# On the Colin de Verdière graph parameter 

Notes for our seminar
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This is an attempt to define the Colin de Verdière graph parameter purely in terms of the nullspace embedding.

## 1. The Colin de Verdière graph parameter

Colin de Verdière [1]
Let $G=([n], E)$ be an undirected graph. The corank of a matrix $M$ is the dimension of its nullspace $\operatorname{ker}(M)$.

The Colin de Verdière parameter $\mu(G)$ [1] is defined to be the maximal corank of any symmetric $n \times n$ matrix $M$ with $M_{i, j}<0$ if $i j \in E$ and $M_{i, j}=0$ if $i \neq j$ and $i j \notin E$, with precisely one negative eigenvalue and having the Strong Arnold Property:
there is no nonzero real symmetric $n \times n$ matrix $X$ with $M X=0$ and $X_{i j}=0$ whenever $i$ and $j$ are equal or adjacent.

## 2. The Strong Arnold Property and quadrics

The Strong Arnold Property of $M$ can be formulated in terms of the nullspace embedding defined by $M$. Let $G=([n], E)$ be an undirected graph and let $M$ be a symmetric $n \times n$ matrix with $M_{i, j}<0$ if $i j \in E$ and $M_{i, j}=0$ if $i \neq j$ and $i j \notin E$, with corank $d$, and with precisely one negative eigenvalue. Let $b_{1}, \ldots, b_{d} \in \mathbb{R}^{n}$ be a basis of $\operatorname{ker}(M)$. Define, for each $i \in[n]$, the vector $u_{i} \in \mathbb{R}^{d}$ by: $\left(u_{i}\right)_{j}:=\left(b_{j}\right)_{i}$, for $j=1, \ldots, d$. So we have $u:[n] \rightarrow \mathbb{R}^{d}$. Then $u$ is called the nullspace embedding of $G$ defined by $M$. Note that $u$ is unique up to linear transformations of $\mathbb{R}^{d}$.

The Strong Arnold Property of $M$ is in fact a property only of the graph $G$ and the function $i \mapsto\left\langle u_{i}\right\rangle$. (Throughout, $\langle\ldots\rangle$ denotes the linear space spanned by ....) When we have $u:[n] \rightarrow \mathbb{R}^{d}$, define $|G|$ to be the following subset of $\mathbb{R}^{d}$ :

$$
\begin{equation*}
|G|:=\bigcup\left\{\left\langle u_{i}\right\rangle \mid i \in[n]\right\} \cup \bigcup\left\{\left\langle u_{i}, u_{j}\right\rangle \mid i j \in E\right\} . \tag{2}
\end{equation*}
$$

A subset $Q$ of $\mathbb{R}^{d}$ is called a homogeneous quadric if it is the solution set of a nonzero homogeneous quadratic equation. The following was observed in [3]:

Proposition 1. M has the Strong Arnold Property if and only $|G|$ is not contained in any homogeneous quadric.

Proof. Let $U$ be the $d \times n$ matrix with as columns the vectors $u_{i}$ for $i \in[n]$.
Suppose that some homogeneous quadric $Q=\left\{y \mid y^{\top} N y=0\right\}$ contains $|G|$, where $N$ is a nonzero symmetric $d \times d$ matrix. Then $X:=U^{\top} N U$ is a nonzero symmetric $n \times n$ matrix that contradicts the Strong Arnold Property (11).

Conversely, suppose that $M$ has not the Strong Arnold Property. Let $X$ be a matrix as in (1). As $M X=0$ and as $X$ is symmetric, we have $X=U^{\top} N U$ for some nonzero symmetric $d \times d$ matrix $N$. Then $Q:=\left\{y \mid y^{\top} N y=0\right\}$ is a homogeneous quadric containing $|G|$.

## 3. $M$ exists iff ...

Having characterized the Strong Arnold Property in terms of the nullspace embedding, we consider in how much the existence of the corresponding matrix $M$ can be expressed in terms of the nullspace embedding $u_{1}, \ldots, u_{n} \in \mathbb{R}^{d}$.

We can assume that $M$ has eigenvalue -1 with eigenvector 1. Indeed, by Brouwer's fixed point theorem, there exists $x \geq 0$ with $\sum_{i=1}^{n} x_{i}=1$ and $\Delta_{x}^{2} M x=\lambda x$ for some $\lambda<0$. So $\left(\Delta_{x} M \Delta_{x}\right) \mathbf{1}=\lambda \mathbf{1}$ for some $\lambda<0$. As $\Delta_{x} M \Delta_{x}$ has precisely one eigenvalue and the same rank as $M$, we can replace $M$ by $\Delta_{x} M \Delta_{x}$. Scaling $M$ then yields eigenvalue -1 .

Fix $G=([n], E)$, a positive $a \in \mathbb{R}^{n}, U \in \mathbb{R}^{n \times d}$, and $W \in \mathbb{R}^{n \times(n-d-1)}$ such that the matrix $[a, U, W]$ is orthogonal. (So $U$ takes the role of $\left[u_{1}, \ldots, u_{n}\right]^{\top}$.)

Define $p_{i}:=a_{i}^{-1} u_{i}, v_{i}:=a_{i}^{-1} w_{i}$, and $\beta_{i}:=a_{i}^{2}$ for each $i$. Then $\sum_{i=1}^{n} \beta_{i} p_{i}=0$ and $\sum_{i=1}^{n} \beta_{i}=1$.

## Proposition 2.

There exists a symmetric matrix $M \in \mathbb{R}^{n \times n}$ of corank $d$, with precisely one negative eigenvalue, with eigenvector $a$, and satisfying $M U=0, M_{i, j}=0$ if $i \neq j$ and $i j \notin E, M_{i, j}<0$ if $i j \in E$,
if and only if
for all $x_{1}, \ldots, x_{n} \in \mathbb{R}^{d}$ and positive semidefinite $P \in \mathbb{R}^{(n-d-1) \times(n-d-1)}$ : if

$$
\begin{equation*}
\left(p_{i}-p_{j}\right)^{\top}\left(x_{i}-x_{j}\right)+\frac{1}{2}\left(v_{i}-v_{j}\right)^{\top} P\left(v_{i}-v_{j}\right) \leq 0 \tag{4}
\end{equation*}
$$

for each $i j \in E$, then

$$
\sum_{i=1}^{n} \beta_{i}\left(p_{i}^{\top} x_{i}+\frac{1}{2} v_{i}^{\top} P v_{i}\right) \leq 0
$$

equality implying that $P=0$ and $\left(p_{i}-p_{j}\right)^{\top}\left(x_{i}-x_{j}\right)=0$ for all $i j \in E$.
Proof. Define
$\mathcal{K}:=\left\{K \in \mathbb{R}^{(n-d-1) \times(n-d-1)} \mid K\right.$ symmetric, $\left(W K W^{\boldsymbol{\top}}\right)_{i, j}=a_{i} a_{j}$ if $i \neq j$ and $i j \notin E$ and $\left(W K W^{\boldsymbol{\top}}\right)_{i, j}<a_{i} a_{j}$ if $\left.i j \in E\right\}$
$=\left\{K \in \mathbb{R}^{(n-d-1) \times(n-d-1)} \mid K\right.$ symmetric, $\operatorname{tr}\left(K W^{\top} E_{i, j} W\right)=a_{i} a_{j}$ if $i \neq j$ and
$i j \notin E$ and $\operatorname{tr}\left(K W^{\top} E_{i, j} W\right)<a_{i} a_{j}$ if $\left.i j \in E\right\}$.
Then (3) is equivalent to: $\mathcal{K}$ contains a positive definite matrix $K$.
Indeed, we can assume that $M$ has negative eigenvalue -1 . Then

$$
\left[\begin{array}{c}
a^{\top}  \tag{6}\\
U^{\top} \\
W^{\top}
\end{array}\right] M[a, U, W]=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & K
\end{array}\right]
$$

for some positive definite $K$. Then

$$
M=[a, U, W]\left[\begin{array}{ccc}
-1 & 0 & 0  \tag{7}\\
0 & 0 & 0 \\
0 & 0 & K
\end{array}\right]\left[\begin{array}{c}
a^{\top} \\
U^{\top} \\
W^{\top}
\end{array}\right]=-a a^{\top}+W K W^{\top} .
$$

So $M$ as in (3) exists if and only if $\mathcal{K}$ contains a positive definite matrix. By convexity, this last is equivalent to: there is no nonzero positive semidefinite matrix $P \in \mathbb{R}^{(n-d-1) \times(n-d-1)}$ such that $\operatorname{tr}(P K) \leq 0$ for all $K \in \mathcal{K}$, that is, $\operatorname{tr}(P K)>0$ for some $K \in \mathcal{K}$; equivalently: the following system of linear inequalities has a solution $K$ :
(i) $-\operatorname{tr}(P K)<0$,
(ii) $\operatorname{tr}\left(K W^{\top} E_{i, j} W\right)<a_{i} a_{j}$ for each $i j \in E$,
(iii) $\operatorname{tr}\left(K W^{\top} E_{i, j} W\right)=a_{i} a_{j}$ for each $i j \notin E$ with $i \neq j$.

By Motzkin's transposition theorem (see Corollary 7.1k in [2]), this is equivalent to: for each nonzero positive semidefinite matrix $P \in \mathbb{R}^{(n-d-1) \times(n-d-1)}$ : if $\mu \geq 0$ and $B \in \mathbb{R}^{n \times n}$ is symmetric and satisfies $B_{i, i}=0$ for all $i$, and $B_{i, j} \geq 0$ if $i j \in E$, and $-\mu P+W^{\top} B W=0$, then
(i) $a^{\top} B a \geq 0$,
(ii) if $a^{\top} B a=0$, then $\mu=0$ and $B_{i, j}=0$ if $i j \in E$.

Since the conditions are homogeneous, we can assume $\mu=0$ or $\mu=1$. So the existence of $M$ is equivalent to: for each symmetric $B \in \mathbb{R}^{n \times n}$ with $B_{i, i}=0$ for all $i$, and $B_{i, j} \geq 0$ if $i j \in E$ :
(i) if $W^{\top} B W=0$, then $a^{\top} B a \geq 0$,
(ii) if $W^{\top} B W=0$ and $a^{\top} B a=0$, then $B_{i, j}=0$ for all $i j \in E$,
(iii) if $W^{\top} B W$ is nonzero and positive semidefinite, then $a^{\top} B a>0$.
[Conditions 10](i) and (ii) (i.e., the case $\mu=0$ ) are in fact equivalent to: $\mathcal{K} \neq \emptyset$. That is, to the existence of a symmetric matrix $M \in \mathbb{R}^{n \times n}$ with $M_{i, j}=0$ if $i \neq j$ and $i j \notin E$ and $M_{i, j}<0$ if $i j \in E$, and such that $M a=-a$ and $M U=0$. (So no condition on the other eigenvalues.)]

For any symmetric $B \in \mathbb{R}^{n \times n}$ there exist unique $y \in \mathbb{R}^{n}, Z \in \mathbb{R}^{n \times d}$, and a symmetric $P \in \mathbb{R}^{(n-d-1) \times(n-d-1)}$ with

$$
B=[a, U]\left[\begin{array}{c}
y^{\top}  \tag{11}\\
Z^{\top}
\end{array}\right]+[y, Z]\left[\begin{array}{c}
a^{\top} \\
U^{\top}
\end{array}\right]+W P W^{\top} .
$$

Note $P=W^{\top} B W$.
If $B_{i, i}=0$ we can eliminate $y_{i}$ : since

$$
\begin{equation*}
a_{i} y_{i}+u_{i}^{\top} z_{i}+a_{i} y_{i}+z_{i}^{\top} u_{i}+w_{i}^{\top} P w_{i}=B_{i, i}=0, \tag{12}
\end{equation*}
$$

we have

$$
\begin{equation*}
y_{i}=-a_{i}^{-1}\left(u_{i}^{\top} z_{i}+\frac{1}{2} w_{i}^{\top} P w_{i}\right) . \tag{13}
\end{equation*}
$$

Therefore, for all $i, j$ :

$$
\begin{align*}
& B_{i, j}=a_{i} y_{j}+u_{i}^{\top} z_{j}+a_{j} y_{i}+z_{i}^{\top} u_{j}+w_{i}^{\top} P w_{j}=  \tag{14}\\
& a_{i}\left(-a_{j}^{-1}\left(u_{j}^{\top} z_{j}+\frac{1}{2} w_{j}^{\top} P w_{j}\right)\right)+u_{i}^{\top} z_{j}+a_{j}\left(-a_{i}^{-1}\left(u_{i}^{\top} z_{i}+\frac{1}{2} w_{i}^{\top} P w_{i}\right)\right)+z_{i}^{\top} u_{j}+w_{i}^{\top} P w_{j}= \\
& a_{i} a_{j}\left(-p_{j}^{\top} x_{j}+p_{i}^{\top} x_{j}-p_{i}^{\top} x_{i}+x_{i}^{\top} p_{j}-\frac{1}{2}\left(v_{i}-v_{j}\right)^{\top} P\left(v_{i}-v_{j}\right)\right)= \\
& -a_{i} a_{j}\left(\left(p_{i}-p_{j}\right)^{\top}\left(x_{i}-x_{j}\right)+\frac{1}{2}\left(v_{i}-v_{j}\right)^{\top} P\left(v_{i}-v_{j}\right)\right) .
\end{align*}
$$

where $x_{i}:=a_{i}^{-1} z_{i}$ for each $i$. Then, with 11) and 13), since $a^{\top} U=0$ and $a^{\top} W=0$ :

$$
\begin{align*}
& a^{\top} B a=a^{\top} a y^{\top} a+a^{\top} y a^{\top} a=2 y^{\top} a=-\sum_{i=1}^{n}\left(2 u_{i}^{\top} z_{i}+w_{i}^{\top} P w_{i}\right)=  \tag{15}\\
& -\sum_{i=1}^{n} \beta_{i}\left(2 p_{i}^{\top} x_{i}+v_{i}^{\top} P v_{i}\right) .
\end{align*}
$$

Therefore, the condition: for each symmetric $B \in \mathbb{R}^{n \times n}$ with $B_{i, i}=0$ for all $i$, and $B_{i, j} \geq 0$ if $i j \in E$ (10) holds, is equivalent to (4).

Set $P=Q^{\top} Q$ for some matrix $Q \in \mathbb{R}^{s \times(n-d-1)}$ for some $s$. Consider $p_{i}(t):=\left(p_{i}+\right.$ $\left.t x_{i}, \sqrt{t} Q v_{i}\right) \in \mathbb{R}^{d+s}$ for $t \in \mathbb{R}$, for each $i$.

If $P=0$, then

$$
\begin{equation*}
\left.x_{i}=\frac{d}{d t} p_{i}(t)\right\rfloor_{t=0} . \tag{16}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\left.\left(p_{i}-p_{j}\right)^{\mathrm{T}}\left(x_{i}-x_{j}\right)=\frac{1}{2} \frac{d}{d t}\left|p_{i}(t)-p_{j}(t)\right|^{2}\right\rfloor_{t=0} \tag{17}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\sum_{i=1}^{n} \beta_{i} p_{i}^{\top} x_{i}=\frac{1}{2} \frac{d}{d t} \sum_{i=1}^{n} \beta_{i}\left|p_{i}(t)\right|^{2}\right\rfloor_{t=0} \tag{18}
\end{equation*}
$$

## References

[1] Y. Colin de Verdière, Sur un nouvel invariant des graphes et un critère de planarité, Journal of Combinatorial Theory, Series B 50 (1990) 11-21 [English translation: On a new graph invariant and a criterion for planarity, in: Graph Structure Theory (N. Robertson, P. Seymour, eds.), American Mathematical Society, Providence, Rhode Island, 1993, pp. 137-147].
[2] A. Schrijver, Theory of Linear and Integer Programming, Wiley, Chichester, 1986.
[3] A. Schrijver, B. Sevenster, The Strong Arnold Property for 4-connected flat graphs, Linear Algebra and Its Applications 522 (2017) 153-160.

