
ELSEVIER
Volume 117 Issue 2 February 2010 ISSN 0097-3165
Journal of Combinatorial Theory
Series A
Editor-in-Chief: S. Ole Warnaar
Advisory Board: Hélène Barcelo, Charles J. Colbourn, Bernard Leclerc, Dhruv Mubayi, Gilles Schaeffer, Volkmar Welker
Editorial Board:
George E. Andrews
Mireille Bousquet-Mélou
Peter Frankl
Adriano M. Garsia
Basil Gordon
Ronald L. Graham
Jonathan I. Hall
Alexander A. Ivanov
David Jackson
Gil Kalai
William M. Kantor
Christian Krattenthaler
Andrew Odlyzko
Vera Pless
Arun Ram
Bruce Rothschild
Vera T. Sós
Richard Stanley
Bernd Sturmfels
Van H. Vu
Richard M. Wilson
Günter M. Ziegler

Journal of Combinatorial Theory, Series A 117 (2010) 216–222

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Note

Dual graph homomorphism functions

László Lovász [a,1], Alexander Schrijver [b]

[a] Institute of Mathematics, Eötvös Loránd University, Budapest, Hungary
[b] CWI and University of Amsterdam, Amsterdam, The Netherlands

A R T I C L E I N F O

Article history:
Received 3 July 2008
Available online 3 June 2009

Keywords:
Graph homomorphism
Positive semidefinite
Graph algebra
Quantum graph
Graph parameter

A B S T R A C T

For any two graphs F and G, let $\hom(F,G)$ denote the number of homomorphisms $F \to G$, that is, adjacency preserving maps $V(F) \to V(G)$ (graphs may have loops but no multiple edges). We characterize graph parameters f for which there exists a graph F such that $f(G) = \hom(F,G)$ for each graph G.
The result may be considered as a certain dual of a characterization of graph parameters of the form $\hom(.,H)$, given by Freedman, Lovász and Schrijver [M. Freedman, L. Lovász, A. Schrijver, Reflection positivity, rank connectivity, and homomorphisms of graphs, J. Amer. Math. Soc. 20 (2007) 37–51]. The conditions amount to the multiplicativity of f and to the positive semidefiniteness of certain matrices $N(f,k)$.

Contents

E-mail addresses: lovasz@cs.elte.hu (L. Lovász), lex@cwi.nl (A. Schrijver).
[1] Research sponsored by OTKA Grant No. 67867.

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2009.04.006

1. Introduction

In this paper, the graphs we consider are finite, undirected and have no parallel edges, but they may have loops. A *graph parameter* is a real valued function defined on graphs, invariant under isomorphisms.

For two graphs F and G, let $\hom(F,G)$ denote the number of homomorphisms $F \to G$, that is, adjacency preserving maps $V(F) \to V(G)$.

The definition can be extended to weighted graphs (when the nodes and edges of G have real weights). In [1] multigraph parameters of the form $\hom(\cdot,G)$ were characterized, where G is a weighted graph. Several variants of this result have been obtained, characterizing graph parameters $\hom(\cdot,G)$ where all nodeweights of G are 1 [6], such graph parameters defined on simple graphs where G is weighted [5], and also when G is an infinite object called a "graphon" [4], and graph parameters defined in a dual setting where the roles of nodes and edges are interchanged [7]. These characterizations involve certain infinite matrices, called *connection matrices*, which are required to be positive semidefinite. Often they also have to satisfy a condition on their rank.

The goal of this paper is to study the dual question, and characterize graph parameters of the form $\hom(F,\cdot)$, where F is an (unweighted) graph. It turns out that reversing the arrows in the category of graphs gives the right hints for the condition, and the characterization involves the dually defined connection matrices.

It is unclear what a weighted version of this dual theorem might mean. That is, if F is a weighted graph (with real weights on its edges), then there is no clear definition for $\hom(F,G)$. On the other hand, the authors recently have obtained a more general theorem in terms of categories that combines both the primal and dual (unweighted) cases — see [3].

For two graphs G and H, the *product* $G \times H$ is the graph with node set $V(G) \times V(H)$, two nodes (u,v) and (u',v') being adjacent if and only if $uu' \in E(G)$ and $vv' \in E(H)$. Then

$$\hom(F, G \times H) = \hom(F,G)\hom(F,H). \tag{1}$$

An *S-colored* graph is a pair (G,ϕ), where G is a graph and $\phi : V(G) \to S$, where S is a finite set. We call (G,ϕ) *colored* if it is S-colored for some S. We call $\phi(v)$ the *color* of v.

The *product* $(G,\phi) \times (H,\psi)$ of two colored graphs is the colored graph (J,ϑ), where J is the subgraph of $G \times H$ induced by the set of nodes (u,v) with $\phi(u) = \psi(v)$, and where $\vartheta(u,v) := \phi(u)$ $(= \psi(v))$.

For two colored graphs (F,ϕ) and (G,ψ), a homomorphism $h : V(F) \to V(G)$ is *color-preserving* if $\phi = \psi h$. Let $\hom_c((F,\phi),(G,\psi))$ denote the number of color-preserving homomorphisms $F \to G$.

It is easy to see that for any three colored graphs F, G and H,

$$\hom_c(F, G \times H) = \hom_c(F,G)\hom_c(F,H).$$

(Eq. (1) is the special case where all nodes have color 1.) Moreover, if both G and H are S-colored, then for any uncolored graph F,

$$\hom(F, G \times H) = \sum_{\phi : V(F) \to S} \hom_c\big((F,\phi),G\big)\hom_c\big((F,\phi),H\big). \tag{2}$$

Here $\hom(F,(G,\phi)) := \hom(F,G)$ for any colored graph (G,ϕ). More generally, we extend any graph parameter f to colored graphs by defining $f(G,\phi) := f(G)$ for any colored graph (G,ϕ).

For every graph parameter f and $k \geqslant 1$, we define an (infinite) matrix $N(f,k)$ as follows. The rows and columns are indexed by $[k]$-colored graphs (where $[k] = \{1,\dots,k\}$), and the entry in row G and column H (where G and H are $[k]$-colored graphs) is $f(G \times H)$.

Eq. (2) implies that for any graph F and any $k \geqslant 1$, the matrix $N(f,k)$ belonging to $f = \hom(F,\cdot)$ is positive semidefinite. Moreover, if $f = \hom(F,\cdot)$, then f is *multiplicative*, that is, $f(\widetilde{K}_1) = 1$ and $f(G \times H) = f(G)f(H)$ for any two (uncolored) graphs. Here $\widetilde{K}_n$ denotes the complete graph with n vertices and with a loop attached at each vertex. The main result of this paper is that these properties characterize such graph parameters:

Theorem 1. *Let f be a graph parameter. Then $f = \hom(F, \cdot)$ for some graph F if and only if f is multiplicative and, for each $k \geqslant 1$, the matrix $N(f, k)$ is positive semidefinite.*

The proof will require a development of an algebraic machinery, similar to the one used in [1] (but the details are different).

2. The algebras $\mathcal{A}$ and $\mathcal{A}_S$

The colored graphs form a semigroup under multiplication $\times$. Let $\mathcal{G}$ denote its semigroup algebra (the elements of $\mathcal{G}$ are formal linear combinations of colored graphs with real coefficients, also called *quantum colored graphs*). Let $\mathcal{G}_S$ denote the semigroup algebra of the semigroup of S-colored graphs. For each S, the graph $\widetilde{K}_S$ is the unit element of $\mathcal{G}_S$, where $\widetilde{K}_S$ is the S-colored graph whose underlying graph is the complete graph on S, with a loop at each vertex, and where vertex $s \in S$ has color s. The function f can be extended linearly to $\mathcal{G}$ and $\mathcal{G}_S$.

By the positive semidefiniteness of $N(f, k)$, the function

$$\langle G, H\rangle := f(G \times H)$$

defines a semidefinite (but not necessarily definite) inner product on $\mathcal{G}$. The set

$$I := \bigl\{g \in \mathcal{G} \bigm| \langle g, g\rangle = 0\bigr\} = \bigl\{g \in \mathcal{G} \bigm| \langle g, x\rangle = 0 \text{ for all } x \in \mathcal{G}\bigr\}$$

is an ideal in $\mathcal{G}$. (This follows essentially from the fact that $\langle G \times H, L\rangle = \langle G, H \times L\rangle$ for all graphs G, H, L.) Hence the quotient $\mathcal{A} = \mathcal{G}/I$ is a commutative algebra with (definite) inner product. We denote multiplication in $\mathcal{A}$ by concatenation. Since $g \in I$ implies $f(g) = 0$, we can define f on $\mathcal{A}$ by $f(g + I) := f(g)$ for $g \in \mathcal{G}$.

It is easy to check that

$$I \cap \mathcal{G}_S = \bigl\{g \in \mathcal{G}_S \bigm| \langle g, x\rangle = 0 \text{ for all } x \in \mathcal{G}_S\bigr\}$$

is an ideal in $\mathcal{G}_S$, and hence the quotient $\mathcal{A}_S = \mathcal{G}_S/(I \cap \mathcal{G}_S)$ is also a commutative algebra with a (definite) inner product. This algebra can be identified with $\mathcal{G}_S/I$ in a natural way.

Note that $1_S := \widetilde{K}_S + I$ is the unit element of $\mathcal{A}_S$ and that $\mathcal{A}_S$ is an ideal in $\mathcal{A}$. Moreover, $\mathcal{A}_S \subseteq \mathcal{A}_T$ if $S \subseteq T$. In fact, the stronger relation $\mathcal{A}_S \cap \mathcal{A}_T = \mathcal{A}_S\mathcal{A}_T = \mathcal{A}_{S\cap T}$ holds. To see this, we show that

$$\mathcal{A}_S \cap \mathcal{A}_T \subseteq \mathcal{A}_S\mathcal{A}_T \subseteq \mathcal{A}_{S\cap T} \subseteq \mathcal{A}_S \cap \mathcal{A}_T.$$

Indeed, if $x \in \mathcal{A}_S \cap \mathcal{A}_T$ then $x = x1_T \in \mathcal{A}_S\mathcal{A}_T$, which proves the first inclusion. If $g \in \mathcal{G}_S$ and $h \in \mathcal{G}_T$, then $(g + I)(h + I) = gh + I \in \mathcal{A}_{S\cap T}$, which proves the second. The third inclusion is trivial.

3. Finite-dimensionality of $\mathcal{A}_S$

Proposition 2. *For each S, $\mathcal{A}_S$ has finite dimension, and* $\dim(\mathcal{A}_S) \leqslant f(\widetilde{K}_S)$.

Proof. Choose elements $e_1, \dots, e_n \in \mathcal{G}_S$ with $\langle e_i, e_j\rangle = \delta_{i,j}$ for $i, j = 1, \dots, n$. We show $n \leqslant f(\widetilde{K}_S)$, which proves the proposition.

For S-colored graphs (G, ϕ) and (H, ψ), let $(G, \phi) \cdot (H, \psi)$ be the $S \times S$-colored graph $(G \times H, \phi \times \psi)$, where $G \times H$ is the product of G and H as uncolored graphs. This extends bilinearly to $\mathcal{G}_S \times \mathcal{G}_S \to \mathcal{G}_{S\times S}$. Let K be the $S \times S$-colored graph whose underlying graph is the complete graph on S, with a loop at each vertex, and where any vertex $s \in S$ has color (s, s). Define the quantum $S \times S$-colored graph x by

$$x := K - \sum_{i=1}^{n} e_i \cdot e_i.$$

We evaluate $\langle x, x\rangle$. First,

$$\langle e_i \cdot e_i, e_j \cdot e_j\rangle = \langle e_i, e_j\rangle^2 = \delta_{i,j}$$

for all $i, j = 1, \dots, n$. Here we use that for any S-colored graphs G, G', H, H', $(G \cdot H) \times (G' \cdot H') = (G \times G') \cdot (H \times H')$ and $f(G \cdot H) = f(G) f(H)$. Moreover we have

$$\langle e_i \cdot e_i, K\rangle = \langle e_i, e_i\rangle = 1$$

for all $i = 1, \dots, n$. Finally, $\langle K, K\rangle = f(\widetilde{K}_S)$. Concluding,

$$\langle x, x\rangle = f(\widetilde{K}_S) - 2n + n = f(\widetilde{K}_S) - n.$$

Since $\langle x, x\rangle \geqslant 0$, this proves $n \leqslant f(\widetilde{K}_S)$. $\square$

As the inner product $\langle\cdot,\cdot\rangle$ satisfies $\langle xy, z\rangle = \langle x, yz\rangle$ for all $x, y, z \in \mathcal{A}_S$, $\mathcal{A}_S$ has a unique orthogonal basis $\mathcal{M}_S$ consisting of idempotents, called the *basic idempotents* of $\mathcal{A}_S$. Every idempotent in $\mathcal{A}_S$ is the sum of a subset of $\mathcal{M}_S$, and in particular

$$1_S = \sum_{p \in \mathcal{M}_S} p. \tag{3}$$

For every nonzero idempotent p we have $f(p) = f(p^2) = \langle p, p\rangle > 0$.

4. Maps between algebras with different color sets

Let S and T be finite subsets of $\mathbb{Z}$, and let $\alpha : S \to T$. We define a linear function $\check{\alpha} : \mathcal{G}_S \to \mathcal{G}_T$ by

$$\check{\alpha}(G, \phi) := (G, \alpha\phi)$$

for any S-colored graph (G, ϕ). We define another linear map $\hat{\alpha} : \mathcal{G}_T \to \mathcal{G}_S$ as follows. Let (G, ϕ) be a T-colored graph. For any node v of G, split v into $|\alpha^{-1}(\phi(v))|$ copies, adjacent to any copy of any neighbor of v in G. Give these copies of v distinct colors from $\alpha^{-1}(\phi(v))$, to get the colored graph $\hat{\alpha}(G, \phi)$.

It is easy to see that the map $\hat{\alpha}$ is an algebra homomorphism, while in general the map $\check{\alpha}$ is not. On the other hand, $\check{\alpha}$ is an isomorphism of the underlying uncolored graphs, but in general $\hat{\alpha}$ is not.

For any T-colored graph G and any S-colored graph H, we have

$$\check{\alpha}\big(\hat{\alpha}(G) \times H\big) = G \times \check{\alpha}(H),$$

which implies that the underlying uncolored graphs of $\hat{\alpha}(G) \times H$ and $G \times \check{\alpha}(H)$ are the same. Then $g \in I$ implies $\check{\alpha}(g) \in I$ for any $g \in \mathcal{G}_S$, since $\langle \check{\alpha}(g), \check{\alpha}(g)\rangle = \langle g, \hat{\alpha}\check{\alpha}(g)\rangle = 0$. Hence $\check{\alpha}$ quotients to a linear function $\mathcal{A}_S \to \mathcal{A}_T$. Similarly, $g \in I$ implies $\hat{\alpha}(g) \in I$ for any $g \in \mathcal{G}_T$, hence $\hat{\alpha}$ quotients to an algebra homomorphism $\mathcal{A}_T \to \mathcal{A}_S$. We abuse notation and denote these induced maps also by $\check{\alpha}$ and $\hat{\alpha}$.

Then

$$\check{\alpha}\big(\hat{\alpha}(x) y\big) = x \check{\alpha}(y)$$

and hence

$$\big\langle \hat{\alpha}(x), y\big\rangle = \big\langle x, \check{\alpha}(y)\big\rangle \tag{4}$$

for all $x \in \mathcal{A}_T$ and $y \in \mathcal{A}_S$.

It is easy to see that if $\alpha : S \to T$ is surjective, then $\check{\alpha} : \mathcal{G}_S \to \mathcal{G}_T$ is surjective and so is the map $\mathcal{A}_S \to \mathcal{A}_T$ it induces. On the other hand, if again $\alpha : S \to T$ is surjective, then $\hat{\alpha} : \mathcal{G}_T \to \mathcal{G}_S$ is injective, and so is the map $\mathcal{A}_T \to \mathcal{A}_S$ it induces.

Since $\hat{\alpha}$ is an algebra homomorphism, $\hat{\alpha}(p)$ is an idempotent in $\mathcal{A}_S$ for any idempotent $p \in \mathcal{A}_T$, and $\hat{\alpha}(1_T) = 1_S$. So (3) implies that

$$\sum_{p \in \mathcal{M}_T} \hat{\alpha}(p) = \hat{\alpha}(1_T) = 1_S = \sum_{q \in M_S} q. \tag{5}$$

Define for any $p \in \mathcal{M}_T$ and $\alpha : S \to T$:

$$\mathcal{M}_{\alpha,p} := \bigl\{ q \in \mathcal{M}_S \bigm| \hat{\alpha}(p) q = q \bigr\}.$$

By (5),

$$\hat{\alpha}(p) = \sum_{q \in \mathcal{M}_{\alpha,p}} q.$$

This implies that if α is surjective, then $\mathcal{M}_{\alpha,p} \neq \emptyset$.

Proposition 3. *Let $p \in \mathcal{M}_T$, $\alpha : S \to T$, and $q \in \mathcal{M}_{\alpha,p}$. Then*

$$\check{\alpha}(q) = \frac{f(q)}{f(p)} p.$$

Proof. If $p' \in \mathcal{M}_T \setminus \{p\}$, then

$$\bigl\langle \check{\alpha}(q), p' \bigr\rangle = \bigl\langle q, \hat{\alpha}(p') \bigr\rangle = 0 = \left\langle \frac{f(q)}{f(p)} p, p' \right\rangle,$$

since $\langle p, p' \rangle = 0$. Moreover,

$$\bigl\langle \check{\alpha}(q), p \bigr\rangle = \bigl\langle q, \hat{\alpha}(p) \bigr\rangle = f\bigl(\hat{\alpha}(p) q\bigr) = f(q) = \left\langle \frac{f(q)}{f(p)} p, p \right\rangle,$$

since $\langle p, p \rangle = f(p)$. □

5. Maximal basic idempotents

For each $x \in \mathcal{A}$, let $C(x)$ be the minimal set S of colors for which $x \in \mathcal{A}_S$. This is well defined because $\mathcal{A}_S \cap \mathcal{A}_T = \mathcal{A}_{S \cap T}$.

Proposition 4. $|C(p)| \leqslant \log_2 f(\widetilde{K}_2)$ *for each basic idempotent p.*

Proof. Let $S := C(p)$. Suppose $|S| > \log_2 f(\widetilde{K}_2)$. Then for t large enough

$$\binom{2^t}{|S|} > \bigl(2^t\bigr)^{\log_2 f(\widetilde{K}_2)} = f(\widetilde{K}_2)^t = f(\widetilde{K}_{2^t}).$$

Now choose T with $|T| = 2^t$. Then $\mathcal{A}_T$ has at least $\binom{2^t}{|S|}$ basic idempotents, since for each subset S' of T of size $|S|$ we can choose a bijection $\alpha : S \to S'$. Then $\check{\alpha}(p)$ belongs to $\mathcal{A}_T$, and they are all distinct.

So $\dim(\mathcal{A}_T) \geqslant \binom{2^t}{|S|} > f(\widetilde{K}_{2^t})$, contradicting Proposition 2. □

This proposition implies that we can choose a basic idempotent p with $|C(p)|$ maximal, which we fix from now on. Define $S := C(p)$.

Proposition 5. *Let $\alpha : T \to S$ be surjective. Then*

$$\hat{\alpha}(p) = \sum_{\substack{\beta:S\to T\\ \alpha\beta=\mathrm{id}_S}} \check{\beta}(p).$$

Note that the maps β in the summation are necessarily injections.

Proof of Proposition 5. Consider any $q \in \mathcal{M}_{\alpha,p}$. By Proposition 3, $\check{\alpha}(q)$ is a nonzero multiple of p. This implies $C(p) = C(\check{\alpha}(q)) \subseteq \alpha(C(q))$. So $|C(q)| \geqslant |C(p)|$, hence by the maximality of $|C(p)|$, $|C(q)| = |C(p)|$. So $\alpha|_{C(q)}$ is a bijection between $C(q)$ and $C(p)$. Setting $\beta = (\alpha|_{C(q)})^{-1}$, we get $q = \check{\beta}(p)$. By symmetry, $\check{\phi}(p)$ occurs in the sum (1) for every injective $\phi : S \to T$ such that $\alpha\phi = \mathrm{id}_S$. □

Proposition 6. *For any finite set T,*

$$\sum_{\alpha:S\to T} \check{\alpha}(p) = f(p)1_T. \tag{6}$$

Proof. Let σ and τ be the projections of $S \times T$ on S and on T, respectively. Then for any S-colored graph G and any T-colored graph H one has that $\hat{\sigma}(G) \times \hat{\tau}(H)$ is, as uncolored graph, equal to the product of the underlying uncolored graphs of G and H. Hence, since f is multiplicative,

$$f\bigl(\hat{\sigma}(G) \times \hat{\tau}(H)\bigr) = f(G)f(H). \tag{7}$$

Now for each $\alpha : S \to T$, there is a unique $\beta : S \to S \times T$ with $\sigma\beta = \mathrm{id}_S$ and $\tau\beta = \alpha$. Hence, with Proposition 5,

$$\sum_{\alpha:S\to T} \check{\alpha}(p) = \sum_{\substack{\beta:S\to S\times T\\ \sigma\beta=\mathrm{id}_S}} \check{\tau}\check{\beta}(p) = \check{\tau} \sum_{\substack{\beta:S\to S\times T\\ \sigma\beta=\mathrm{id}_S}} \check{\beta}(p) = \check{\tau}\hat{\sigma}(p).$$

So for any $x \in \mathcal{A}_T$, with (4) and (7),

$$\bigl\langle \check{\tau}\hat{\sigma}(p), x\bigr\rangle = \bigl\langle \hat{\sigma}(p), \hat{\tau}(x)\bigr\rangle = f\bigl(\hat{\sigma}(p)\hat{\tau}(x)\bigr) = f(p)f(x) = \bigl\langle f(p)1_T, x\bigr\rangle.$$

This implies that $\check{\tau}\hat{\sigma}(p) = f(p)1_T$. □

Remark 7. While it follows from the theorem, it may be worthwhile to point out that the maximal basic idempotent p is unique up to renaming the colors, and all other basic idempotents arise from it by merging and renaming colors. Indeed, we know by Proposition 3 that every term in (6) is a positive multiple of a basic idempotent in $\mathcal{A}_T$, and so it follows that every basic idempotent in $\mathcal{A}_T$ is a positive multiple of $\check{\alpha}(p)$ for an appropriate map α. In particular, if p' is another basic idempotent with $|C(p')| = |C(p)|$, then it follows that $p' = \check{\alpha}(p)$ for some bijective map $\alpha : C(p) \to C(p')$.

6. Möbius transforms

For any colored graph H, define the quantum graph $\mu(H)$ (the *Möbius transform*) by

$$\mu(H) := \sum_{Y\subseteq E(H)} (-1)^{|Y|}(H - Y).$$

We call a colored graph G *flat*, if $V(G) = T$, and the color of node t is t. For any T-colored graph G and any finite set S, define

$$\lambda_S(G) := \sum_{\alpha:S\to T} \hat{\alpha}(G).$$

Proposition 8. *Let F and G be flat colored graphs, and $S := V(F)$, $T := V(G)$. Then*

$$\lambda_S(G) \times \mu(F) = \hom(F, G)\mu(F). \tag{8}$$

Proof. Consider any map $\alpha : S \to T$. If α is a homomorphism $F \to G$, then $\hat{\alpha}(G) \times \mu(F)$ is equal to $\mu(F)$. If α is not a homomorphism $F \to G$, then $\hat{\alpha}(G) \times \mu(F) = 0$, since F contains edges that are not represented in $\hat{\alpha}(G) \times F$. □

7. Completing the proof

Since $\widetilde{K}_S = \sum_F \mu(F)$, where F ranges over all flat S-colored graphs, and since $\widetilde{K}_S p = p$, there exists a flat S-colored graph F with $\mu(F)p \neq 0$. (Here we denote the image in $\mathcal{A}$ of any element g of $\mathcal{G}$ just by g.) We prove that $f = \hom(F, \cdot)$.

Choose a flat T-colored graph G. As p is a basic idempotent, $\mu(F)p = \gamma p$ for some real $\gamma \neq 0$. So p is in the ideal generated by $\mu(F)$. Hence, by (8), $\lambda_S(G)p = \hom(F, G)p$. Then by (6) and (4):

$$f(p)f(G) = f(p)\langle G, 1_T\rangle = \sum_{\alpha:S\to T} \big\langle G, \check{\alpha}(p)\big\rangle = \sum_{\alpha:S\to T} \big\langle \hat{\alpha}(G), p\big\rangle$$
$$= \big\langle \lambda_S(G), p\big\rangle = f\big(\lambda_S(G)p\big) = \hom(F, G) f(p).$$

Since $f(p) \neq 0$, this gives $f = \hom(F, \cdot)$.

8. Concluding remarks

For a fixed finite set S of colors, colored graphs can be thought of as arrows $G \to \widetilde{K}_S$ in the category of graph homomorphisms. The product of two colored graphs is pullback of the corresponding pair of maps. The setup in [1,6] can be described by reversing the arrows. This raises the possibility that there is a common generalization in terms of categories, which is handled in [3].

The methods from [1] have been applied in extremal graph theory and elsewhere (see [2] for a survey). Are there similar applications of the methods used in this paper?

References

[1] M. Freedman, L. Lovász, A. Schrijver, Reflection positivity, rank connectivity, and homomorphisms of graphs, J. Amer. Math. Soc. 20 (2007) 37–51.
[2] L. Lovász, Very large graphs, in: Current Developments in Mathematics, 2008, in press.
[3] L. Lovász, A. Schrijver, Semidefinite functions on categories, Electron. J. Combin., in press http://www.cwi.nl/~lex/files/category-semidef.pdf.
[4] L. Lovász, B. Szegedy, Limits of dense graph sequences, J. Combin. Theory Ser. B 96 (2006) 933–957.
[5] L. Lovász, B. Szegedy, The moment problem for 2-variable functions and reflection positive graph parameters, manuscript, http://www.cs.elte.hu/~lovasz/moment.pdf.
[6] A. Schrijver, Graph invariants in the spin model, J. Combin. Theory Ser. B 99 (2009) 502–511.
[7] B. Szegedy, Edge coloring models and reflection positivity, J. Amer. Math. Soc. 20 (2007) 969–988.