7. Cliques, cocliques, and colourings

7.1. Introduction

We have seen in Chapter 5 that in any graph $G = (V, E)$, a matching of maximum cardinality can be found in polynomial time. Similarly, an edge-cover of minimum cardinality can be found in polynomial time.

On the other hand, it is NP-complete to find a maximum-cardinality coclique in a graph. That is, determining $\alpha(G)$ is NP-complete. To be more precise, the problem COCLIQUE is:

(1) given: a graph G and a natural number k,
decide: if $\alpha(G) \geq k$.

Then:

Theorem 7.1. *The problem* COCLIQUE *is* NP*-complete.*

Proof. We reduce SAT to COCLIQUE. Let $C_1 \wedge \cdots \wedge C_k$ be a boolean expression in the variables $x_1, \ldots, x_n$. Let $x_1, \neg x_1, \ldots, x_n, \neg x_n$ be the *literals*. Consider the graph $G = (V, E)$ with $V := \{(\sigma, i) | \sigma \text{ is a literal in } C_i\}$ and $E := \{\{(\sigma, i), (\tau, j)\} | i = j \text{ or } \sigma = \neg\tau\}$. Then the expression is satisfiable if and only if G has a coclique of size k. ∎

Since by Gallai's theorem Theorem 3.1, $\alpha(G) = |V| - \tau(G)$, also determining the vertex-cover number $\tau(G)$ is NP-complete.

A *clique* in a graph $G = (V, E)$ is a subset C of V such that u and w are adjacent for any two distinct u, w in C. The *clique number* of G, denoted by $\omega(G)$, is the maximum cardinality of any clique in G.

Observe that a subset C of V is a clique in G if and only if C is a coclique in the complementary graph $\overline{G}$. So finding a maximum-cardinality clique in G is equivalent to finding a maximum-cardinality coclique in $\overline{G}$, and $\omega(G) = \alpha(\overline{G})$. As determining $\alpha(G)$ is NP-complete, also determining $\omega(G)$ is NP-complete.

A *(vertex-)colouring* of a graph $G = (V, E)$ is a partition of V into cocliques $C_1, \ldots, C_k$. The sets $C_1, \ldots, C_k$ are called the *colours* of the colouring. The *(vertex-)colouring number*, or *(vertex-)chromatic number*, of G, denoted by $\gamma(G)$, is the minimum number of colours in any vertex-colouring of G. A graph G is called *k-colourable* if $\gamma(G) \leq k$.

Well-known is the *four-colour conjecture* (*4CC*), stating that $\gamma(G) \leq 4$ for each planar graph G. This conjecture was proved by Appel and Haken [1977] and Appel, Haken, and Koch [1977], and is now called the *four-colour theorem* (*4CT*).

Again, it is NP-complete to decide if a graph is k-colourable. In fact, it is NP-complete to decide if a planar graph is 3-colourable. [Note that one can decide in polynomial time if a graph G is 2-colourable, as bipartiteness can be checked in polynomial time.]

These NP-completeness results imply that if NP$\neq$co-NP, then one may not expect a min-max relation characterizing the coclique number $\alpha(G)$, the vertex-cover number $\tau(G)$, the clique number $\omega(G)$, or the colouring number $\gamma(G)$ of a graph G.

There is a trivial upper bound on the colouring number:

(2) $$\gamma(G) \leq \Delta(G) + 1,$$

where $\Delta(G)$ denotes the maximum valency of G. Brooks [1941] sharpened this inequality as follows:

Theorem 7.2 (Brooks' theorem). *For any connected graph G one has $\gamma(G) \leq \Delta(G)$, except if $G = K_n$ or $G = C_{2n+1}$ for some $n \geq 1$.*[18]

Another inequality relates the clique number and the colouring number:

(3) $$\omega(G) \leq \gamma(G).$$

This is easy, since in any clique all vertices should have different colours.

But there are several graphs which have strict inequality in (3). We mention the odd circuits C_{2k+1}, with $2k+1 \geq 5$: then $\omega(C_{2k+1}) = 2$ and $\gamma(C_{2k+1}) = 3$. Moreover, for the complement $\overline{C_{2k+1}}$ of any such graph we have: $\omega(\overline{C_{2k+1}}) = k$ and $\gamma(\overline{C_{2k+1}}) = k+1$.

It is a conjecture of Berge [1963] that these graphs are crucial: [19]

Strong perfect graph conjecture: Let G be a graph. If $\omega(G) < \gamma(G)$ then G contains C_n or $\overline{C_n}$, for some odd $n \geq 5$, as an *induced* subgraph.

Another conjecture is due to Hadwiger [1943]. Since there exist graphs with $\omega(G) < \gamma(G)$, it is not true that if $\gamma(G) \geq n$ then G contains the complete graph K_n on n vertices as a subgraph. However, Hadwiger conjectured the following, where a graph H is called a *minor* of a graph G if H arises from some subgraph of G by contracting some (possible none) edges.

Hadwiger's conjecture: If $\gamma(G) \geq n$ then G contains K_n as a minor.

In other words, for each n, the graph K_n is the only graph G with the property that G is not $(n-1)$-colourable and each proper minor of G is $(n-1)$-colourable.

Hadwiger's conjecture is trivial for $n = 1, 2, 3$, and was shown by Hadwiger for $n = 4$ (see Exercise 7.8). As planar graphs do not contain K_5 as a minor, Hadwiger's conjecture for $n = 5$ implies the four-colour theorem. In fact, Wagner [1937] showed that Hadwiger's conjecture for $n = 5$ is equivalent to the four-colour conjecture. Recently, Robertson, Seymour, and Thomas [1993] showed that Hadwiger's conjecture is true also for $n = 6$, by showing that in that case it is equivalent to the four-colour theorem. For $n \geq 7$ Hadwiger's conjecture is unsettled.

Application 7.1: Map colouring. A well-known application of colouring the vertices of a graph is that of colouring the countries in a map in such a way that adjacent countries obtain different colours. So the four-colour theorem implies that if each country is connected, then the map can be

[18] Here C_k denotes the circuit with k vertices.

[19] Let $G = (V, E)$ be a graph and let $V' \subseteq V$. Then the subgraph of G *induced by* V', denoted by $G|V'$ is the graph (V', E'), where E' equals the set of all edges in E contained in V'. The graph $G|V'$ is called an *induced* subgraph of G.

coloured using not more than four colours. (One should not consider countries as 'adjacent' if they have a common boundary of measure 0 only.)

There are several other cases where colouring a map amounts to finding a minimum vertex-colouring in a graph. For instance, consider a map of the Paris Métro network (Figure 7.1).

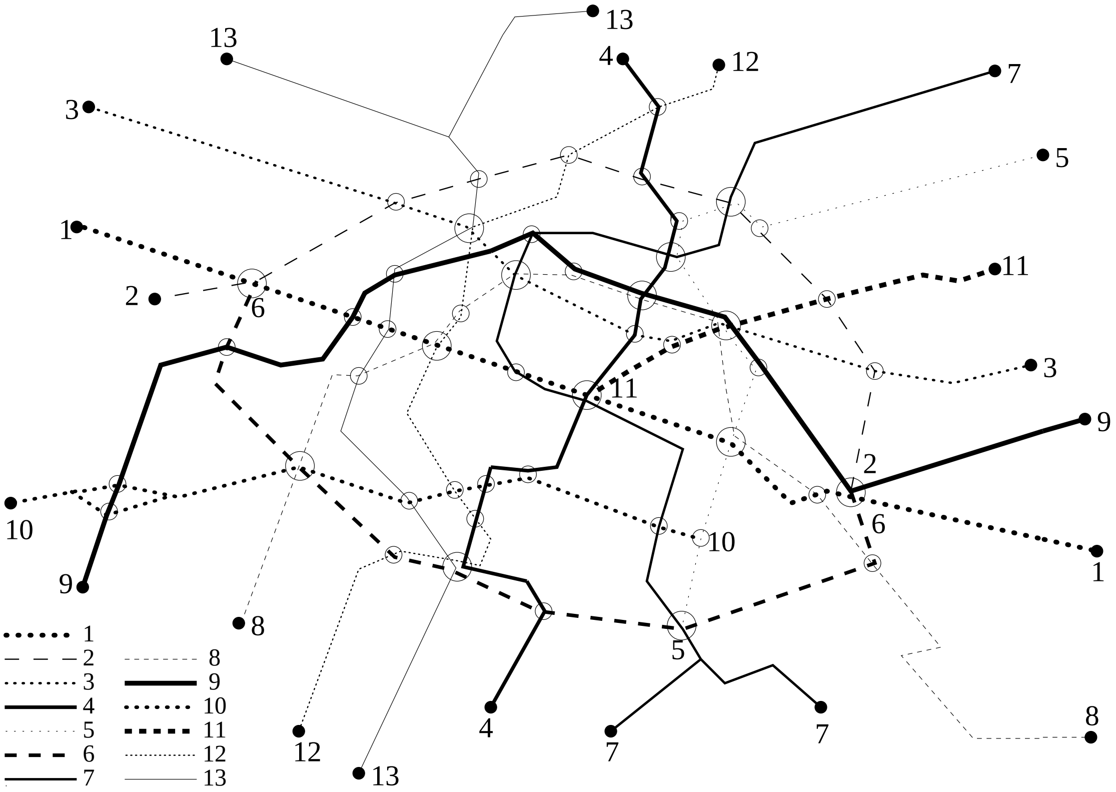


Figure 7.1

Suppose now that you want to print a coloured map of the network, indicating each of the 13 lines by a colour, in such a way that lines that cross each other or meet each other in a station, are indicated by different colours and in such a way that a minimum number of colours is used. This easily reduces to a graph colouring problem.

Application 7.2: Storage of goods, etc. Suppose you are the director of a circus and wish to transport your animals in a number of carriages, in such a way that no two of the animals put into one carriage eat each other, and in such a way that you use a minimum number of carriages.

This trivially reduces to a graph colouring problem. A similar problem is obtained if you have to store a number of chemicals in a minimum number of rooms of a storehouse, in such a way that no two of the chemicals stored in one room react upon each other in an unwanted way.

This problem may also occur when assigning multiple-bed rooms to school boys on a school trip.

Application 7.3: Assigning frequencies to radio stations, car phones, etc. Suppose one has to assign frequencies to radio stations in a certain area. Certain pairs of radio stations that are too close to each other cannot be assigned the same frequency as it would cause mutual interference.

Such pairs of radio stations form the edge set of a graph G, with vertex set the set of radio stations. The chromatic number of G is equal to the minimum number of different frequencies that one needs in order to assign a frequency to each of the stations.

The problem occurs also when assigning frequencies to car phones, where often in a very short time new frequencies should be determined.

Exercises

7.1. Determine $\omega(G)$ and $\gamma(G)$ for the graph G obtained from the Paris Métro map given in Application 7.1.

7.2. Colour the map of Figure 7.2 (from the April 1975 issue of *Scientific American*).

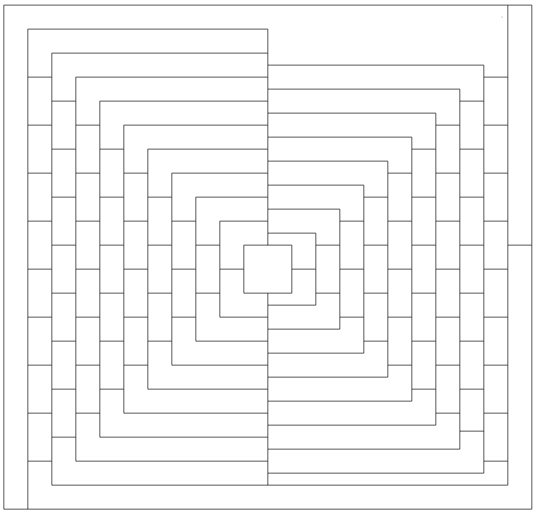


Figure 7.2

7.3. Show that if G is a bipartite graph, then $\omega(G) = \gamma(G)$.

7.4. Derive from Kőnig's edge cover theorem (Corollary 3.2a) that if G is the complement of a bipartite graph, then $\omega(G) = \gamma(G)$.

7.5. Derive Kőnig's edge cover theorem (Corollary 3.2a) from the strong perfect graph conjecture.

7.6. Let H be a bipartite graph and let G be the complement of the line-graph of H. Derive from Kőnig's matching theorem (Theorem 3.2) that $\omega(G) = \gamma(G)$.

7.7. Derive Kőnig's matching theorem (Theorem 3.2) from the strong perfect graph conjecture.

7.8. Let $G = (V, E)$ be a simple graph such that no minor of G is isomorphic to K_4. Show that $\gamma(G) \leq 3$.

[*Hint:* One may assume that G is not a forest or a circuit. Then G has a circuit not covering all vertices of G. As G has no K_4-minor, G is not 3-connected, that is, G has a vertex cut set of size less than 3; then $\gamma(G) \leq 3$ follows by induction.]

7.2. Edge-colourings of bipartite graphs

For any graph $G = (V, E)$, an *edge-colouring* is a partition $\Pi = \{M_1, \ldots, M_p\}$ of the edge set E, where each M_i is a matching. Each of these matchings is called a *colour*. Define the *edge-colouring number* or *edge-chromatic number* $\chi(G)$ by

(4) $$\chi(G) := \min\{|\Pi| \; |\Pi \text{ is an edge-colouring of } G\}.$$

Let $\Delta(G)$ denote the maximum degree of (the vertices of) G. Clearly,

(5) $$\chi(G) \geq \Delta(G),$$

since at each vertex v, the edges incident with v should have different colours. Again the triangle K_3 has strict inequality. Kőnig [1916] showed that for bipartite graphs the two numbers are equal.

Theorem 7.3 (Kőnig's edge-colouring theorem). *For any bipartite graph $G = (V, E)$ one has*

(6) $$\chi(G) = \Delta(G).$$

That is, the edge-colouring number of a bipartite graph is equal to its maximum degree.

Proof. First notice that the theorem is easy if $\Delta(G) \leq 2$. In that case, G consists of a number of vertex-disjoint paths and even circuits.

In the general case, colour as many edges of G as possible with $\Delta(G)$ colours, without giving the same colour to two intersecting edges. If all edges are coloured we are done, so suppose some edge $e = \{u, w\}$ is not coloured. At least one colour, say *red*, does not occur among the colours given to the edges incident with u. Similarly, there is a colour, say *blue*, not occurring at w. (Clearly, *red*$\neq$*blue*, since otherwise we could give edge e the colour *red*.)

Let H be the subgraph of G having as edges all *red* and *blue* edges of G, together with the edge e. Now $\Delta(H) = 2$, and hence $\chi(H) = \Delta(H) = 2$. So all edges occurring in H can be (re)coloured with *red* and *blue*. In this way we colour more edges of G than before. This contradicts the maximality assumption. ∎

This proof also gives a polynomial-time algorithm to find an edge-colouring with $\Delta(G)$ colours.

We remark here that Vizing [1964] proved that for general simple graphs G one has

(7) $$\Delta(G) \leq \chi(G) \leq \Delta(G) + 1.$$

Here 'simple' cannot be deleted, as is shown by the graph G with three vertices, where any two vertices are connected by two parallel edges: then $\Delta(G) = 4$ while $\chi(G) = 6$.

A theorem 'dual' to Kőnig's edge-colouring theorem was also shown by Kőnig. Note that the edge-colouring number $\chi(G)$ of a graph G is the minimum number of matchings needed to cover the edges of a bipartite graph. Dually, one can define:

(8) $$\xi(G) := \text{the maximum number of pairwise disjoint edge covers in } G.$$

So, in terms of colours, $\xi(G)$ is the maximum number of colours that can be used in colouring the edges of G in such a way that at each vertex all colours occur. Hence, if $\delta(G)$ denotes the minimum degree of G, then

(9) $$\xi(G) \leq \delta(G).$$

The triangle K_3 again is an example having strict inequality. For bipartite graphs however:

Corollary 7.3a. *For any bipartite graph $G = (V, E)$ one has*

(10) $$\xi(G) = \delta(G).$$

That is, the maximum number of pairwise disjoint edge covers is equal to the minimum degree.

Proof. One may derive from G a bipartite graph H, each vertex of which has degree $\delta(G)$ or 1, by repeated application of the following procedure:

(11) for any vertex v of degree larger than $\delta(G)$, add a new vertex u, and replace one of the edges incident with v, $\{v, w\}$ say, by $\{u, w\}$.

So there is a one-to-one correspondence between the edges of the final graph H and the edges of G. Since H has maximum degree $\delta(G)$, by Theorem 7.3 the edges of H can be coloured with $\delta(G)$ colours such that no two edges of the same colour intersect. So at any vertex of H of degree $\delta(G)$ all colours occur. This gives a colouring of the edges of G with $\delta(G)$ colours such that at any vertex of G all colours occur. ∎

Application 7.4: Scheduling classes. Suppose we have n classes and m teachers. In the following scheme it is indicated by an X which classes should be taught by which teachers (one lesson of one hour a day):

class:	1	2	3	4	5	6
teacher: a	X	X		X		
b	X		X	X	X	
c		X		X	X	
d		X				X
e	X		X		X	X
f	X		X	X		X
g		X	X		X	X

The question is: What is the minimum timespan in which all lessons can be scheduled?

Theorem 7.3 tells us that all lessons can be scheduled within a timespan of 4 hours. Indeed, make a bipartite graph G with colour classes $T :=$ set of teachers and $C :=$ set of classes, where $t \in T$ and $c \in C$ are connected if and only if teacher t should teach class c; that is, if there is an X in position (t, c) in the scheme.

In the above example G will have maximum degree $\Delta(G)$ equal to 4. Hence according to Theorem 7.3, the edge-colouring number $\chi(G)$ of G is also equal to 4. So we can colour the edges of G by 4 colours so that no two edges of the same colour have a vertex in common. That is, we can colour the X's in the scheme by 4 colours so that there are no two crosses of the same colour in any row or column. If every colour represent one hour, we obtain a schedule spanning 4 hours.

This application can be extended to the case where teachers can give more than one lesson a day to a class. In that case we obtain a bipartite graph with multiple edges.

For any k-edge-colouring of a graph $G = (V, E)$, we can assume that any two colours differ by at most 1 in size (if they differ more, one can exchange the two colours on one of the path components of the union of the two colours, to bring their cardinalities closer together). That is, each colour has size $\lfloor |E|/k \rfloor$ or $\lceil |E|/k \rceil$. It implies that there is a schedule in which no more than $\lceil |E|/k \rceil$ lessons are scheduled simultaneously. So the number of classrooms needed is $\lceil |E|/k \rceil$, which is clearly best possible if we want to schedule $|E|$ lessons within k hours.

Exercises

7.9. Determine a schedule for the following scheduling problems:

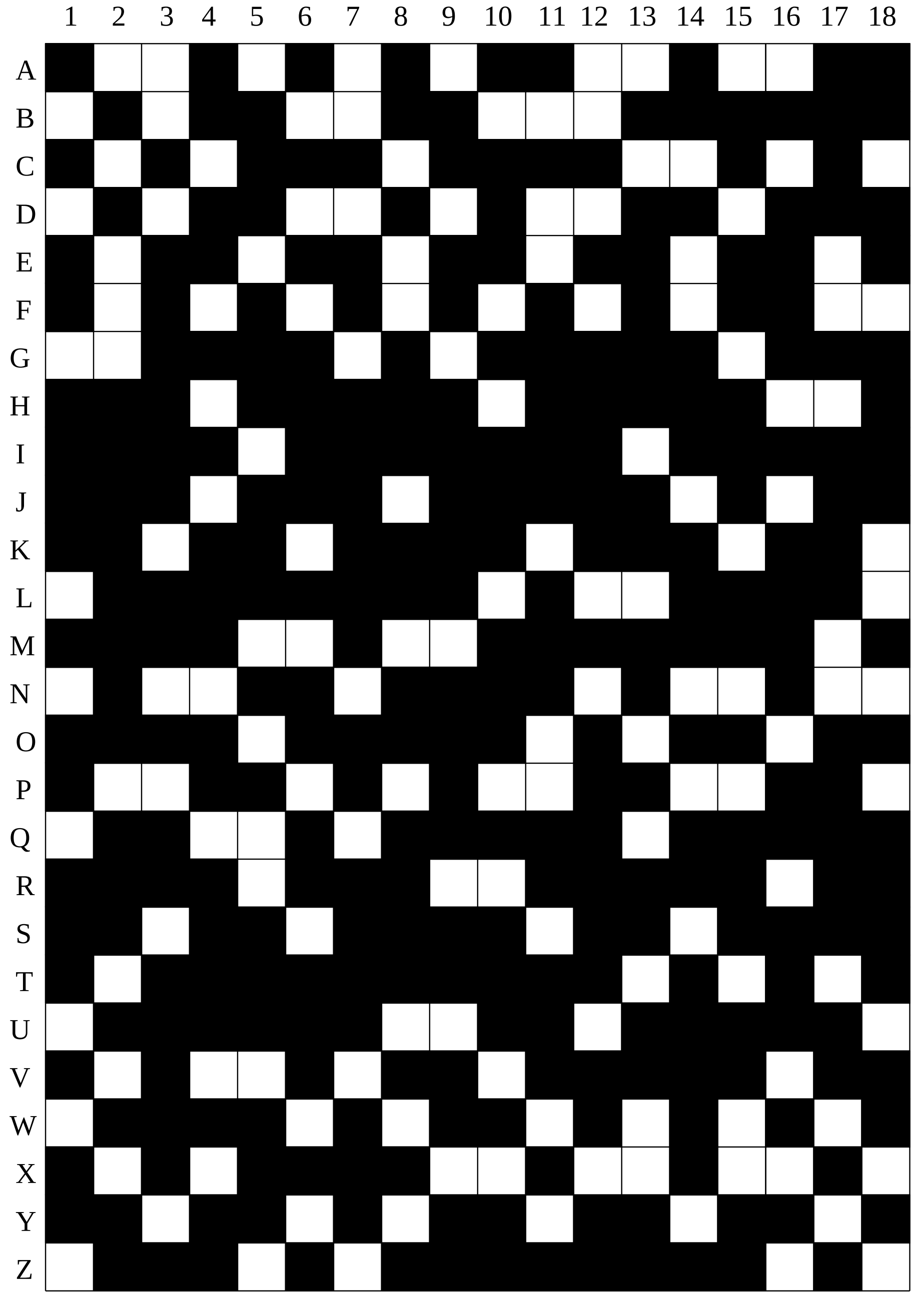
(i)

X	X	X	X	
X	X		X	X
X	X	X		X
	X	X	X	X
X		X	X	X

(ii)

X		X	X	X		
X	X	X			X	
			X	X	X	X
X	X				X	X
X		X	X	X		
	X		X		X	X
	X	X		X		X

(iii)



(Here the slots to be scheduled are indicated by open cells.)

7.10. Let G be the line-graph of some bipartite graph H. Derive from Kőnig's edge-colouring theorem (Theorem 7.3) that $\omega(G) = \gamma(G)$.

7.11. Derive Kőnig's edge-colouring theorem (Theorem 7.3) from the strong perfect graph conjecture.

7.12. Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be partitions of a finite set X such that $|A_1| = \cdots = |A_n| = |B_1| = \cdots = |B_n| = k$. Show that $\mathcal{A}$ and $\mathcal{B}$ have k pairwise disjoint common transversals.

7.13. Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a finite set X.

(i) Let $k \in \mathbb{N}$. Suppose that X can be partitioned into k partial SDR's of $\mathcal{A}$, and that X also can be partitioned into k partial SDR's of $\mathcal{B}$. Derive that X can be partitioned into k *common* partial SDR's for $\mathcal{A}$ and $\mathcal{B}$.

(ii) Show that the minimum number of common partial SDR's of $\mathcal{A}$ and $\mathcal{B}$ needed to cover X is equal to

$$(12) \qquad \lceil \max_{Y \subseteq X} \max\{\frac{|Y|}{|\{i | A_i \cap Y \neq \emptyset\}|}, \frac{|Y|}{|\{i | B_i \cap Y \neq \emptyset\}|}\} \rceil.$$

(*Hint:* Use Exercise 3.8.)

7.14. Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a finite set X and let $k \in \mathbb{N}$. Suppose that X has a partition $(Y_1, \ldots, Y_k)$ such that each Y_i is an SDR of $\mathcal{A}$. Suppose moreover that X has a partition $(Z_1, \ldots, Z_k)$ such that each Z_i is an SDR of $\mathcal{B}$. Derive that X has a partition $(X_1, \ldots, X_k)$ such that each X_i is an SDR both of $\mathcal{A}$ and of $\mathcal{B}$.

7.15. Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a finite set X and let $k \in \mathbb{N}$. Suppose that X has a partition $(Y_1, \ldots, Y_n)$ such that $|Y_i| = k$ and $Y_i \subseteq A_i$ for $i = 1, \ldots, n$. Suppose moreover that X has a partition $(Z_1, \ldots, Z_n)$ such that $|Z_i| = k$ and $Z_i \subseteq B_i$ for $i = 1, \ldots, n$. Derive that X has a partition $(X_1, \ldots, X_k)$ such that each X_i is an SDR both of $\mathcal{A}$ and of $\mathcal{B}$.

7.16. Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_m)$ be families of subsets of a finite set and let k be a natural number. Prove that $\mathcal{A}$ and $\mathcal{B}$ have k pairwise disjoint common SDR's, if and only if for all $I, J \subseteq \{1, \ldots, n\}$:

$$(13) \qquad |\bigcup_{i \in I} A_i \cap \bigcup_{j \in J} B_j| \geq k(|I| + |J| - n).$$

(*Hint:* Use Exercise 7.15.)

7.17. Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a finite set X.

(i) Let $k \in \mathbb{N}$. Suppose that $\mathcal{A}$ has k pairwise disjoint SDR's and that also $\mathcal{B}$ has k pairwise disjoint SDR's. Derive that X can be partitioned into k subsets $X_1, \ldots, X_k$ such that each X_i contains an SDR of $\mathcal{A}$ and contains an SDR of $\mathcal{B}$.

(ii) Show that the maximum number k for which there exists a partition as in (i) is equal to

$$(14) \qquad \lfloor \min_{\emptyset \neq I \subseteq \{1, \ldots, n\}} \min\{\frac{|\bigcup_{i \in I} A_i|}{|I|}, \frac{|\bigcup_{i \in I} |B_i|}{|I|}\} \rfloor.$$

(*Hint:* Use Exercise 3.7.)

7.3. Partially ordered sets

A *partially ordered set* is a pair $(X, \leq)$ where X is a set and where $\leq$ is a relation on X satisfying:

(15)
(i) $x \leq x$ for each $x \in X$;
(ii) if $x \leq y$ and $y \leq x$ then $x = y$;
(iii) if $x \leq y$ and $y \leq z$ then $x \leq z$.

A subset C of X is called a *chain* if for all $x, y \in C$ one has $x \leq y$ or $y \leq x$. A subset A of X is called an *antichain* if for all $x, y \in A$ with $x \neq y$ one has $x \not\leq y$ and $y \not\leq x$. Note that if C is a chain and A is an antichain then

(16) $$|C \cap A| \leq 1.$$

First we observe the following easy min-max relation:

Theorem 7.4. *Let $(X, \leq)$ be a partially ordered set, with X finite. Then the minimum number of antichains needed to cover X is equal to the maximum cardinality of any chain.*

Proof. The fact that the maximum cannot be larger then the minimum follows easily from (16). To see that the two numbers are equal, define for any element $x \in X$ the *height* of x as the maximum cardinality of any chain in X with maximum x. For any $i \in \mathbb{N}$, let A_i denote the set of all elements of height i.

Let k be the maximum height of the elements of X. Then $A_1, \ldots, A_k$ are antichains covering X, and moreover there exists a chain of size k. ∎

Dilworth [1950] proved that the same theorem also holds when we interchange the words 'chain' and 'antichain':

Theorem 7.5 (Dilworth's decomposition theorem). *Let $(X, \leq)$ be a partially ordered set, with X finite. Then the minimum number of chains needed to cover X is equal to the maximum cardinality of any antichain.*

Proof. We apply induction on $|X|$. The fact that the maximum cannot be larger then the minimum follows easily from (16). To see that the two numbers are equal, let α be the maximum cardinality of any antichain and let A be an antichain of cardinality α. Define

(17) $$\begin{aligned} A^{\downarrow} &:= \{x \in X \mid \exists y \in A : x \leq y\}, \\ A^{\uparrow} &:= \{x \in X \mid \exists y \in A : x \geq y\}. \end{aligned}$$

Then $A^{\downarrow} \cup A^{\uparrow} = X$ (since A is a maximum antichain) and $A^{\downarrow} \cap A^{\uparrow} = A$.

First assume $A^{\downarrow} \neq X$ and $A^{\uparrow} \neq X$. Then by induction $A^{\downarrow}$ can be covered with α chains. Since $A \subseteq A^{\downarrow}$, each of these chains contains exactly one element in A. For each $x \in A$, let C_x denote the chain containing x. Similarly, there exist α chains C'_x (for $x \in A$) covering $A^{\uparrow}$, where C'_x contains x. Then for each $x \in A$, $C_x \cup C'_x$ forms a chain in X, and moreover these chains cover X.

So we may assume that for each antichain A of cardinality α one has $A^{\downarrow} = X$ or $A^{\uparrow} = X$. It means that each antichain A of cardinality α is either the set of minimal elements of X or the set of maximal elements of X. Now choose a minimal element x and a maximal element y of X such that $x \leq y$. Then the maximum cardinality of an antichain in $X \setminus \{x, y\}$ is equal to $\alpha - 1$ (since each antichain in X of cardinality α contains x or y). By induction, $X \setminus \{x, y\}$ can be covered with $\alpha - 1$ chains. Adding the chain $\{x, y\}$ yields a covering of X with α chains. ∎

Application 7.5: Project scheduling. Suppose you have to perform a project consisting of several jobs. Each job takes one time-unit, say one hour. Certain jobs have to be done before other jobs; this relation is given by a partial order on the jobs. Assuming that you have sufficient workers, the time required to finish the project is equal to the size γ of the longest chain. Indeed, by Theorem 7.4, the jobs can be split into γ antichains $A_1, \ldots, A_\gamma$; in fact, these antichains can be chosen such that if $x \in A_i$ and $y \in A_j$ and $x < y$ then $i < j$. As in each of these antichains, the jobs can be done simultaneously, we obtain a feasible schedule.

This is an application quite similar to PERT-CPM (Application 1.4).

Application 7.6: Bungalow assignment. Suppose you are the manager of a bungalow park, with bungalows that can be rented out during the holiday season. There have been made a number of reservations, each for a connected period of some weeks, like in Figure 7.3. If the number of

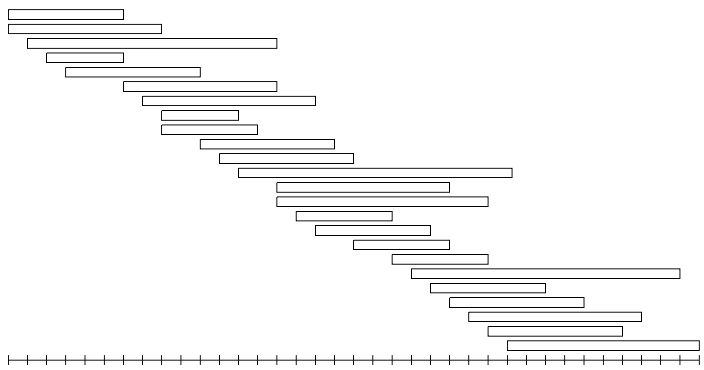


Figure 7.3

reservations during any of the weeks in the holiday season is not larger than the total number of bungalows available, then there exists an allocation of customers to bungalows, in such a way that no renter has to switch bungalows during his/her stay. This rule well-known to bungalow park managers, is a special case of Dilworth's decomposition theorem.

Indeed, one can make a partial order as follows. Let X be the set of reservations made, and for any $x, y \in X$ let $x < y$ if the last day for reservation x is earlier than or equal to the first day of reservation y.

Then the maximum size of any antichain of $(X, \leq)$ is equal to the maximum number n of reservations made for any week in the season. By Dilworth's decomposition theorem, X can be split into n chains. Each chain now gives a series of reservations that can be assigned to one and the same bungalow.

A similar problem occurs when assigning hotel rooms to hotel guests.

Application 7.7: Terminal and platform assignment. A similar problem as in Application 7.6 occurs when one has to assign airplanes to terminals at an airport, or trains or buses to platforms in a train or bus station. The model has to be adapted however, if one requires a *periodic* assignment; this occurs for instance if the trains or buses run a periodic timetable, say with period one hour.

Exercises

7.18. Let $(X, \leq)$ be a partially ordered set. Call a chain *maximal* if it is not contained in any other chain. Prove that the maximum number of pairwise disjoint maximal chains is equal to the minimum cardinality of a set intersecting all maximal chains.

7.19. Derive Kőnig's edge cover theorem from Dilworth's decomposition theorem.

7.20. Let $G = (V, E)$ be a bipartite graph, with colour classes V_1 and V_2, with $|V_1| = |V_2| = n$. Let k be a natural number. Derive from Dilworth's decomposition theorem that the edges of G can be covered by k perfect matchings, if and only if for each vertex cover $W \subseteq V$ the number of edges contained in W is at most $k(|W| - n)$.

7.21. Let $\mathcal{I} = (I_1, \ldots, I_n)$ be a family of intervals on $\mathbb{R}$, in such a way that each $x \in \mathbb{R}$ is contained in at most k of these intervals. Show that $\mathcal{I}$ can be partitioned into k classes $\mathcal{I}_1, \ldots, \mathcal{I}_k$ so that each $\mathcal{I}_j$ consists of pairwise disjoint intervals.

7.22. Let $D = (V, A)$ be an acyclic directed graph and let r and s be vertices of D such that each arc of D occurs in at least one $r - s$ path. Derive from Dilworth's decomposition theorem that the minimum number of $r - s$ paths needed to cover all arcs is equal to the maximum cardinality of $\delta^{\text{out}}(U)$, where U ranges over all subsets of V satisfying $r \in U, s \notin U$ and $\delta^{\text{in}}(U) = \emptyset$.

7.23. A graph $G = (V, E)$ is called a *comparability graph* if there exists a partial order $\leq$ on V such that for all u, w in V with $u \neq w$ one has:

$$\{u, w\} \in E \Leftrightarrow u \leq w \text{ or } w \leq u. \tag{18}$$

(i) Show that if G is a comparability graph, then $\omega(G) = \gamma(G)$.

(ii) Show that if G is the complement of a comparability graph, then $\omega(G) = \gamma(G)$.
(*Hint:* Use Dilworth's decomposition theorem (Theorem 7.5).)

7.24. Let $(X, \leq)$ be a partially ordered set, with X finite. Let $\mathcal{C}$ and $\mathcal{A}$ denote the collections of chains and antichains in $(X, \leq)$, respectively. Let $w : X \to \mathbb{Z}_+$ be a 'weight' function.

(i) Show that the maximum weight $w(C)$ of any chain is equal to the minimum value of $\sum_{A \in \mathcal{A}} \lambda(A)$, where the $\lambda(A)$ range over all nonnegative integers satisfying

$$\sum_{A \in \mathcal{A}, x \in A} \lambda(A) = w(x) \tag{19}$$

for each $x \in X$.

(ii) Show that the maximum weight $w(A)$ of any antichain is equal to the minimum value of $\sum_{C \in \mathcal{C}} \lambda(C)$, where the $\lambda(C)$ range over all nonnegative integers satisfying

$$\sum_{C \in \mathcal{C}, x \in C} \lambda(C) = w(x) \tag{20}$$

for each $x \in X$.

(iii) Derive that the convex hull of the incidence vectors of antichains (as vectors in $\mathbb{R}^X$) is equal to the set of all vectors $f \in \mathbb{R}^X_+$ satisfying $f(C) \leq 1$ for each chain C.
[For any finite set X and any subset Y of X, define the *incidence vector* $\chi^Y \in \mathbb{R}^X$ of Y as:

$$\begin{aligned} \chi^Y_x &:= 1 \quad \text{if } x \in Y; \\ &:= 0 \quad \text{if } x \notin Y.] \end{aligned} \tag{21}$$

(iv) Derive also that the convex hull of the incidence vectors of chains (as vectors in $\mathbb{R}^X$) is equal to the set of all vectors $f \in \mathbb{R}^X_+$ satisfying $f(A) \leq 1$ for each antichain A.

7.25. Derive Dilworth's decomposition theorem (Theorem 7.5) from the strong perfect graph conjecture.

7.4. Perfect graphs

We now consider a general class of graphs, the 'perfect' graphs, that turn out to unify several results in combinatorial optimization, in particular, min-max relations and polyhedral characterizations.

As we saw before, the clique number $\omega(G)$ and the colouring number $\gamma(G)$ of a graph $G = (V, E)$ are related by the inequality:

(22) $$\omega(G) \leq \gamma(G).$$

There are graphs that have strict inequality; for instance, the circuit C_5 on five vertices.

Having equality in (22) does not say that much about the internal structure of a graph: any graph $G = (V, E)$ can be extended to a graph $G' = (V', E')$ satisfying $\omega(G') = \gamma(G')$, simply by adding to G a clique of size $\gamma(G)$, disjoint from V.

However, if we require that equality in (22) holds for each induced subgraph of G, we obtain a much more powerful condition. The idea for this was formulated by Berge [1963]. He defined a graph $G = (V, E)$ te be *perfect* if $\omega(G') = \gamma(G')$ holds for each induced subgraph G' of G.

Several classes of graphs could be shown to be perfect, and Berge [1961,1963] observed the important phenomenon that for several classes of graphs that were shown to be perfect, also the class of complementary graphs is perfect. (The *complement* or the *complementary graph* $\overline{G}$ of a graph $G = (V, E)$ is the graph with vertex set V, where any two distinct vertices in V are adjacent in $\overline{G}$ if and only if they are nonadjacent in G.)

Berge therefore conjectured that the complement of any perfect graph is perfect again. This conjecture was proved by Lovász [1972b], and his *perfect graph theorem* forms the kernel of perfect graph theory. It has several other theorems in graph theory as consequence. Lovász [1972a] gave the following stronger form of the conjecture, which we show with the elegant linear-algebraic proof found by Gasparian [1996].

Theorem 7.6. *A graph G is perfect if and only if $\omega(G')\alpha(G') \geq |V(G')|$ for each induced subgraph G' of G.*

Proof. Necessity is easy, since if G is perfect, then $\omega(G') = \gamma(G')$ for each induced subgraph G' of G, and since $\gamma(G')\alpha(G') \geq |V(G')|$ for any graph G'.

To see sufficiency, suppose to the contrary that there exists an imperfect graph G satisfying the condition, and choose such a graph with $|V(G)|$ minimal. So $\gamma(G) > \omega(G)$, while $\gamma(G') = \omega(G')$ for each induced subgraph $G' \neq G$ of G.

Let $\omega := \omega(G)$ and $\alpha := \alpha(G)$. We can assume that $V(G) = \{1, \ldots, n\}$.

We first construct

(23) cocliques $C_0, \ldots, C_{\alpha\omega}$ such that each vertex is covered by exactly α of the C_i.

Let C_0 be a coclique in G of size α. By the minimality of G, we know that for each $v \in C_0$, the subgraph of G induced by $V(G) \setminus \{v\}$ is perfect, and that hence its colouring number is at most ω (as its clique number is at most ω); therefore $V(G) \setminus \{v\}$ can be partitioned into ω cocliques. Doing this for each $v \in C_0$, we obtain cocliques as in (23).

Now for each $i = 0, \ldots, \alpha\omega$, there exists a clique K_i of size ω with $K_i \cap C_i = \emptyset$. Otherwise, the subgraph G' of G induced by $V(G) \setminus C_i$ would have $\omega(G') < \omega$, and hence it has colouring number at most $\omega - 1$. Adding C_i as a colour would give an ω-vertex colouring of G, contradicting the assumption that $\gamma(G) > \omega(G)$.

Then, if $i \neq j$ with $0 \leq i, j \leq \alpha\omega$, we have $|K_j \cap C_i| = 1$. This follows from the fact that K_j has size ω and intersects each C_i in at most one vertex, and hence, by (23), it intersects $\alpha\omega$ of the C_i. As $K_j \cap C_j = \emptyset$, we have that $|K_j \cap C_i| = 1$ if $i \neq j$.

Now consider the $(\alpha\omega + 1) \times n$ incidence matrices $M = (m_{i,j})$ and $N = (n_{i,j})$ of $C_0, \ldots, C_{\alpha\omega}$ and $K_0, \ldots, K_{\alpha\omega}$ respectively. So M and N are $0,1$ matrices, with $m_{i,j} = 1 \Leftrightarrow j \in C_i$, and $n_{i,j} = 1 \Leftrightarrow j \in K_i$, for $i = 0, \ldots, \alpha\omega$ and $j = 1, \ldots, n$. By the above, $MN^T = J - I$, where J is the $\alpha\omega \times \alpha\omega$ all-1 matrix, and I the $\alpha\omega \times \alpha\omega$ identity matrix. As $J - I$ has rank $\alpha\omega + 1$, we have $n \geq \alpha\omega + 1$. This contradicts the condition given in the theorem. ∎

This implies:

Corollary 7.6a ((Lovász's) perfect graph theorem). *The complement of a perfect graph is perfect again.*

Proof. Directly from Theorem 7.6, as the condition given in it is maintained under taking the complementary graph. ∎

In fact, Berge [1963] also made an even stronger conjecture, which is still unproved (we mentioned this conjecture in Section 7.1 in a different but equivalent form):

Strong perfect graph conjecture. A graph G is perfect if and only if G does not contain any odd circuit C_{2k+1} with $k \geq 2$ or its complement as an induced subgraph.

We now show how several theorems we have seen before follow as consequences from the perfect graph theorem. First observe that trivially, any bipartite graph G is perfect. This implies Kőnig's edge cover theorem (Theorem 3.2a):

Corollary 7.6b (Kőnig's edge cover theorem). *The complement of a bipartite graph is perfect. Equivalently, the edge cover number of any bipartite graph (without isolated vertices) is equal to its coclique number.*

Proof. Directly from the perfect graph theorem. Note that if G is a bipartite graph, then its cliques have size at most 2; hence $\gamma(\overline{G})$ is equal to the edge cover number of G if G has no isolated vertices.

Note moreover that the class of complements of bipartite graphs is closed under taking induced subgraphs. Hence the second statement in the Corollary indeed is equivalent to the first. ∎

We saw in Section 3.2 that by Gallai's theorem (Theorem 3.1), Kőnig's edge cover theorem directly implies Kőnig's matching theorem (Theorem 3.2), saying that the matching number of a bipartite graph G is equal to its vertex cover number. That is, the coclique number of the line graph $L(G)$ of G is equal to the minimum number of cliques of $L(G)$ that cover all vertices of $L(G)$. As this is true for any induced subgraph of $L(G)$ we know that the complement $\overline{L(G)}$ of the line graph $L(G)$ of any bipartite graph G is perfect.

Hence with the perfect graph theorem we obtain Kőnig's edge-colouring theorem (Theorem 7.3):

Corollary 7.6c (Kőnig's edge-colouring theorem). *The line graph of a bipartite graph is perfect. Equivalently, the edge-colouring number of any bipartite graph is equal to its maximum degree.*

Proof. Again directly from Kőnig's matching theorem and the perfect graph theorem. ∎

We can also derive Dilworth's decomposition theorem (Theorem 7.5) easily from the perfect graph theorem. Let $(V, \leq)$ be a partially ordered set. Let $G = (V, E)$ be the graph with:

(24) $uv \in E$ if and only if $u < v$ or $v < u$.

Any graph G obtained in this way is called a *comparability graph*.

As Theorem 7.4 we saw the following easy 'dual' form of Dilworth's decomposition theorem:

Theorem 7.7. *In any partially ordered set $(V, \leq)$, the maximum size of any chain is equal to the minimum number of antichains needed to cover V.*

Proof. For any $v \in V$ define the *height* of v as the maximum size of any chain in V with maximum element v. Let k be the maximum height of any element $v \in V$. For $i = 1, \ldots, k$ let A_i be the set of elements of height i. Then $A_1, \ldots, A_k$ are antichains covering V, and moreover, there is a chain of size k, since there is an element of height k. ∎

Equivalently, we have $\omega(G) = \gamma(G)$ for any comparability graph. As the class of comparability graphs is closed under taking induced subgraphs we have:

Corollary 7.7a. *Any comparability graph is perfect.*

Proof. Directly from Theorem 7.7. ∎

So by the perfect graph theorem:

Corollary 7.7b. *The complement of any comparability graph is perfect.*

Proof. Directly from Corollary 7.7a and the perfect graph theorem (Corollary 7.6a). ∎

That is:

Corollary 7.7c (Dilworth's decomposition theorem). *In any partially ordered set $(V, \leq)$,* *the maximum size of any antichain is equal to the minimum number of chains needed to cover V.*

Proof. Directly from Corollary 7.7b. ∎

A further application of the perfect graph theorem is to 'chordal graphs', which we describe in the next section.

We note here that it was shown with the help of the 'ellipsoid method' that there exists a polynomial-time algorithm for finding a maximum clique and a minimum vertex-colouring in any perfect graph (Grötschel, Lovász, and Schrijver [1981]). However no *combinatorial* polynomial-time algorithm is known for these problems.

Exercises

7.26. Show that the graph obtained from the Paris Métro network (see Application 7.1) is perfect.

7.27. Show that Theorem 7.6 is implied by the strong perfect graph conjecture.

7.5. Chordal graphs

We finally consider a further class of perfect graphs, the 'chordal graphs' (or 'rigid circuit graphs' or 'triangulated graphs'). A graph G is called *chordal* if each circuit in G of length at least 4 has a chord. (A *chord* is an edge connecting two vertices of the circuit that do not form two neighbours in the circuit.)

For any set A of vertices let $N(A)$ denote the set of vertices not in A that are adjacent to at least one vertex in A. Call a vertex v *simplicial* if $N(\{v\})$ is a clique in G.

Dirac [1961] showed the following basic property of chordal graphs:

Theorem 7.8. *Each chordal graph G contains a simplicial vertex.*

Proof. We may assume that G has at least two nonadjacent vertices a, b. Let A be a maximal nonempty subset of V such that $G|A$ is connected and such that $A \cup N(A) \neq V$. Such a subset A exists as $G|\{a\}$ is connected and $\{a\} \cup N(\{a\}) \neq V$.

Let $B := V \setminus (A \cup N(A))$. Then each vertex v in $N(A)$ is adjacent to each vertex in B, since otherwise we could increase A by v. Moreover, $N(A)$ is a clique, for suppose that $u, w \in N(A)$ are nonadjacent. Choose $v \in B$. Let P be a shortest path in $A \cup N(A)$ connecting u and w. Then $P \cup \{u, v, w\}$ would form a circuit of length at least 4 without chords, a contradiction.

Now inductively we know that $G|B$ contains a vertex v that is simplicial in $G|B$. Since $N(A)$ is a clique and since each vertex in B is connected to each vertex in $N(A)$, v is also simplicial in G. ∎

This implies a result of Hajnal and Surányi [1958]:

Theorem 7.9. *The complement of any chordal graph is perfect.*

Proof. Let $G = (V, E)$ be a chordal graph. Since the class of chordal graphs is closed under taking induced subgraphs, it suffices to show $\omega(\overline{G}) \geq \gamma(\overline{G})$.

By Theorem 7.1, G has a simplicial vertex v. So $K := \{v\} \cup N(\{v\})$ is a clique. Let G' be the subgraph of G induced by $V \setminus K$. By induction we have $\omega(\overline{G'}) = \gamma(\overline{G'})$.

Now $\omega(\overline{G}) \geq \omega(\overline{G'}) + 1$, since we can add v to any clique of $\overline{G'}$. Similarly, $\gamma(\overline{G}) \leq \gamma(\overline{G'}) + 1$, since we can add K to any colouring of $\overline{G'}$. Hence $\omega(\overline{G}) \geq \gamma(\overline{G})$. ∎

With Lovász's perfect graph theorem, this implies the result of Berge [1960]:

Corollary 7.9a. *Any chordal graph is perfect.*

Proof. Directly from Theorem 7.9 and the perfect graph theorem (Corollary 7.6a). ∎

We can characterize chordal graphs in terms of subtrees of a tree T. Let $\mathcal{S}$ be a collection of nonempty subtrees of a tree T. The *intersection graph* of $\mathcal{S}$ is the graph with vertex set $\mathcal{S}$, where two vertices S, S' are adjacent if and only if they intersect (in at least one vertex).

The class of graphs obtained in this way coincides with the class of chordal graphs. To see this, we first show the following elementary lemma:

Lemma 7.1. *Let $\mathcal{S}$ be a collection of pairwise intersecting subtrees of a tree T. Then there is a vertex of T contained in all subtrees in $\mathcal{S}$.*

Proof. By induction on $|VT|$. If $|VT| = 1$ the lemma is trivial, so assume $|VT| \geq 2$. Let t be an end vertex of T. If there exists a subtree in $\mathcal{S}$ consisting only of t, the lemma is trivial. Hence we may assume that each subtree in $\mathcal{S}$ containing t also contains the neighbour of t. So deleting t from T and from all subtrees in $\mathcal{S}$ gives the lemma by induction. ∎

Then:

Theorem 7.10. *A graph is chordal if and only if it is isomorphic to the intersection graph of a collection of subtrees of some tree.*

Proof. *Necessity.* Let $G = (V, E)$ be chordal. By Theorem 7.8, G contains a simplicial vertex v. By induction, the subgraph $G - v$ of G is the intersection graph of a collection $\mathcal{S}$ of subtrees of some tree T. Let $\mathcal{S}'$ be the subcollection of $\mathcal{S}$ corresponding to the set N of neighbours of v in G. As N is a clique, $\mathcal{S}'$ consists of pairwise intersecting subtrees. Hence, by Lemma 7.1 these subtrees have a vertex t of T in common. Now we extend T and all subtrees in $\mathcal{S}'$ with a new vertex t' and a new edge tt'. Moreover, we introduce a new subtree $\{t'\}$ representing v. In this way we obtain a subtree representation for G.

Sufficiency. Let G be the intersection graph of some collection $\mathcal{S}$ of subtrees of some tree T. Suppose that G contains a chordless circuit C_k with $k \geq 4$. Let C_k be the intersection graph of $S_1, \ldots, S_k \in \mathcal{S}$, with S_1 and S_2 intersecting. Then $S_1, S_2, S_3 \cup \cdots \cup S_k$ are three subtrees of T that are pairwise intersecting. So by Lemma 7.1, T has a vertex v contained in each of these three subtrees. So $v \in S_1 \cap S_2 \cap S_i$ for some $i \in \{3, \ldots, k\}$. This yields a chord in C_k. ∎

This theorem enables us to interpret the perfectness of chordal graphs in terms of trees:

Corollary 7.10a. *Let $\mathcal{S}$ be a collection of nonempty subtrees of a tree T. Then the maximum number of pairwise vertex-disjoint trees in $\mathcal{S}$ is equal to the minimum number of vertices of T intersecting each tree in $\mathcal{S}$.*

Proof. Directly from Theorems 7.9 and 7.10, using Lemma 7.1. ∎

Similarly we have:

Corollary 7.10b. *Let $\mathcal{S}$ be a collection of subtrees of a tree T. Let k be the maximum number of times that any vertex of T is covered by trees in $\mathcal{S}$. Then $\mathcal{S}$ can be partitioned into subcollections $\mathcal{S}_1, \ldots, \mathcal{S}_k$ such that each $\mathcal{S}_i$ consists of pairwise vertex-disjoint trees.*

Proof. Directly from Corollary 7.9a and Theorem 7.10, again using Lemma 7.1. ∎

Exercises

7.28. Show that a graph $G = (V, E)$ is chordal if and only if each induced subgraph has a simplicial vertex.

7.29. Show that a graph is an interval graph if and only if it is chordal and its complement is a comparability graph.

7.30. Derive from the proof of Theorem 7.8 that each chordal graph is either a clique or contains two nonadjacent simplicial vertices.

7.31. Let G be a chordal graph. Derive from the proof of Theorem 7.8 that each vertex v that is nonadjacent to at least one vertex $w \neq v$, is nonadjacent to at least one simplicial vertex $w \neq v$.

7.32. Show that a graph $G = (V, E)$ is chordal if and only if the edges of G can be oriented so as to obtain a directed graph $D = (V, A)$ with the following properties:

(25)
(i) D is acyclic;
(ii) if (u, v) and (u, w) belong to A then (v, w) or (w, v) belongs to A.