On the Colin de Verdière graph parameter

Notes for our seminar Lex Schrijver

This is an attempt to define the Colin de Verdière graph parameter purely in terms of the nullspace embedding.

1. The Colin de Verdière graph parameter

Colin de Verdière [1]

Let G = ([n], E) be an undirected graph. The *corank* of a matrix M is the dimension of its nullspace ker(M).

The Colin de Verdière parameter $\mu(G)$ [1] is defined to be the maximal corank of any symmetric $n \times n$ matrix M with $M_{i,j} < 0$ if $ij \in E$ and $M_{i,j} = 0$ if $i \neq j$ and $ij \notin E$, with precisely one negative eigenvalue and having the Strong Arnold Property:

(1) there is no nonzero real symmetric $n \times n$ matrix X with MX = 0 and $X_{ij} = 0$ whenever i and j are equal or adjacent.

2. The Strong Arnold Property and quadrics

The Strong Arnold Property of M can be formulated in terms of the nullspace embedding defined by M. Let G = ([n], E) be an undirected graph and let M be a symmetric $n \times n$ matrix with $M_{i,j} < 0$ if $ij \in E$ and $M_{i,j} = 0$ if $i \neq j$ and $ij \notin E$, with corank d, and with precisely one negative eigenvalue. Let $b_1, \ldots, b_d \in \mathbb{R}^n$ be a basis of ker(M). Define, for each $i \in [n]$, the vector $u_i \in \mathbb{R}^d$ by: $(u_i)_j := (b_j)_i$, for $j = 1, \ldots, d$. So we have $u : [n] \to \mathbb{R}^d$. Then u is called the *nullspace embedding of G defined by M*. Note that u is unique up to linear transformations of \mathbb{R}^d .

The Strong Arnold Property of M is in fact a property only of the graph G and the function $i \mapsto \langle u_i \rangle$. (Throughout, $\langle \ldots \rangle$ denotes the linear space spanned by \ldots .) When we have $u : [n] \to \mathbb{R}^d$, define |G| to be the following subset of \mathbb{R}^d :

(2)
$$|G| := \bigcup \{ \langle u_i \rangle \mid i \in [n] \} \cup \bigcup \{ \langle u_i, u_j \rangle \mid ij \in E \}.$$

A subset Q of \mathbb{R}^d is called a *homogeneous quadric* if it is the solution set of a nonzero homogeneous quadratic equation. The following was observed in [3]:

Proposition 1. *M* has the Strong Arnold Property if and only |G| is not contained in any homogeneous quadric.

Proof. Let U be the $d \times n$ matrix with as columns the vectors u_i for $i \in [n]$.

Suppose that some homogeneous quadric $Q = \{y \mid y^{\mathsf{T}}Ny = 0\}$ contains |G|, where N is a nonzero symmetric $d \times d$ matrix. Then $X := U^{\mathsf{T}}NU$ is a nonzero symmetric $n \times n$ matrix that contradicts the Strong Arnold Property (1).

Conversely, suppose that M has not the Strong Arnold Property. Let X be a matrix as in (1). As MX = 0 and as X is symmetric, we have $X = U^{\mathsf{T}}NU$ for some nonzero symmetric $d \times d$ matrix N. Then $Q := \{y \mid y^{\mathsf{T}} N y = 0\}$ is a homogeneous quadric containing |G|.

3. M exists iff ...

Having characterized the Strong Arnold Property in terms of the nullspace embedding, we consider in how much the existence of the corresponding matrix M can be expressed in terms of the nullspace embedding $u_1, \ldots, u_n \in \mathbb{R}^d$.

We can assume that M has eigenvalue -1 with eigenvector 1. Indeed, by Brouwer's fixed point theorem, there exists $x \ge 0$ with $\sum_{i=1}^{n} x_i = 1$ and $\Delta_x^2 M x = \lambda x$ for some $\lambda < 0$. So $(\Delta_x M \Delta_x) \mathbf{1} = \lambda \mathbf{1}$ for some $\lambda < 0$. As $\Delta_x M \Delta_x$ has precisely one eigenvalue and the same rank as M, we can replace M by $\Delta_x M \Delta_x$. Scaling M then yields eigenvalue -1.

Fix G = ([n], E), a positive $a \in \mathbb{R}^n$, $U \in \mathbb{R}^{n \times d}$, and $W \in \mathbb{R}^{n \times (n-d-1)}$ such that the

matrix [a, U, W] is orthogonal. (So U takes the role of $[u_1, \ldots, u_n]^{\mathsf{T}}$.) Define $p_i := a_i^{-1}u_i, v_i := a_i^{-1}w_i$, and $\beta_i := a_i^2$ for each *i*. Then $\sum_{i=1}^n \beta_i p_i = 0$ and $\sum_{i=1}^{n} \beta_i = 1.$

Proposition 2.

There exists a symmetric matrix $M \in \mathbb{R}^{n \times n}$ of corank d, with precisely one (3)negative eigenvalue, with eigenvector a, and satisfying MU = 0, $M_{i,j} = 0$ if $i \neq j$ and $ij \notin E$, $M_{i,j} < 0$ if $ij \in E$,

if and only if

(4) for all
$$x_1, \ldots, x_n \in \mathbb{R}^d$$
 and positive semidefinite $P \in \mathbb{R}^{(n-d-1)\times(n-d-1)}$: if
 $(p_i - p_j)^\mathsf{T}(x_i - x_j) + \frac{1}{2}(v_i - v_j)^\mathsf{T}P(v_i - v_j) \le 0$

for each $ij \in E$, then

$$\sum_{i=1}^{n} \beta_i (p_i^{\mathsf{T}} x_i + \frac{1}{2} v_i^{\mathsf{T}} P v_i) \le 0,$$

equality implying that P = 0 and $(p_i - p_j)^{\mathsf{T}}(x_i - x_j) = 0$ for all $ij \in E$.

Proof. Define

(5)
$$\mathcal{K} := \{ K \in \mathbb{R}^{(n-d-1)\times(n-d-1)} \mid K \text{ symmetric, } (WKW^{\mathsf{T}})_{i,j} = a_i a_j \text{ if } i \neq j \text{ and} \\ ij \notin E \text{ and } (WKW^{\mathsf{T}})_{i,j} < a_i a_j \text{ if } ij \in E \} \\ = \{ K \in \mathbb{R}^{(n-d-1)\times(n-d-1)} \mid K \text{ symmetric, } \operatorname{tr}(KW^{\mathsf{T}}E_{i,j}W) = a_i a_j \text{ if } i \neq j \text{ and} \\ ij \notin E \text{ and } \operatorname{tr}(KW^{\mathsf{T}}E_{i,j}W) < a_i a_j \text{ if } ij \in E \}.$$

Then (3) is equivalent to: \mathcal{K} contains a positive definite matrix K. Indeed, we can assume that M has negative eigenvalue -1. Then

(6)
$$\begin{bmatrix} a^{\mathsf{T}} \\ U^{\mathsf{T}} \\ W^{\mathsf{T}} \end{bmatrix} M[a, U, W] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K \end{bmatrix}$$

for some positive definite K. Then

(7)
$$M = [a, U, W] \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} a^{\mathsf{T}} \\ U^{\mathsf{T}} \\ W^{\mathsf{T}} \end{bmatrix} = -aa^{\mathsf{T}} + WKW^{\mathsf{T}}$$

So M as in (3) exists if and only if \mathcal{K} contains a positive definite matrix. By convexity, this last is equivalent to: there is no nonzero positive semidefinite matrix $P \in \mathbb{R}^{(n-d-1)\times(n-d-1)}$ such that $\operatorname{tr}(PK) \leq 0$ for all $K \in \mathcal{K}$, that is, $\operatorname{tr}(PK) > 0$ for some $K \in \mathcal{K}$; equivalently: the following system of linear inequalities has a solution K:

(8) (i)
$$-\operatorname{tr}(PK) < 0$$
,
(ii) $\operatorname{tr}(KW^{\mathsf{T}}E_{i,j}W) < a_ia_j$ for each $ij \in E$,
(iii) $\operatorname{tr}(KW^{\mathsf{T}}E_{i,j}W) = a_ia_j$ for each $ij \notin E$ with $i \neq j$.

By Motzkin's transposition theorem (see Corollary 7.1k in [2]), this is equivalent to: for each nonzero positive semidefinite matrix $P \in \mathbb{R}^{(n-d-1)\times(n-d-1)}$: if $\mu \geq 0$ and $B \in \mathbb{R}^{n\times n}$ is symmetric and satisfies $B_{i,i} = 0$ for all *i*, and $B_{i,j} \geq 0$ if $ij \in E$, and $-\mu P + W^{\mathsf{T}}BW = 0$, then

(9) (i)
$$a^{\mathsf{T}}Ba \ge 0$$
,
(ii) if $a^{\mathsf{T}}Ba = 0$, then $\mu = 0$ and $B_{i,j} = 0$ if $ij \in E$.

Since the conditions are homogeneous, we can assume $\mu = 0$ or $\mu = 1$. So the existence of M is equivalent to: for each symmetric $B \in \mathbb{R}^{n \times n}$ with $B_{i,i} = 0$ for all i, and $B_{i,j} \ge 0$ if $ij \in E$:

(10) (i) if $W^{\mathsf{T}}BW = 0$, then $a^{\mathsf{T}}Ba \ge 0$, (ii) if $W^{\mathsf{T}}BW = 0$ and $a^{\mathsf{T}}Ba = 0$, then $B_{i,j} = 0$ for all $ij \in E$, (iii) if $W^{\mathsf{T}}BW$ is nonzero and positive semidefinite, then $a^{\mathsf{T}}Ba > 0$.

[Conditions (10)(i) and (ii) (i.e., the case $\mu = 0$) are in fact equivalent to: $\mathcal{K} \neq \emptyset$. That is, to the existence of a symmetric matrix $M \in \mathbb{R}^{n \times n}$ with $M_{i,j} = 0$ if $i \neq j$ and $ij \notin E$ and $M_{i,j} < 0$ if $ij \in E$, and such that Ma = -a and MU = 0. (So no condition on the other eigenvalues.)]

For any symmetric $B \in \mathbb{R}^{n \times n}$ there exist unique $y \in \mathbb{R}^n$, $Z \in \mathbb{R}^{n \times d}$, and a symmetric $P \in \mathbb{R}^{(n-d-1) \times (n-d-1)}$ with

(11)
$$B = [a, U] \begin{bmatrix} y^{\mathsf{T}} \\ Z^{\mathsf{T}} \end{bmatrix} + [y, Z] \begin{bmatrix} a^{\mathsf{T}} \\ U^{\mathsf{T}} \end{bmatrix} + WPW^{\mathsf{T}}.$$

Note $P = W^{\mathsf{T}} B W$.

If $B_{i,i} = 0$ we can eliminate y_i : since

(12)
$$a_i y_i + u_i^{\mathsf{T}} z_i + a_i y_i + z_i^{\mathsf{T}} u_i + w_i^{\mathsf{T}} P w_i = B_{i,i} = 0,$$

we have

(13)
$$y_i = -a_i^{-1}(u_i^{\mathsf{T}} z_i + \frac{1}{2} w_i^{\mathsf{T}} P w_i).$$

Therefore, for all i, j:

(14)
$$B_{i,j} = a_i y_j + u_i^{\mathsf{T}} z_j + a_j y_i + z_i^{\mathsf{T}} u_j + w_i^{\mathsf{T}} P w_j = a_i (-a_j^{-1} (u_j^{\mathsf{T}} z_j + \frac{1}{2} w_j^{\mathsf{T}} P w_j)) + u_i^{\mathsf{T}} z_j + a_j (-a_i^{-1} (u_i^{\mathsf{T}} z_i + \frac{1}{2} w_i^{\mathsf{T}} P w_i)) + z_i^{\mathsf{T}} u_j + w_i^{\mathsf{T}} P w_j = a_i a_j (-p_j^{\mathsf{T}} x_j + p_i^{\mathsf{T}} x_j - p_i^{\mathsf{T}} x_i + x_i^{\mathsf{T}} p_j - \frac{1}{2} (v_i - v_j)^{\mathsf{T}} P (v_i - v_j)) = -a_i a_j ((p_i - p_j)^{\mathsf{T}} (x_i - x_j) + \frac{1}{2} (v_i - v_j)^{\mathsf{T}} P (v_i - v_j)).$$

where $x_i := a_i^{-1} z_i$ for each *i*. Then, with (11) and (13), since $a^{\mathsf{T}} U = 0$ and $a^{\mathsf{T}} W = 0$:

(15)
$$a^{\mathsf{T}}Ba = a^{\mathsf{T}}ay^{\mathsf{T}}a + a^{\mathsf{T}}ya^{\mathsf{T}}a = 2y^{\mathsf{T}}a = -\sum_{i=1}^{n} (2u_{i}^{\mathsf{T}}z_{i} + w_{i}^{\mathsf{T}}Pw_{i}) = -\sum_{i=1}^{n} \beta_{i}(2p_{i}^{\mathsf{T}}x_{i} + v_{i}^{\mathsf{T}}Pv_{i}).$$

Therefore, the condition: for each symmetric $B \in \mathbb{R}^{n \times n}$ with $B_{i,i} = 0$ for all i, and $B_{i,j} \ge 0$ if $ij \in E$ (10) holds, is equivalent to (4).

Set $P = Q^{\mathsf{T}}Q$ for some matrix $Q \in \mathbb{R}^{s \times (n-d-1)}$ for some s. Consider $p_i(t) := (p_i + tx_i, \sqrt{t}Qv_i) \in \mathbb{R}^{d+s}$ for $t \in \mathbb{R}$, for each i.

If P = 0, then

(16)
$$x_i = \frac{d}{dt} p_i(t) \big|_{t=0}.$$

Hence

(17)
$$(p_i - p_j)^{\mathsf{T}}(x_i - x_j) = \frac{1}{2} \frac{d}{dt} |p_i(t) - p_j(t)|^2 \Big|_{t=0}$$

and

(18)
$$\sum_{i=1}^{n} \beta_{i} p_{i}^{\mathsf{T}} x_{i} = \frac{1}{2} \frac{d}{dt} \sum_{i=1}^{n} \beta_{i} |p_{i}(t)|^{2} \Big|_{t=0}.$$

References

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