Block diagonalization of matrix $*$-algebras

A *matrix $*$-algebra* is a nonempty collection $\mathcal{A} \subseteq \mathbb{C}^{n\times n}$ closed under sums, scalar and matrix multiplication, and taking the adjoint. Call $\mathcal{A}, \mathcal{A}' \subseteq \mathbb{C}^{n\times n}$ *equivalent* if there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$\mathcal{A}' = \{U^*MU \mid M \in \mathcal{A}\}. \tag{1}$$

For matrices M_1 and M_2, the direct sum is

$$M_1 \oplus M_2 := \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}. \tag{2}$$

The iterated direct sum of $M_1, \ldots, M_n$ is denoted by

$$\bigoplus_{i=1}^{n} M_i. \tag{3}$$

We write

$$t \odot M := \bigoplus_{i=1}^{t} M. \tag{4}$$

Call $\mathcal{A}$ *basic* if

$$\mathcal{A} = t \odot \mathbb{C}^{m\times m} := \{t \odot M \mid M \in \mathbb{C}^{m\times m}\} \tag{5}$$

for some t and m.

The *direct sum* of $\mathcal{A}$ and $\mathcal{A}'$ is

$$\mathcal{A} \oplus \mathcal{A}' := \{M \oplus M' \mid M \in \mathcal{A}, M' \in \mathcal{A}'\}. \tag{6}$$

$\mathcal{A}$ is called a *zero algebra* if $\mathcal{A}$ only consists of the zero matrix.

Theorem. *Each matrix $*$-algebra is equivalent to a direct sum of basic algebras and a zero algebra.*

Proof. We first show:

(7) Each matrix $*$-algebra $\mathcal{A}$ is equivalent to a direct sum of a matrix $*$-algebra containing the identity matrix and a zero algebra. In particular, $\mathcal{A}$ contains a unit.

Let N be a matrix in $\mathcal{A}$ of maximum rank. Then the row space row$N$ of N contains the row space of each matrix in $\mathcal{A}$. For let $M \in \mathcal{A}$. Then $\ker M \supseteq \ker N$, since

(8) $x \in \ker(M^*M + N^*N) \iff (M^*M + N^*N)x = 0 \iff x^*(M^*M + N^*N)x = 0 \iff x^*M^*Mx = 0$ and $x^*N^*Nx = 0 \iff Mx = 0$ and $Nx = 0 \iff x \in \ker M \cap \ker N$.

By the maximality of the rank of N, $\ker(M^*M + N^*N) = \ker N$, hence $\ker(M) \supseteq \ker N$.

So the row space of each matrix in $\mathcal{A}$ is contained in row$N$ = row(N^*N). We can assume that N^*N is a diagonal matrix. (Replace $\mathcal{A}$ by $\{U^*MU \mid M \in \mathcal{A}\}$ for some unitary matrix U.) So $\mathcal{A}$ is a direct sum of a zero algebra and a matrix $*$-algebra containing a nonsingular diagonal matrix Δ. Then I is a linear combination of $\Delta, \Delta^2, \Delta^3, \ldots$ (by the theory of Vandermonde matrices).

This proves (7). Hence to prove the theorem, we can assume that $I \in \mathcal{A}$.

Let $\mathcal{C}_{\mathcal{A}}$ be the center of $\mathcal{A}$; that is,

(9) $\mathcal{C}_{\mathcal{A}} := \{C \in \mathcal{A} \mid CM = MC \text{ for all } M \in \mathcal{A}\}$.

As $\mathcal{C}_{\mathcal{A}}$ is a commutative matrix $*$-algebra, the matrices in $\mathcal{C}_{\mathcal{A}}$ can be simultaneously diagonalized by some unitary matrix U. That is, $\{U^*MU \mid M \in \mathcal{C}_{\mathcal{A}}\}$ consists of diagonal matrices. So we can assume that $\mathcal{C}_{\mathcal{A}}$ consists of diagonal matrices only. Then $\mathcal{C}_{\mathcal{A}}$ is the linear hull of certain diagonal 0,1 matrices $E_1, \ldots, E_t$, with $E_iE_j = 0$ if $i \neq j$.

Then for each $M \in \mathcal{A}$ and each i one has $E_iM = ME_i$. So M is 0 in positions (k, l) with $(E_i)_{k,k} \neq (E_i)_{l.l} = 0$. So $\mathcal{A}$ is the direct sum of matrix $*$-algebras each with the property that the scalar multiples of the identity matrix (of appropriate dimension) are the only matrices commuting with all matrices in the subalgebra.

Hence it suffices to show

(10) if $\mathcal{A}$ is a matrix $*$-algebra with $\mathcal{C}_{\mathcal{A}} = \mathbb{C}I$, then $\mathcal{A}$ is basic.

Let $\mathcal{B}$ be a maximal subset of $\mathcal{A}$ with the property that $A^* = A$ and $AB = BA$ for all $A, B \in \mathcal{B}$. This implies that $\mathcal{B}$ is a commutative $*$-subalgebra of $\mathcal{A}$ and that $I \in \mathcal{B}$. We can assume that $\mathcal{B}$ consists of diagonal matrices only. Let $E_1, \ldots, E_t$ be the minimal idempotents of $\mathcal{B}$. So $\mathcal{B} = \mathbb{C}E_1 + \cdots + \mathbb{C}E_t$ and $E_1 + \cdots + E_t = I$.

Now for each i:

(11) For each $M \in \mathcal{A}$ there is a $\lambda \in \mathbb{C}$ with $E_iME_i = \lambda E_i$. In other words, $E_i\mathcal{A}E_i = \mathbb{C}E_i$.

Suppose not. We can assume that $M^* = M$, since we can replace M by $M + M^*$ or $iM - iM^*$. Then we can add E_iME_i to $\mathcal{B}$, since it commutes with each of $E_1, \ldots, E_t$. This contradicts the maximality of $\mathcal{B}$. This proves (11).

Moreover,

(12) for all i there is an $M_i \in \mathcal{A}$ with $E_1M_iE_i \neq 0$.

To see this, consider the linear space $\mathcal{I}$ generated by $\mathcal{A}E_1\mathcal{A}$. This is a $*$-algebra. Let P be the identity of $\mathcal{I}$ (which exists by (7)). Then for each $M \in \mathcal{A}$, $PM \in \mathcal{I}$ and $MP \in \mathcal{I}$, hence $PM = PMP = MP$. So $P \in \mathcal{C}_{\mathcal{A}}$. Hence by the assumption in (10), $P = I$. So $\mathcal{I} = \mathcal{A}$, and therefore $E_i \in \mathcal{I}$, and hence E_i is a sum of matrices in $\mathcal{A}E_1\mathcal{A}$, hence in $\mathcal{A}E_1\mathcal{A}E_i$. Hence we have (12).

We can assume that $M_i = E_1M_iE_i$ (by resetting $M_i := E_1M_iE_i$). Now $M_i^*M_i$ belongs to $E_i\mathcal{A}E_i$, hence by (11) to $\mathbb{C}E_i$. Similarly, $M_iM_i^* \in \mathbb{C}E_1$. By scaling we can assume that $M_i^*M_i = E_i$ for each i. This implies that $M_iM_i^* = E_1$ for each i, since $M_iM_i^* = \lambda E_1$ for some $\lambda \in \mathbb{C}$. Then

$$(13) \qquad \lambda^2 E_1 = M_iM_i^*M_iM_i^* = M_iE_iM_i^* = M_iM_i^* = \lambda E_1,$$

and hence $\lambda = 1$. Since $M_1 \in E_1\mathcal{A}E_1$, we have by (11) that $M_1 = \lambda E_1$ for some $\lambda \in \mathbb{C}$. As $M_1M_1^* = E_1$, we know $\lambda\overline{\lambda} = 1$. Hence replacing M_1 by $\lambda^{-1}E_1$, we obtain $M_1 = E_1$.

So $\operatorname{rank}(E_1) = \operatorname{rank}(M_iM_i^*) = \operatorname{rank}(M_i^*M_i) = \operatorname{rank}(E_i)$ for each i. Hence all E_i have the same number of 1's. For $A \in \mathcal{A}$ and $i,j = 1,\ldots,t$, let $A_{i,j}$ be the submatrix induced by the rows where E_i has 1's and the columns where E_j has 1's. So E_iAE_j arises from $A_{i,j}$ by adding all-zero rows and columns.

So for each i, the matrix $(M_i)_{1,i}$ is unitary. Let $U_i := (M_i)_{1,i}^*$. Let U be the unitary matrix with $U_{i,i} = U_i$ for each i, and $U_{i,j} = 0$ if $i \neq j$. Replacing $\mathcal{A}$ by $U^*\mathcal{A}U$, we obtain that $(M_i)_{1,i} = I$ for each i.

Now for $i,j = 1,\ldots,t$, let $N_{i,j}$ be be the matrix with $E_iN_{i,j}E_j = N_{i,j}$ and $(N_{i,j})_{i,j} = I$. So $N_{i,i} = E_i$ and $N_{1,i} = M_i$ for each $i = 1,\ldots,t$. So each $N_{1,i}$ belongs to $\mathcal{A}$. Since $N_{i,j} = N_{1,i}^*N_{1,j}$ for all i,j, it follows that $N_{i,j} \in \mathcal{A}$ for all i,j.

We finally show:

$$(14) \qquad \text{for each } M \in \mathcal{A} \text{ and } i,j = 1,\ldots,t,\ E_iME_j = \lambda N_{i,j} \text{ for some } \lambda \in \mathbb{C}.$$

Indeed, using (11),

$$(15) \qquad E_iME_j = E_iME_jN_{j,i}E_iN_{i,j} \in E_i\mathcal{A}E_iN_{i,j} = \mathbb{C}E_iN_{i,j} = \mathbb{C}N_{i,j},$$

as required. So $\mathcal{A}$ is basic. ∎