## 1 Prerequisites

A singleton symmetric congestion game is a tuple $\left(\mathcal{N}, R,\left(d_{r}\right)_{r \in R}\right)$ consisting of the following components.

- A set of players $\mathcal{N}=[n]$ for some natural number $n$.
- A set of resources $R$.
- Non-decreasing delay functions $d_{r}: \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ for each $r \in R$.

Each player $i \in \mathcal{N}$ picks a resource $A_{i} \in R$, giving an action profile $A=\left(A_{1}, \ldots, A_{n}\right) \in$ $R^{n}$. The congestion vector $x(A)=\left(x(A)_{r}\right)_{r \in R}$ specifies the number of players picking each resource:

$$
x(A)_{r}=\left|\left\{i \in \mathcal{N}: r \in A_{i}\right\}\right| .
$$

The cost player $i \in \mathcal{N}$ has to pay is given by

$$
c_{i}(A)=d_{A_{i}}\left(x(A)_{A_{i}}\right),
$$

that is, each player pays the delay / cost of the resource they chose, which is a nondecreasing function of the number of players picking the resource. The goal of each player is to minimise their own cost. One can think for example of jobs (players) choosing a machine (resource) and "paying" the total processing time of the machine.

We call $A \in R^{n}$ a Nash equilibrium if no player has an incentive to change its action, given that the others don't change their action. That is, for each $r, s \in R$ with $r=A_{i}$ for some $i \in \mathcal{N}$,

$$
d_{r}\left(x(A)_{r}\right)=c_{i}(A) \leq c_{i}\left(A_{1}, \ldots, A_{i-1}, s, A_{i+1}, \ldots, A_{n}\right)=d_{s}\left(x(A)_{s}+1\right) .
$$

In a sequential-move version of the game, the players move one after the other and provide a strategy which tells you the resource they will choose given the choices of their predecessors.

Example 1.1. Consider $\mathcal{N}=\{1,2\}, R=\{r, s\}$ and costs $d_{r}\left(x_{r}\right)=x_{r}$ and $d_{s}\left(x_{s}\right)=2 x_{s}$. Below the costs $\left(c_{1}(A), c_{2}(A)\right)$ are given for all possible action profiles $A$.

|  | Player 2 plays $r$ | Player 2 plays $s$ |
| :--- | :---: | :---: |
| Player 1 plays $r$ | $(2,2)$ | $(1,2)$ |
| Player 1 plays $s$ | $(2,1)$ | $(4,4)$ |

There are two possible sequential-move versions of this game: the one in which player 1 moves first and the one in which player 2 moves first. Consider the sequential-move version where player 1 moves first. A strategy $S_{1}$ for player 1 is an element in $\{r, s\}$. A strategy for player 2 is a function

$$
S_{2}:\{r, s\} \rightarrow\{r, s\}
$$

that tells us which action player 2 plays given the action of player 1 . The game tree of the sequential-move version with possible strategies is given in Figure 1.

A subgame-perfect equilibrium $S=\left(S_{i}\right)_{i \in \mathcal{N}}$ consists of a strategy $S_{i}$ for each player $i \in$ $\mathcal{N}$ so that $S$ is a Nash equilibrium in each subgame. For example, since $c_{2}(s, s)>c_{2}(s, r)$, player 2 is strictly better off with playing $r$ if player 1 plays $s$. Hence his only best-response in the subgame given by $(s)$ ("given by forcing player 1 to play $s$ ") is $r$. If player 1 plays


Figure 1: The game trees of sequential-move versions of the game are given. At a leaf of the tree, the costs of both players are given of the corresponding action profile. Two possible subgame-perfect equilibria are drawn red. The corresponding subgame-perfect outcomes are $(r, s)$ and $(s, r)$. In the tree to the right, it is also subgame-perfect for the first player to play $r$, which gives subgame-perfect outcome $(r, r)$.
$r$, player 2 is indifferent between $r$ and $s$, so either choice is subgame-perfect. Strategy $S_{1}=r$ is a best-response to any strategy $S_{2}$ because

$$
c_{1}\left(r, S_{2}(r)\right)=2 \leq c_{1}\left(s, S_{2}(s)\right)
$$

independent of how we define $S_{2}$. However, we may also take $S_{1}=s$ if $S_{2}(r)=r$ and $S_{2}(s)=r$, since then

$$
c_{1}\left(s, S_{2}(s)\right)=c_{1}(s, r)=2 \leq 2=c_{1}(r, s)=c_{1}\left(r, S_{2}(r)\right) .
$$

A subgame-perfect outcome is the action profile of the original game that we find by following the red path in the game tree from the root to a leave. In this case, the set of subgame-perfect outcomes

$$
\{(r, r),(r, s),(s, r)\}
$$

equals the set of Nash equilibria.
Proposition 1.2. Every sequential-move version has a subgame-perfect equilibrium.
In layer $n$ of the tree, one connects each node to a leave node by minimising the cost of the last player. After this is fixed, you know the resulting cost vectors given the actions of the first $n-1$ players, so you are in your initial situation but with one player less. Hence, by "backtracking bottom-up" we will find a subgame-perfect equilibrium.

## 2 Each Nash equilibrium is a subgame-perfect outcome

The worst cost $W=W(G)$ of $G$ is defined as

$$
W:=\min _{A \in R^{n}} W_{A}(G)=\min _{A \in R^{n}} \max _{r \in A} d_{r}\left(x(A)_{r}\right) .
$$

If we order the numbers

$$
d_{r}\left(x_{r}\right) \text { for } x_{r} \in\{1, \ldots, n\} \text { and } r \in R
$$

from low to high, then $W$ will be the $n$th number. The intuition behind the worst cost is that nobody should get anything higher than $W$ (as there exists a profile for which this is the case), yet someone will always have to get cost $W$ (since $W$ is minimal). Define $d_{r}(0)=-\infty$. Let

$$
M_{r}=\max \left\{m \in\{0, \ldots, n\} \mid d_{r}(m) \leq W\right\} \quad(r \in R)
$$

denote the number of times resource $r$ can be chosen while keeping its cost $\leq W$. A resources costs more than $W$ for a new player if and only if it has been chosen at least $M_{r}$ times.

Given the choices of the other players, you can always pick a resource costing at most $W$, because the action profile $A$ achieving the minimum in the definition of $W$ satisfies

$$
\sum_{r \in R} M_{r} \geq \sum_{r \in R}\left|\left\{i \in \mathcal{N}: A_{i}=r\right\}\right|=n
$$

(We see each action profile as a tuple and perform set operations as if it were a multiset.)
Lemma 2.1. Any player has a cost of at most $W$ in a subgame-perfect outcome.
Proof. The proof goes by induction on the number of players $n$. The case $n=1$ is clear. Suppose the claim holds for all games with at most $k$ players for some $k \geq 1$ and consider $n=k+1$. Suppose $A$ is a subgame-perfect outcome; renumber the players so that the corresponding order is the identity. Assume towards contradiction that player $j$ picks a resource costing $>W$. At the point players $1, \ldots, j-1$ have played, there must still be a resource $s$ costing at most $W$ at that moment. If player $j$ picks $s$ instead, then the cost of $s$ will be $>W$ in the outcome of the game (since $A_{j}$ is subgame-perfect for $j$ ), hence one of the successors of $j$ must pick $s$. However, the subgame induced by $\left(A_{1}, \ldots, A_{j-1}, s\right)$ has strictly less than $n$ players, the same or lower worst cost (because $\sum_{r \in R} M_{r} \geq n$ ) and is still a singleton symmetric congestion game, hence all players in this game have cost at most $W$ by the induction hypothesis. This is a contradiction: at least one of the successors of $j$ picks $s$ as well and hence must receive a cost $>W$ as well.

Theorem 2.2. Each Nash equilibrium is a subgame-perfect outcome.
Proof. Let $A=\left(A_{1}, \ldots, A_{n}\right)$ be a Nash equilibrium. Note that by symmetry in the players, it suffices to show that a permutation of $A$ is a subgame-perfect outcome corresponding to the identity.

Consider the game tree of the sequential-move version corresponding to the identity. We are going to draw in a subgame-perfect equilibrium (with a permutation of $A$ on the equilibrium path) in red.

Consider any player $1 \leq j \leq n$ any any sequence of actions $B_{1}, \ldots, B_{j-1}$ of his predecessors. If we are allowed to colour a resource from the multiset $M:=\left\{A_{1}, \ldots, A_{n}\right\} \backslash$ $\left\{B_{1}, \ldots, B_{j-1}\right\}$ red, then we are done: if we keep making such choices, we will get a red path from top to bottom which is a permutation of $A$ (and this suffices by symmetry in the players). We will prove the following.
(1) Any $r \in R \backslash M$ costs at least $W$ if player $j$ picks $r$.
(2) Let $s \in M$. Then $\left(B_{1}, \ldots, B_{j-1}, s\right)$ has only subgame-perfect equilibria in which $s$ costs at most $W$.

We need to pick an arc for player $j$ that leads to a lowest possible cost. If we show the statements above, we are sure that at least one of those "perfect responses" will be in $M$, which completes our proof.

Proof of (1). Let $r \in R \backslash M$. Then resource $r$ has been chosen at least $x(A)_{r}$ times already at the point $j$ makes his choice. Assume $j$ picks $r$ as well. If (1) were false, we would have $d_{r}(x(A)+1)<W$ (since the cost functions are non-decreasing), hence $A$ would not be a Nash equilibrium, as the player paying at least $W$ (which must exist by minimality of $W$ ) would prefer to switch to resource $r$.

Proof of (2). At the point $B_{1}, \ldots, B_{j-1}, s$ are played, there has to be a choice for the other $n-j-1$ players in which all of these get cost $\leq W$. Hence the new $W$ of this subgame is lower or equal to the original $W$, and hence we get from Lemma 2.1 that $\left(B_{1}, \ldots, B_{j-1}, s\right)$ has only subgame-perfect equilibria in which each successor of $j$ has cost at most $W$. Since $s \in M$, it costs at most $W$ before the other players make their move, and because of the fact above, no subgame-perfect move of a successor will make it cost more than $W$.

In fact, it follows from Lemma 2.1 that the set of Nash equilibria coincides with the set of subgame-perfect outcomes, which was already proved by different methods as well $[2,1]$.

## References

[1] J. de Jong. Quality of equilibria in resource allocation games. PhD thesis, University of Twente, 2016.
[2] I. Milchtaich. Crowding games are sequentially solvable. International Journal of Game Theory, 27:501-509, 1998.

