III. Disjoint paths

1. Shortest paths

Let $D = (V, A)$ be a directed graph, and let $s, t \in V$.[1] A *path* is a sequence $P = (v_0, a_1, v_1, \ldots, a_m, v_m)$ where a_i is an arc from v_{i-1} to v_i for $i = 1, \ldots, m$ and where $v_0, \ldots, v_m$ all are different. The path P is called an $s-t$ *path* if $v_0 = s$ and $v_m = t$. The *length* of P is m. Here m is allowed to be 0. The *distance* from s to t is the minimum length of any $s-t$ path. (If no $s-t$ path exists, we set the distance from s to t equal to ∞.) A shortest $s-t$ path can easily be found by breadth-first search.

There is a trivial min-max relation characterizing the minimum length of an $s-t$ path. To this end, call a subset C of A an $s-t$ *cut* if $C = \delta^{\text{out}}(U)$ for some subset U of V satisfying $s \in U$ and $t \notin U$.[2] Throughout, *disjoint* means *pairwise disjoint*. Then the following was observed by Robacker [8]:

Theorem 1. *The minimum length of an $s-t$ path is equal to the maximum number of disjoint $s-t$ cuts.*

Proof. Trivially, the minimum is at least the maximum, since each $s-t$ path intersects each $s-t$ cut in an arc. To see equality, let d be the distance from s to t, and let U_i be the set of vertices at distance less than i from s, for $i = 1, \ldots, d$. Taking $C_i := \delta^{\text{out}}(U_i)$, we obtain disjoint $s-t$ cuts $C_1, \ldots, C_d$. ∎

2. Length functions

This can be generalized to the case where arcs have a certain 'length'. Let $l : A \to \mathbb{R}_+$, called a *length function*. For any path $P = (v_0, a_1, v_1, \ldots, a_m, v_m)$, let $l(P)$ be the length of P. That is:

$$(1) \qquad l(P) := \sum_{i=1}^{m} l(a_i).$$

Now the *distance* from s to t (with respect to l) is equal to the minimum length of any $s-t$ path. If no $s-t$ path exists, the distance is ∞.

Then a weighted version of Theorem 1 is as follows:

Theorem 2. *Let $D = (V, A)$ be a directed graph, let $s, t \in V$, and let $l : A \to \mathbb{Z}_+$. Then the minimum length of an $s-t$ path is equal to the maximum number k of $s-t$ cuts $C_1, \ldots, C_k$ (repetition allowed) such that each arc a is in at most $l(a)$ of the cuts C_i.*

[1] A *directed graph* or *digraph* is a pair (V, A), where V is a finite set and $A \subseteq V \times V$. The elements of A are called the *arcs* of D. If $a = (u, v)$, then u is called the *tail* of a and v is called the *head* of a.

[2] $\delta^{\text{out}}(U)$ and $\delta^{\text{in}}(U)$ denote the sets of arcs leaving and entering U, respectively.

Proof. Again, the minimum is not smaller than the maximum, since if P is any $s-t$ path and $C_1, \ldots, C_k$ is any collection as described in the theorem:[3]

$$(2) \qquad l(P) = \sum_{a \in AP} l(a) \geq \sum_{a \in AP} (\text{ number of } i \text{ with } a \in C_i) = \sum_{i=1}^{k} |C_i \cap AP| \geq \sum_{i=1}^{k} 1 = k.$$

To see equality, let d be the distance from s to t, and let U_i be the set of vertices at distance less than i from s, for $i = 1, \ldots, d$. Taking $C_i := \delta^{\text{out}}(U_i)$, we obtain a collection $C_1, \ldots, C_d$ as required. ∎

3. Menger's theorem

In this section we study the maximum number k of disjoint paths in a graph connecting two vertices, or two sets of vertices.

Let $D = (V, A)$ be a directed graph and let S and T be subsets of V. A path is called an $S-T$ *path* if it runs from a vertex in S to a vertex in T.

Menger [7] gave a min-max theorem for the maximum number of disjoint $S-T$ paths. We follow the proof given by Göring [6].

Call a set of paths *vertex-disjoint* if no two of them have vertices in common. (Hence they also have no arcs in common.) A set C of vertices is called $S-T$ *disconnecting* if C intersects each $S-T$ path (C may intersect $S \cup T$).

Theorem 3 (Menger's theorem (directed vertex-disjoint version))**.** *Let $D = (V, A)$ be a digraph and let $S, T \subseteq V$. Then the maximum number of vertex-disjoint $S-T$ paths is equal to the minimum size of an $S-T$ disconnecting vertex set.*

Proof. Obviously, the maximum does not exceed the minimum. Equality is shown by induction on $|A|$, the case $A = \emptyset$ being trivial.

Let k be the minimum size of an $S-T$ disconnecting vertex set. Choose $a = (u, v) \in A$. Let $D' := (V, A \setminus \{a\})$. If each $S-T$ disconnecting vertex set in D' has size at least k, then inductively there exist k vertex-disjoint $S-T$ paths in D', hence in D.

So we can assume that D' has an $S-T$ disconnecting vertex set C of size $\leq k-1$. Then $C \cup \{u\}$ and $C \cup \{v\}$ are $S-T$ disconnecting vertex sets of D of size k.

Now each $S - (C \cup \{u\})$ disconnecting vertex set B of D' has size at least k, as it is $S-T$ disconnecting in D. Indeed, each $S-T$ path P in D intersects $C \cup \{u\}$, and hence P contains an $S - (C \cup \{u\})$ path in D'. So P intersects B.

So by induction, D' contains k disjoint $S - (C \cup \{u\})$ paths. Similarly, D' contains k disjoint $(C \cup \{v\}) - T$ paths. Any path in the first collection intersects any path in the second collection only in C, since otherwise D' contains an $S-T$ path avoiding C.

Hence, as $|C| = k-1$, we can pairwise concatenate these paths to obtain disjoint $S-T$ paths, inserting arc a between the path ending at u and the path starting at v. ∎

A consequence of this theorem is a variant on *internally vertex-disjoint* $s-t$ paths, that

[3] AP denotes the set of arcs traversed by P.

is, $s - t$ paths no two of which have a vertex in common except for s and t. A set U of vertices is called an $s - t$ *vertex-cut* if $s, t \notin U$ and each $s - t$ path intersects U.

Corollary 3a (Menger's theorem (directed internally vertex-disjoint version))**.** *Let $D = (V, A)$ be a digraph and let s and t be two nonadjacent vertices of D. Then the maximum number of internally vertex-disjoint $s - t$ paths is equal to the minimum size of an $s - t$ vertex-cut.*

Proof. Let $D' := D - s - t$ and let S and T be the sets of outneighbours of s and of inneighbours of t, respectively. Then Theorem 3 applied to D', S, T gives the corollary. ∎

In turn, Theorem 3 follows from Corollary 3a by adding two new vertices s and t and arcs (s, v) for all $v \in S$ and (v, t) for all $v \in T$.

Also an arc-disjoint version can be derived, where paths are *arc-disjoint* if they have no arc in common. Recall that a set C of arcs is an $s - t$ *cut* if $C = \delta^{\text{out}}(U)$ for some subset U of V with $s \in U$ and $t \notin U$.

Corollary 3b (Menger's theorem (directed arc-disjoint version))**.** *Let $D = (V, A)$ be a digraph and let $s, t \in V$. Then the maximum number of arc-disjoint $s - t$ paths is equal to the minimum size of an $s - t$ cut.*

Proof. Let $L(D)$ be the line digraph of D.[4] Let $S := \delta^{\text{out}}_A(s)$ and $T := \delta^{\text{in}}_A(t)$. Then Theorem 3 for $L(D), S, T$ implies the corollary. Note that a minimum-size set of arcs intersecting each $s - t$ path necessarily is an $s - t$ cut. ∎

The internally vertex-disjoint version of Menger's theorem can be derived in turn from the arc-disjoint version: make a digraph D' as follows from D: replace any vertex v by two vertices v', v'' and make an arc (v', v''); moreover, replace each arc (u, v) by (u'', v'). Then Corollary 3b for D', s'', t' gives Corollary 3a for D, s, t.

Similar theorems hold for *undirected* graphs. They can be derived from the directed case by replacing each undirected edge uw by two opposite arcs (u, w) and (w, u).

Exercises

3.1. Derive Kőnig's matching theorem from Theorem 3.

3.2. Let $D = (V, A)$ be a directed graph and let $s, t_1, \ldots, t_k$ be vertices of D. Prove that there exist arc-disjoint paths $P_1, \ldots, P_k$ such that P_i is an $s - t_i$ path ($i = 1, \ldots, k$) if and only if for each $U \subseteq V$ with $s \in U$ one has

(3) $$|\delta^{\text{out}}(U)| \geq |\{i \mid t_i \notin U\}|.$$

3.3. Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a finite set. Show that $\mathcal{A}$ and $\mathcal{B}$ have a common SDR if and only if for all $I, J \subseteq \{1, \ldots, n\}$ one has

(4) $$\Big|\bigcup_{i \in I} A_i \cap \bigcup_{j \in J} B_j\Big| \geq |I| + |J| - n.$$

[4]The *line digraph* of a digraph $D = (V, A)$ is the digraph with vertex set A and arcs set $\{(a, a') \mid a, a' \in A, \text{head}(a) = \text{tail}(a')\}$.

4. Flows in networks

Other consequences of Menger's theorem concern 'flows in networks'. Let $D = (V, A)$ be a directed graph and let $s, t \in V$. A function $f : A \to \mathbb{R}$ is called an $s-t$ *flow* if:[5]

$$\text{(5)} \qquad \begin{array}{llcll} \text{(i)} & f(a) & \geq & 0 & \text{for each } a \in A; \\ \text{(ii)} & \sum_{a \in \delta^{\text{in}}(v)} f(a) & = & \sum_{a \in \delta^{\text{out}}(v)} f(a) & \text{for each } v \in V \setminus \{s,t\}. \end{array}$$

Condition (5)(ii) is called the *flow conservation law*: the amount of flow entering a vertex $v \neq s, t$ should be equal to the amount of flow leaving v.

The *value* of an $s-t$ flow f is, by definition:

$$\text{(6)} \qquad \text{value}(f) := \sum_{a \in \delta^{\text{out}}(s)} f(a) - \sum_{a \in \delta^{\text{in}}(s)} f(a).$$

So the value is the net amount of flow leaving s. It can be shown that it is equal to the net amount of flow entering t.

Let $c : A \to \mathbb{R}_+$, called a *capacity function*. We say that a flow f is *under* c (or *subject to* c) if

$$\text{(7)} \qquad f(a) \leq c(a) \text{ for each } a \in A;$$

that is, if $f \leq c$. The *maximum flow problem* now is to find an $s-t$ flow under c, of maximum value.

To formulate a min-max relation, define the *capacity* of a cut $\delta^{\text{out}}(U)$ by:

$$\text{(8)} \qquad c(\delta^{\text{out}}(U)) := \sum_{a \in \delta^{\text{out}}(U)} c(a).$$

Then:

Proposition 1. *For every $s-t$ flow f under c and every $s-t$ cut $\delta^{\text{out}}(U)$ one has:*

$$\text{(9)} \qquad \text{value}(f) \leq c(\delta^{\text{out}}(U)).$$

Proof.

$$\text{(10)} \qquad \begin{aligned} \text{value}(f) &= \sum_{a \in \delta^{\text{out}}(s)} f(a) - \sum_{a \in \delta^{\text{in}}(s)} f(a) \\ &= \sum_{a \in \delta^{\text{out}}(s)} f(a) - \sum_{a \in \delta^{\text{in}}(s)} f(a) + \sum_{v \in U \setminus \{s\}} \Big(\sum_{a \in \delta^{\text{out}}(v)} f(a) - \sum_{a \in \delta^{\text{in}}(v)} f(a) \Big) \end{aligned}$$

[5] $\delta^{\text{out}}(v)$ and $\delta^{\text{in}}(v)$ denote the sets of arcs leaving v and entering v, respectively.

$$= \sum_{v\in U}\Big(\sum_{a\in\delta^{\text{out}}(v)} f(a) - \sum_{a\in\delta^{\text{in}}(v)} f(a)\Big) = \sum_{a\in\delta^{\text{out}}(U)} f(a) - \sum_{a\in\delta^{\text{in}}(U)} f(a)$$
$$\overset{\star}{\leq} \sum_{a\in\delta^{\text{out}}(U)} f(a) \overset{\star\star}{\leq} \sum_{a\in\delta^{\text{out}}(U)} c(a) = c(\delta^{\text{out}}(U)).$$
∎

It is convenient to note the following:

(11) $$\text{equality holds in (9)} \iff \begin{array}{l}\forall a \in \delta^{\text{in}}(U) : f(a) = 0 \text{ and}\\ \forall a \in \delta^{\text{out}}(U) : f(a) = c(a).\end{array}$$

This follows directly from the inequalities $\star$ and $\star\star$ in (10).

Now from Menger's theorem one can derive that equality can be attained in (9), which is a theorem of Ford and Fulkerson [4]:

Theorem 4 (max-flow min-cut theorem)**.** *For any directed graph $D = (V, A)$, $s, t \in V$, and $c : A \to \mathbb{R}_+$, the maximum value of an $s-t$ flow under c is equal to the minimum capacity of an $s-t$ cut. In formula:*

(12) $$\max_{\substack{f\ s-t\text{-flow}\\ f\leq c}} \text{value}(f) = \min_{\delta^{\text{out}}(U)\ s-t\text{-cut}} c(\delta^{\text{out}}(U)).$$

Proof. If c is integer-valued, the corollary follows from Menger's theorem by replacing each arc a by $c(a)$ parallel arcs. If c is rational-valued, there exists a natural number N such that $Nc(a)$ is integer for each $a \in A$. This resetting multiplies both the maximum and the minimum by N. So the equality follows from the case where c is integer-valued.

If c is real-valued, we can derive the corollary from the case where c is rational-valued, by continuity and compactness arguments, as follows. Suppose that

(13) $$\max_{\substack{f\ s-t\text{ flow}\\ f\leq c}} \text{value}(f) < \min_{\delta^{\text{out}}(U)\ s-t\text{ cut}} c(\delta^{\text{out}}(U)).$$

(The maximum exists, as the set of $s-t$ flows f with $f \leq c$ is compact.)

Then we can choose a rational-valued $c' \leq c$ close enough to c such that

(14) $$\max_{\substack{f\ s-t\text{ flow}\\ f\leq c}} \text{value}(f) < \min_{\delta^{\text{out}}(U)\ s-t\text{ cut}} c'(\delta^{\text{out}}(U)).$$

So

(15) $$\max_{\substack{f\ s-t\text{ flow}\\ f\leq c'}} \text{value}(f) \leq \max_{\substack{f\ s-t\text{ flow}\\ f\leq c}} \text{value}(f) < \min_{\delta^{\text{out}}(U)\ s-t\text{ cut}} c'(\delta^{\text{out}}(U)).$$

This contradicts the above, as c' is rational. ∎

Moreover, one has (Dantzig [1]):

Corollary 4a (Integrity theorem)**.** *If c is integer-valued, there exists an integer-valued maximum-value flow $f \leq c$.*

Proof. Directly from Menger's theorem. ∎

Exercises

4.1. Let $D = (V, A)$ be a directed graph and let $s, t \in V$. Let $f : A \to \mathbb{R}_+$ be an $s-t$ flow of value β. Show that there exists an $s-t$ flow $f' : A \to \mathbb{Z}_+$ of value $\lceil \beta \rceil$ such that $\lfloor f(a) \rfloor \leq f'(a) \leq \lceil f(a) \rceil$ for each arc a.

5. Finding a maximum flow

Let $D = (V, A)$ be a directed graph, let $s, t \in V$, and let $c : A \to \mathbb{Q}_+$ be a 'capacity' function. We now describe the algorithm of Ford and Fulkerson [4] to find an $s - t$ flow of maximum value under c.

By *flow* we will mean an $s - t$ flow under c, and by *cut* an $s - t$ cut. A *maximum flow* is a flow of maximum value.

We now describe the algorithm of Ford and Fulkerson [5] to determine a maximum flow. We assume that $c(a) > 0$ for each arc a. First we give an important subroutine:

Flow augmenting algorithm

input: a flow f.
output: either (i) a flow f' with value$(f') >$ value(f),
or (ii) a cut $\delta^{\text{out}}(U)$ with $c(\delta^{\text{out}}(U)) =$ value(f).
description of the algorithm: For any pair $a = (v, w)$ define $a^{-1} := (w, v)$. Make an auxiliary graph $D_f = (V, A_f)$ by the following rule: for any arc $a \in A$,

(16) if $f(a) < c(a)$ then $a \in A_f$,
if $f(a) > 0$ then $a^{-1} \in A_f$.

So if $0 < f(a) < c(a)$ then both a and a^{-1} are arcs of A_f.

Now there are two possibilities:

(17) **Case 1:** *There exists an $s - t$ path in D_f.*
Case 2: *There is no $s - t$ path in D_f.*

Case 1: *There exists an $s - t$ path $P = (v_0, a_1, v_1, \ldots, a_k, v_k)$ in $D_f = (V, A_f)$.*
So $v_0 = s$ and $v_k = t$. As $a_1, \ldots, a_k$ belong to A_f, we know by (16) that for each $i = 1, \ldots, k$:

(18) either (i) $a_i \in A$ and $\sigma_i := c(a_i) - f(a_i) > 0$
or (ii) $a_i^{-1} \in A$ and $\sigma_i := f(a_i^{-1}) > 0$.

Define $\alpha := \min\{\sigma_1, \ldots, \sigma_k\}$. So $\alpha > 0$. Let $f' : A \to \mathbb{R}_+$ be defined by, for $a \in A$:

$$(19) \qquad f'(a) := \begin{cases} f(a) + \alpha & \text{if } a = a_i \text{ for some } i = 1, \ldots, k; \\ f(a) - \alpha & \text{if } a = a_i^{-1} \text{ for some } i = 1, \ldots, k; \\ f(a) & \text{for all other } a. \end{cases}$$

Then f' again is an $s-t$ flow under c. The inequalities $0 \leq f'(a) \leq c(a)$ hold because of our choice of α. It is easy to check that also the flow conservation law (5)(ii) is maintained.

Moreover,

$$(20) \qquad \text{value}(f') = \text{value}(f) + \alpha,$$

since either $(v_0, v_1) \in A$, in which case the outgoing flow in s is increased by α, or $(v_1, v_0) \in A$, in which case the ingoing flow in s is decreased by α.

Path P is called a *flow augmenting path.*

Case 2: *There is no $s-t$ path in $D_f = (V, A_f)$.*
Now define:

$$(21) \qquad U := \{u \in V \mid \text{there exists a path in } D_f \text{ from } s \text{ to } u\}.$$

Then $s \in U$ while $t \notin U$, and so $\delta^{\text{out}}(U)$ is an $s-t$ cut.

By definition of U, if $u \in U$ and $v \notin U$, then $(u, v) \notin A_f$ (as otherwise also v would belong to U). Therefore:

$$(22) \qquad \begin{array}{l} \text{if } (u,v) \in \delta^{\text{out}}(U), \text{ then } (u,v) \notin A_f, \text{ and so (by (16)): } f(u,v) = c(u,v), \\ \text{if } (u,v) \in \delta^{\text{in}}(U), \text{ then } (v,u) \notin A_f, \text{ and so (by (16)): } f(u,v) = 0. \end{array}$$

Then (11) gives:

$$(23) \qquad c(\delta^{\text{out}}(U)) = \text{value}(f). \qquad \blacksquare$$

This finishes the description of the flow augmenting algorithm. The description of the *(Ford-Fulkerson) maximum flow algorithm* is now simple:

Maximum flow algorithm

input: directed graph $D = (V, A), s, t \in V, c : A \to \mathbb{R}_+$.
output: a maximum flow f and a cut $\delta^{\text{out}}(U)$ of minimum capacity, with $\text{value}(f) = c(\delta^{\text{out}}(U))$.
description of the algorithm: Let f_0 be the 'null flow' (that is, $f_0(a) = 0$ for each arc a). Determine with the flow augmenting algorithm flows $f_1, f_2, \ldots, f_N$ such that $f_{i+1} = f_i'$, until, in the Nth iteration, say, we obtain output (ii) of the flow augmenting algorithm. Then we have flow f_N and a cut $\delta^{\text{out}}(U)$ with the given properties. $\blacksquare$

We show that the algorithm terminates, provided that all capacities are rational.

Theorem 5. *If all capacities $c(a)$ are rational, the algorithm terminates.*

Proof. If all capacities are rational, there exists a natural number K so that $Kc(a)$ is an integer for each $a \in A$. (We can take for K the l.c.m. of the denominators of the $c(a)$.)

Then in the flow augmenting iterations, every flow $f_i(a)$ and every α is a multiple of $1/K$. So at each iteration, the flow value increases by at least $1/K$. Since the flow value cannot exceed $c(\delta^{\text{out}}(s))$, we can have only finitely many iterations. ∎

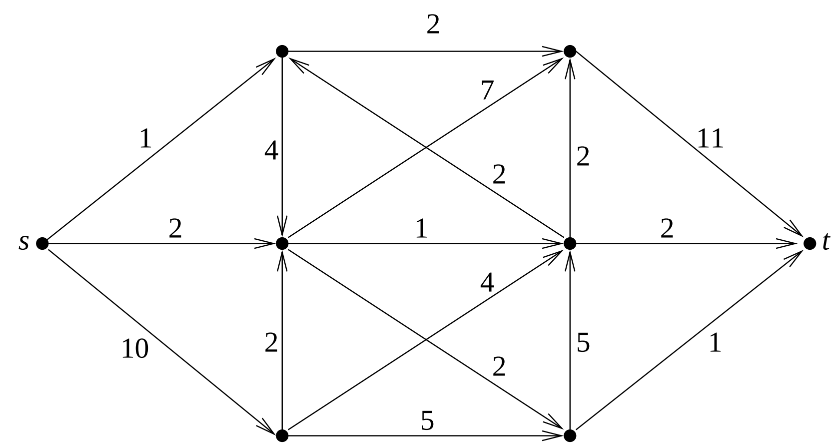
We note here that this theorem is not true if we allow general real-valued capacities. On the other hand, it was shown by Dinits [2] and Edmonds and Karp [3] that if we choose always a shortest path as flow augmenting path, then the algorithm has polynomially bounded running time (also in the case of irrational capacities).

Note that the algorithm also implies the max-flow min-cut theorem (Theorem 4). Note moreover that in the maximum flow algorithm, if all capacities are integer, then the maximum flow found will also be integer-valued. So it also implies the integrity theorem (Corollary 4a).

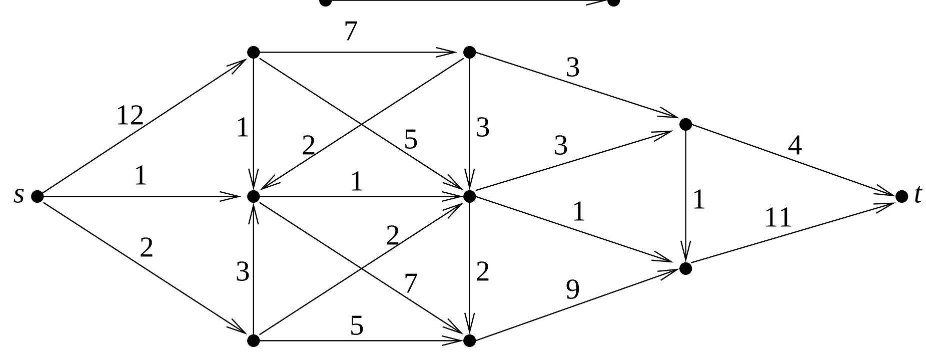
Exercises

5.1. Determine with the maximum flow algorithm an $s - t$ flow of maximum value and an $s - t$ cut of minimum capacity in the following graphs (where the numbers at the arcs give the capacities):

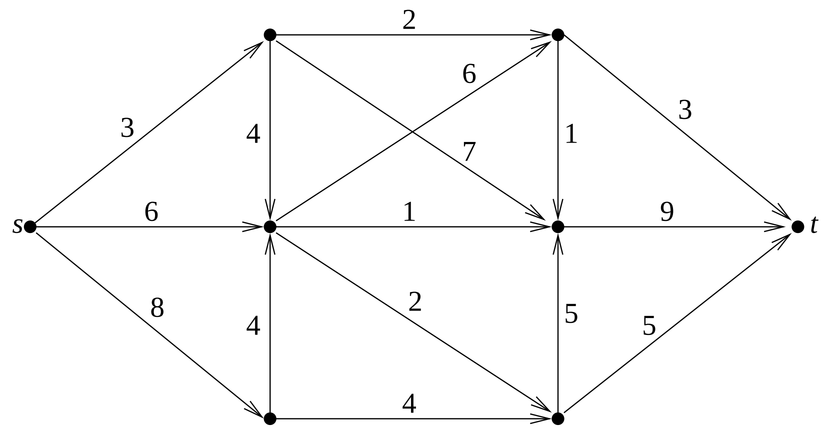
(i)



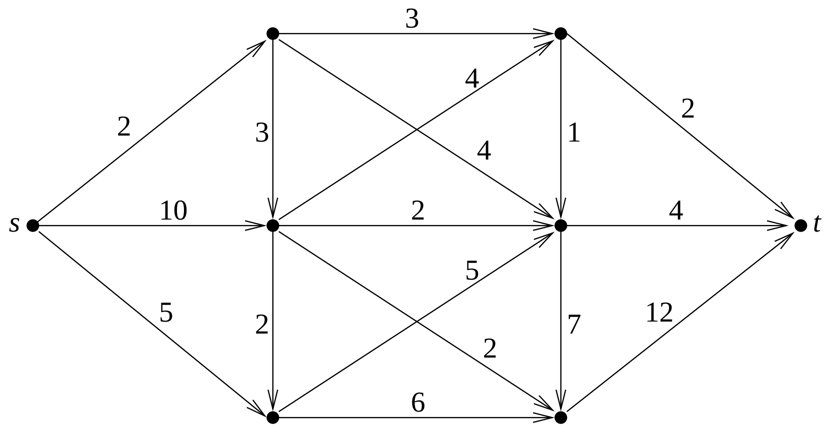
(ii)



(iii)



(iv)



5.2. Describe the problem of finding a maximum-size matching in a bipartite graph as a maximum integer flow problem.

5.3. Let $D = (V, A)$ be a directed graph, let $s, t \in V$ and let $f : A \to \mathbb{Q}_+$ be an $s - t$ flow of value b. Show that for each $U \subseteq V$ with $s \in U, t \notin U$ one has:

$$\sum_{a \in \delta^{\text{out}}(U)} f(a) - \sum_{a \in \delta^{\text{in}}(U)} f(a) = b. \tag{24}$$

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