lovász, On the Shannon capacity of a graph, IEEE Transactions on Information Theory IT-25 (1979) 1-7.
[1956] C.E. Shannon, The zero error capacity of a noisy channel, IRE Transactions on Information Theory IT-2 (1956) 8-19.
[2022] A. Wigderson and J. Zuiddam, Asymptotic spectra: theory, applications and extensions, manuscript, 2022.


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