

# III. Disjoint paths

## 1. Shortest paths

Let  $D = (V, A)$  be a directed graph, and let  $s, t \in V$ .<sup>1</sup> A *path* is a sequence  $P = (v_0, a_1, v_1, \dots, a_m, v_m)$  where  $a_i$  is an arc from  $v_{i-1}$  to  $v_i$  for  $i = 1, \dots, m$  and where  $v_0, \dots, v_m$  all are different. The path  $P$  is called an  $s - t$  *path* if  $v_0 = s$  and  $v_m = t$ . The *length* of  $P$  is  $m$ . Here  $m$  is allowed to be 0. The *distance* from  $s$  to  $t$  is the minimum length of any  $s - t$  path. (If no  $s - t$  path exists, we set the distance from  $s$  to  $t$  equal to  $\infty$ .) A shortest  $s - t$  path can easily be found by breadth-first search.

There is a trivial min-max relation characterizing the minimum length of an  $s - t$  path. To this end, call a subset  $A'$  of  $A$  an  $s - t$  *cut* if  $A' = \delta^{\text{out}}(U)$  for some subset  $U$  of  $V$  satisfying  $s \in U$  and  $t \notin U$ .<sup>2</sup> Throughout, *disjoint* means *pairwise disjoint*. Then the following was observed by Robacker [8]:

**Theorem 1.** *The minimum length of an  $s - t$  path is equal to the maximum number of disjoint  $s - t$  cuts.*

**Proof.** Trivially, the minimum is at least the maximum, since each  $s - t$  path intersects each  $s - t$  cut in an arc. To see equality, let  $d$  be the distance from  $s$  to  $t$ , and let  $U_i$  be the set of vertices at distance less than  $i$  from  $s$ , for  $i = 1, \dots, d$ . Taking  $C_i := \delta^{\text{out}}(U_i)$ , we obtain disjoint  $s - t$  cuts  $C_1, \dots, C_d$ . ■

## 2. Length functions

This can be generalized to the case where arcs have a certain ‘length’. Let  $l : A \rightarrow \mathbb{R}_+$ , called a *length function*. For any path  $P = (v_0, a_1, v_1, \dots, a_m, v_m)$ , let  $l(P)$  be the length of  $P$ . That is:

$$(1) \quad l(P) := \sum_{i=1}^m l(a_i).$$

Now the *distance* from  $s$  to  $t$  (with respect to  $l$ ) is equal to the minimum length of any  $s - t$  path. If no  $s - t$  path exists, the distance is  $\infty$ .

Then a weighted version of Theorem 1 is as follows:

**Theorem 2.** *Let  $D = (V, A)$  be a directed graph, let  $s, t \in V$ , and let  $l : A \rightarrow \mathbb{Z}_+$ . Then the minimum length of an  $s - t$  path is equal to the maximum number  $k$  of  $s - t$  cuts  $C_1, \dots, C_k$  (repetition allowed) such that each arc  $a$  is in at most  $l(a)$  of the cuts  $C_i$ .*

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<sup>1</sup>A *directed graph* or *digraph* is a pair  $(V, A)$ , where  $V$  is a finite set and  $A \subseteq V \times V$ . The elements of  $A$  are called the *arcs* of  $D$ . If  $a = (u, v)$ , then  $u$  is called the *tail* of  $a$  and  $v$  is called the *head* of  $a$ .

<sup>2</sup> $\delta^{\text{out}}(U)$  and  $\delta^{\text{in}}(U)$  denote the sets of arcs leaving and entering  $U$ , respectively.

**Proof.** Again, the minimum is not smaller than the maximum, since if  $P$  is any  $s - t$  path and  $C_1, \dots, C_k$  is any collection as described in the theorem:<sup>3</sup>

$$(2) \quad l(P) = \sum_{a \in AP} l(a) \geq \sum_{a \in AP} (\text{number of } i \text{ with } a \in C_i) = \sum_{i=1}^k |C_i \cap AP| \geq \sum_{i=1}^k 1 = k.$$

To see equality, let  $d$  be the distance from  $s$  to  $t$ , and let  $U_i$  be the set of vertices at distance less than  $i$  from  $s$ , for  $i = 1, \dots, d$ . Taking  $C_i := \delta^{\text{out}}(U_i)$ , we obtain a collection  $C_1, \dots, C_d$  as required.  $\blacksquare$

### 3. Menger's theorem

In this section we study the maximum number  $k$  of disjoint paths in a graph connecting two vertices, or two sets of vertices.

Let  $D = (V, A)$  be a directed graph and let  $S$  and  $T$  be subsets of  $V$ . A path is called an  $S - T$  path if it runs from a vertex in  $S$  to a vertex in  $T$ .

Menger [7] gave a min-max theorem for the maximum number of disjoint  $S - T$  paths. We follow the proof given by Göring [6].

Call a set of paths *vertex-disjoint* if no two of them have vertices in common. (Hence they also have no arcs in common.) A set  $C$  of vertices is called  $S - T$  *disconnecting* if  $C$  intersects each  $S - T$  path ( $C$  may intersect  $S \cup T$ ).

**Theorem 3** (Menger's theorem (directed vertex-disjoint version)). *Let  $D = (V, A)$  be a digraph and let  $S, T \subseteq V$ . Then the maximum number of vertex-disjoint  $S - T$  paths is equal to the minimum size of an  $S - T$  disconnecting vertex set.*

**Proof.** Obviously, the maximum does not exceed the minimum. Equality is shown by induction on  $|A|$ , the case  $A = \emptyset$  being trivial.

Let  $k$  be the minimum size of an  $S - T$  disconnecting vertex set. Choose  $a = (u, v) \in A$ . Let  $D' := (V, A \setminus \{a\})$ . If each  $S - T$  disconnecting vertex set in  $D'$  has size at least  $k$ , then inductively there exist  $k$  vertex-disjoint  $S - T$  paths in  $D'$ , hence in  $D$ .

So we can assume that  $D'$  has an  $S - T$  disconnecting vertex set  $C$  of size  $\leq k - 1$ . Then  $C \cup \{u\}$  and  $C \cup \{v\}$  are  $S - T$  disconnecting vertex sets of  $D$  of size  $k$ .

Now each  $S - (C \cup \{u\})$  disconnecting vertex set  $B$  of  $D'$  has size at least  $k$ , as it is  $S - T$  disconnecting in  $D$ . Indeed, each  $S - T$  path  $P$  in  $D$  intersects  $C \cup \{u\}$ , and hence  $P$  contains an  $S - (C \cup \{u\})$  path in  $D'$ . So  $P$  intersects  $B$ .

So by induction,  $D'$  contains  $k$  disjoint  $S - (C \cup \{u\})$  paths. Similarly,  $D'$  contains  $k$  disjoint  $(C \cup \{v\}) - T$  paths. Any path in the first collection intersects any path in the second collection only in  $C$ , since otherwise  $D'$  contains an  $S - T$  path avoiding  $C$ .

Hence, as  $|C| = k - 1$ , we can pairwise concatenate these paths to obtain disjoint  $S - T$  paths, inserting arc  $a$  between the path ending at  $u$  and the path starting at  $v$ .  $\blacksquare$

A consequence of this theorem is a variant on *internally vertex-disjoint*  $s - t$  paths, that

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<sup>3</sup> $AP$  denotes the set of arcs traversed by  $P$ .

is,  $s - t$  paths no two of which have a vertex in common except for  $s$  and  $t$ . A set  $U$  of vertices is called an  $s - t$  *vertex-cut* if  $s, t \notin U$  and each  $s - t$  path intersects  $U$ .

**Corollary 3a** (Menger's theorem (directed internally vertex-disjoint version)). *Let  $D = (V, A)$  be a digraph and let  $s$  and  $t$  be two nonadjacent vertices of  $D$ . Then the maximum number of internally vertex-disjoint  $s - t$  paths is equal to the minimum size of an  $s - t$  vertex-cut.*

**Proof.** Let  $D' := D - s - t$  and let  $S$  and  $T$  be the sets of outneighbours of  $s$  and of inneighbours of  $t$ , respectively. Then Theorem 3 applied to  $D', S, T$  gives the corollary. ■

In turn, Theorem 3 follows from Corollary 3a by adding two new vertices  $s$  and  $t$  and arcs  $(s, v)$  for all  $v \in S$  and  $(v, t)$  for all  $v \in T$ .

Also an arc-disjoint version can be derived, where paths are *arc-disjoint* if they have no arc in common. Recall that a set  $C$  of arcs is an  $s - t$  *cut* if  $C = \delta^{\text{out}}(U)$  for some subset  $U$  of  $V$  with  $s \in U$  and  $t \notin U$ .

**Corollary 3b** (Menger's theorem (directed arc-disjoint version)). *Let  $D = (V, A)$  be a digraph and let  $s, t \in V$ . Then the maximum number of arc-disjoint  $s - t$  paths is equal to the minimum size of an  $s - t$  cut.*

**Proof.** Let  $L(D)$  be the line digraph of  $D$ .<sup>4</sup> Let  $S := \delta_A^{\text{out}}(s)$  and  $T := \delta_A^{\text{in}}(t)$ . Then Theorem 3 for  $L(D), S, T$  implies the corollary. Note that a minimum-size set of arcs intersecting each  $s - t$  path necessarily is an  $s - t$  cut. ■

The internally vertex-disjoint version of Menger's theorem can be derived in turn from the arc-disjoint version: make a digraph  $D'$  as follows from  $D$ : replace any vertex  $v$  by two vertices  $v', v''$  and make an arc  $(v', v'')$ ; moreover, replace each arc  $(u, v)$  by  $(u'', v')$ . Then Corollary 3b for  $D', s'', t'$  gives Corollary 3a for  $D, s, t$ .

Similar theorems hold for *undirected* graphs. They can be derived from the directed case by replacing each undirected edge  $uw$  by two opposite arcs  $(u, w)$  and  $(w, u)$ .

## Exercises

- 3.1. Let  $D = (V, A)$  be a directed graph and let  $s, t_1, \dots, t_k$  be vertices of  $D$ . Prove that there exist arc-disjoint paths  $P_1, \dots, P_k$  such that  $P_i$  is an  $s - t_i$  path ( $i = 1, \dots, k$ ) if and only if for each  $U \subseteq V$  with  $s \in U$  one has

$$(3) \quad |\delta^{\text{out}}(U)| \geq |\{i \mid t_i \notin U\}|.$$

- 3.2. Let  $\mathcal{A} = (A_1, \dots, A_n)$  and  $\mathcal{B} = (B_1, \dots, B_n)$  be families of subsets of a finite set. Show that  $\mathcal{A}$  and  $\mathcal{B}$  have a common SDR if and only if for all  $I, J \subseteq \{1, \dots, n\}$  one has

$$(4) \quad \left| \bigcup_{i \in I} A_i \cap \bigcup_{j \in J} B_j \right| \geq |I| + |J| - n.$$

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<sup>4</sup>The *line digraph* of a digraph  $D = (V, A)$  is the digraph with vertex set  $A$  and arcs set  $\{(a, a') \mid a, a' \in A, \text{head}(a) = \text{tail}(a')\}$ .

## 4. Flows in networks

Other consequences of Menger's theorem concern 'flows in networks'. Let  $D = (V, A)$  be a directed graph and let  $s, t \in V$ . A function  $f : A \rightarrow \mathbb{R}$  is called an  $s - t$  flow if:<sup>5</sup>

$$(5) \quad \begin{aligned} (i) \quad & f(a) \geq 0 && \text{for each } a \in A; \\ (ii) \quad & \sum_{a \in \delta^{\text{in}}(v)} f(a) = \sum_{a \in \delta^{\text{out}}(v)} f(a) && \text{for each } v \in V \setminus \{s, t\}. \end{aligned}$$

Condition (5)(ii) is called the *flow conservation law*: the amount of flow entering a vertex  $v \neq s, t$  should be equal to the amount of flow leaving  $v$ .

The *value* of an  $s - t$  flow  $f$  is, by definition:

$$(6) \quad \text{value}(f) := \sum_{a \in \delta^{\text{out}}(s)} f(a) - \sum_{a \in \delta^{\text{in}}(s)} f(a).$$

So the value is the net amount of flow leaving  $s$ . It can be shown that it is equal to the net amount of flow entering  $t$ .

Let  $c : A \rightarrow \mathbb{R}_+$ , called a *capacity function*. We say that a flow  $f$  is *under*  $c$  (or *subject to*  $c$ ) if

$$(7) \quad f(a) \leq c(a) \text{ for each } a \in A;$$

that is, if  $f \leq c$ . The *maximum flow problem* now is to find an  $s - t$  flow under  $c$ , of maximum value.

To formulate a min-max relation, define the *capacity* of a cut  $\delta^{\text{out}}(U)$  by:

$$(8) \quad c(\delta^{\text{out}}(U)) := \sum_{a \in \delta^{\text{out}}(U)} c(a).$$

Then:

**Proposition 1.** *For every  $s - t$  flow  $f$  under  $c$  and every  $s - t$  cut  $\delta^{\text{out}}(U)$  one has:*

$$(9) \quad \text{value}(f) \leq c(\delta^{\text{out}}(U)).$$

**Proof.**

$$(10) \quad \begin{aligned} \text{value}(f) &= \sum_{a \in \delta^{\text{out}}(s)} f(a) - \sum_{a \in \delta^{\text{in}}(s)} f(a) \\ &= \sum_{a \in \delta^{\text{out}}(s)} f(a) - \sum_{a \in \delta^{\text{in}}(s)} f(a) + \sum_{v \in U \setminus \{s\}} \left( \sum_{a \in \delta^{\text{out}}(v)} f(a) - \sum_{a \in \delta^{\text{in}}(v)} f(a) \right) \end{aligned}$$

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<sup>5</sup> $\delta^{\text{out}}(v)$  and  $\delta^{\text{in}}(v)$  denote the sets of arcs leaving  $v$  and entering  $v$ , respectively.

$$\begin{aligned}
&= \sum_{v \in U} \left( \sum_{a \in \delta^{\text{out}}(v)} f(a) - \sum_{a \in \delta^{\text{in}}(v)} f(a) \right) = \sum_{a \in \delta^{\text{out}}(U)} f(a) - \sum_{a \in \delta^{\text{in}}(U)} f(a) \\
&\stackrel{\star}{\leq} \sum_{a \in \delta^{\text{out}}(U)} f(a) \stackrel{\star\star}{\leq} \sum_{a \in \delta^{\text{out}}(U)} c(a) = c(\delta^{\text{out}}(U)). \quad \blacksquare
\end{aligned}$$

It is convenient to note the following:

$$(11) \quad \text{equality holds in (9)} \iff \begin{aligned} &\forall a \in \delta^{\text{in}}(U) : f(a) = 0 \text{ and} \\ &\forall a \in \delta^{\text{out}}(U) : f(a) = c(a). \end{aligned}$$

This follows directly from the inequalities  $\star$  and  $\star\star$  in (10).

Now from Menger's theorem one can derive that equality can be attained in (9), which is a theorem of Ford and Fulkerson [4]:

**Theorem 4** (max-flow min-cut theorem). *For any directed graph  $D = (V, A)$ ,  $s, t \in V$ , and  $c : A \rightarrow \mathbb{R}_+$ , the maximum value of an  $s - t$  flow under  $c$  is equal to the minimum capacity of an  $s - t$  cut. In formula:*

$$(12) \quad \max_{\substack{f \text{ } s-t \text{ flow} \\ f \leq c}} \text{value}(f) = \min_{\delta^{\text{out}}(U) \text{ } s-t \text{ cut}} c(\delta^{\text{out}}(U)).$$

**Proof.** If  $c$  is integer-valued, the corollary follows from Menger's theorem by replacing each arc  $a$  by  $c(a)$  parallel arcs. If  $c$  is rational-valued, there exists a natural number  $N$  such that  $Nc(a)$  is integer for each  $a \in A$ . This resetting multiplies both the maximum and the minimum by  $N$ . So the equality follows from the case where  $c$  is integer-valued.

If  $c$  is real-valued, we can derive the corollary from the case where  $c$  is rational-valued, by continuity and compactness arguments, as follows. Suppose that

$$(13) \quad \max_{\substack{f \text{ } s-t \text{ flow} \\ f \leq c}} \text{value}(f) < \min_{\delta^{\text{out}}(U) \text{ } s-t \text{ cut}} c(\delta^{\text{out}}(U)).$$

(The maximum exists, as the set of  $s - t$  flows  $f$  with  $f \leq c$  is compact.)

Then we can choose a rational-valued  $c' \leq c$  close enough to  $c$  such that

$$(14) \quad \max_{\substack{f \text{ } s-t \text{ flow} \\ f \leq c}} \text{value}(f) < \min_{\delta^{\text{out}}(U) \text{ } s-t \text{ cut}} c'(\delta^{\text{out}}(U)).$$

So

$$(15) \quad \max_{\substack{f \text{ } s-t \text{ flow} \\ f \leq c'}} \text{value}(f) \leq \max_{\substack{f \text{ } s-t \text{ flow} \\ f \leq c}} \text{value}(f) < \min_{\delta^{\text{out}}(U) \text{ } s-t \text{ cut}} c'(\delta^{\text{out}}(U)).$$

This contradicts the above, as  $c'$  is rational. \blacksquare

Moreover, one has (Dantzig [1]):

**Corollary 4a** (Integrity theorem). *If  $c$  is integer-valued, there exists an integer-valued maximum-value flow  $f \leq c$ .*

**Proof.** Directly from Menger's theorem. ■

### Exercises

- 4.1. Let  $D = (V, A)$  be a directed graph and let  $s, t \in V$ . Let  $f : A \rightarrow \mathbb{R}_+$  be an  $s-t$  flow of value  $\beta$ . Show that there exists an  $s-t$  flow  $f' : A \rightarrow \mathbb{Z}_+$  of value  $\lceil \beta \rceil$  such that  $\lfloor f(a) \rfloor \leq f'(a) \leq \lceil f(a) \rceil$  for each arc  $a$ .

## 5. Finding a maximum flow

Let  $D = (V, A)$  be a directed graph, let  $s, t \in V$ , and let  $c : A \rightarrow \mathbb{Q}_+$  be a 'capacity' function. We now describe the algorithm of Ford and Fulkerson [4] to find an  $s-t$  flow of maximum value under  $c$ .

By *flow* we will mean an  $s-t$  flow under  $c$ , and by *cut* an  $s-t$  cut. A *maximum flow* is a flow of maximum value.

We now describe the algorithm of Ford and Fulkerson [5] to determine a maximum flow. We assume that  $c(a) > 0$  for each arc  $a$ . First we give an important subroutine:

### Flow augmenting algorithm

**input:** a flow  $f$ .

**output:** either (i) a flow  $f'$  with  $\text{value}(f') > \text{value}(f)$ ,  
or (ii) a cut  $\delta^{\text{out}}(U)$  with  $c(\delta^{\text{out}}(U)) = \text{value}(f)$ .

**description of the algorithm:** For any pair  $a = (v, w)$  define  $a^{-1} := (w, v)$ . Make an auxiliary graph  $D_f = (V, A_f)$  by the following rule: for any arc  $a \in A$ ,

$$(16) \quad \begin{array}{l} \text{if } f(a) < c(a) \text{ then } a \in A_f, \\ \text{if } f(a) > 0 \text{ then } a^{-1} \in A_f. \end{array}$$

So if  $0 < f(a) < c(a)$  then both  $a$  and  $a^{-1}$  are arcs of  $A_f$ .

Now there are two possibilities:

(17) **Case 1:** *There exists an  $s-t$  path in  $D_f$ .*

**Case 2:** *There is no  $s-t$  path in  $D_f$ .*

**Case 1:** *There exists an  $s-t$  path  $P = (v_0, a_1, v_1, \dots, a_k, v_k)$  in  $D_f = (V, A_f)$ .*

So  $v_0 = s$  and  $v_k = t$ . We may assume that  $P$  is a simple path. As  $a_1, \dots, a_k$  belong to  $A_f$ , we know by (16) that for each  $i = 1, \dots, k$ :

$$(18) \quad \begin{array}{l} \text{either (i) } a_i \in A \text{ and } \sigma_i := c(a_i) - f(a_i) > 0 \\ \text{or (ii) } a_i^{-1} \in A \text{ and } \sigma_i := f(a_i^{-1}) > 0. \end{array}$$

Define  $\alpha := \min\{\sigma_1, \dots, \sigma_k\}$ . So  $\alpha > 0$ . Let  $f' : A \rightarrow \mathbb{R}_+$  be defined by, for  $a \in A$ :

$$(19) \quad f'(a) := \begin{cases} f(a) + \alpha & \text{if } a = a_i \text{ for some } i = 1, \dots, k; \\ f(a) - \alpha & \text{if } a = a_i^{-1} \text{ for some } i = 1, \dots, k; \\ f(a) & \text{for all other } a. \end{cases}$$

Then  $f'$  again is an  $s-t$  flow under  $c$ . The inequalities  $0 \leq f'(a) \leq c(a)$  hold because of our choice of  $\alpha$ . It is easy to check that also the flow conservation law (5)(ii) is maintained. Moreover,

$$(20) \quad \text{value}(f') = \text{value}(f) + \alpha,$$

since either  $(v_0, v_1) \in A$ , in which case the outgoing flow in  $s$  is increased by  $\alpha$ , or  $(v_1, v_0) \in A$ , in which case the ingoing flow in  $s$  is decreased by  $\alpha$ .

Path  $P$  is called a *flow augmenting path*.

**Case 2:** *There is no  $s-t$  path in  $D_f = (V, A_f)$ .*

Now define:

$$(21) \quad U := \{u \in V \mid \text{there exists a path in } D_f \text{ from } s \text{ to } u\}.$$

Then  $s \in U$  while  $t \notin U$ , and so  $\delta^{\text{out}}(U)$  is an  $s-t$  cut.

By definition of  $U$ , if  $u \in U$  and  $v \notin U$ , then  $(u, v) \notin A_f$  (as otherwise also  $v$  would belong to  $U$ ). Therefore:

$$(22) \quad \begin{aligned} &\text{if } (u, v) \in \delta^{\text{out}}(U), \text{ then } (u, v) \notin A_f, \text{ and so (by (16)): } f(u, v) = c(u, v), \\ &\text{if } (u, v) \in \delta^{\text{in}}(U), \text{ then } (v, u) \notin A_f, \text{ and so (by (16)): } f(u, v) = 0. \end{aligned}$$

Then (11) gives:

$$(23) \quad c(\delta^{\text{out}}(U)) = \text{value}(f). \quad \blacksquare$$

This finishes the description of the flow augmenting algorithm. The description of the (*Ford-Fulkerson*) *maximum flow algorithm* is now simple:

### Maximum flow algorithm

**input:** directed graph  $D = (V, A)$ ,  $s, t \in V$ ,  $c: A \rightarrow \mathbb{R}_+$ .

**output:** a maximum flow  $f$  and a cut  $\delta^{\text{out}}(U)$  of minimum capacity, with  $\text{value}(f) = c(\delta^{\text{out}}(U))$ .

**description of the algorithm:** Let  $f_0$  be the ‘null flow’ (that is,  $f_0(a) = 0$  for each arc  $a$ ). Determine with the flow augmenting algorithm flows  $f_1, f_2, \dots, f_N$  such that  $f_{i+1} = f'_i$ , until, in the  $N$ th iteration, say, we obtain output (ii) of the flow augmenting algorithm. Then we have flow  $f_N$  and a cut  $\delta^{\text{out}}(U)$  with the given properties.  $\blacksquare$

We show that the algorithm terminates, provided that all capacities are rational.

**Theorem 5.** *If all capacities  $c(a)$  are rational, the algorithm terminates.*

**Proof.** If all capacities are rational, there exists a natural number  $K$  so that  $Kc(a)$  is an integer for each  $a \in A$ . (We can take for  $K$  the l.c.m. of the denominators of the  $c(a)$ .)

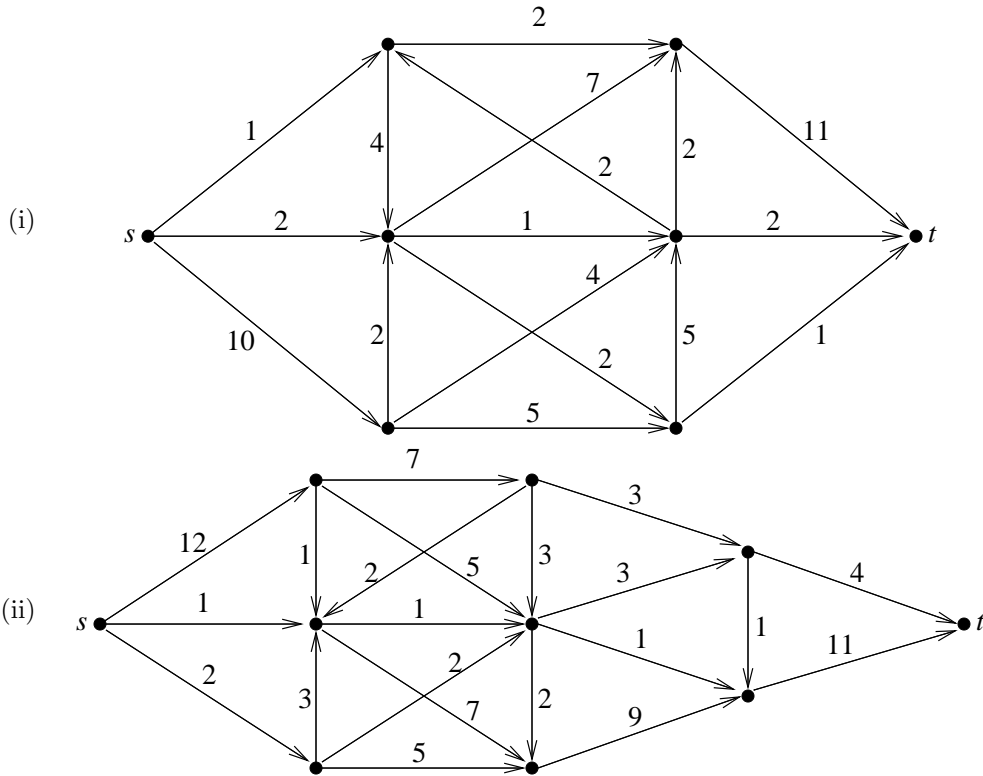
Then in the flow augmenting iterations, every flow  $f_i(a)$  and every  $\alpha$  is a multiple of  $1/K$ . So at each iteration, the flow value increases by at least  $1/K$ . Since the flow value cannot exceed  $c(\delta^{\text{out}}(s))$ , we can have only finitely many iterations. ■

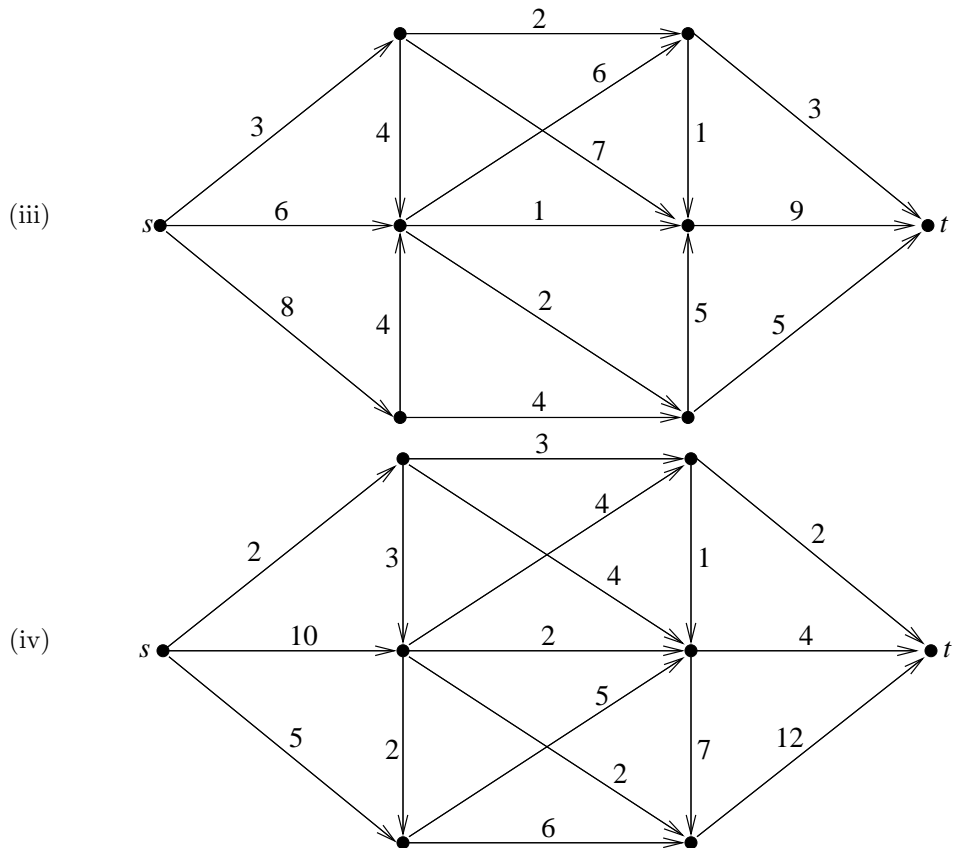
We note here that this theorem is not true if we allow general real-valued capacities. On the other hand, it was shown by Dinits [2] and Edmonds and Karp [3] that if we choose always a shortest path as flow augmenting path, then the algorithm has polynomially bounded running time (also in the case of irrational capacities).

Note that the algorithm also implies the max-flow min-cut theorem (Theorem 4). Note moreover that in the maximum flow algorithm, if all capacities are integer, then the maximum flow found will also be integer-valued. So it also implies the integrity theorem (Corollary 4a).

### Exercises

- 5.1. Determine with the maximum flow algorithm an  $s - t$  flow of maximum value and an  $s - t$  cut of minimum capacity in the following graphs (where the numbers at the arcs give the capacities):





- 5.2. Describe the problem of finding a maximum-size matching in a bipartite graph as a maximum integer flow problem.
- 5.3. Let  $D = (V, A)$  be a directed graph, let  $s, t \in V$  and let  $f : A \rightarrow \mathbb{Q}_+$  be an  $s - t$  flow of value  $b$ . Show that for each  $U \subseteq V$  with  $s \in U, t \notin U$  one has:

$$(24) \quad \sum_{a \in \delta^{\text{out}}(U)} f(a) - \sum_{a \in \delta^{\text{in}}(U)} f(a) = b.$$

## References

- [1] G.B. Dantzig, Application of the simplex method to a transportation problem, in: *Activity Analysis of Production and Allocation — Proceedings of a Conference* (Proceedings Conference on Linear Programming, Chicago, Illinois, 1949; Tj.C. Koopmans, ed.), Wiley, New York, 1951, pp. 359–373.
- [2] E.A. Dinits, Algoritm resheniya zadachi o maksimal'nom potoke v seti so stepennoĭ otsenkoĭ [Russian], *Doklady Akademii Nauk SSSR* 194 (1970) 754–757 [English translation: Algorithm for solution of a problem of maximum flow in a network with power estimation *Soviet Mathematics Doklady* 11 (1970) 1277–1280].
- [3] J. Edmonds, R.M. Karp, Theoretical improvements in algorithmic efficiency for network flow problems, *Journal of the Association for Computing Machinery* 19 (1972) 248–264.
- [4] L.R. Ford, Jr, D.R. Fulkerson, Maximal flow through a network, *Canadian Journal of Mathematics* 8 (1956) 399–404.

- [5] L.R. Ford, Jr, D.R. Fulkerson, A simple algorithm for finding maximal network flows and an application to the Hitchcock problem, *Canadian Journal of Mathematics* 9 (1957) 210–218.
- [6] F. Göring, Short proof of Menger’s theorem, *Discrete Mathematics* 219 (2000) 295–296.
- [7] K. Menger, Zur allgemeinen Kurventheorie, *Fundamenta Mathematicae* 10 (1927) 96–115.
- [8] J.T. Robacker, *Min-Max Theorems on Shortest Chains and Disjoint Cuts of a Network*, Research Memorandum RM-1660, The RAND Corporation, Santa Monica, California, [12 January] 1956.