1 Prerequisites

A singleton symmetric congestion game is a tuple $(\mathcal{N}, R, (d_r)_{r \in R})$ consisting of the following components.

- A set of players $\mathcal{N} = [n]$ for some natural number n.
- A set of **resources** R.
- Non-decreasing **delay functions** $d_r : \mathbb{N} \to \mathbb{R}_{>0}$ for each $r \in R$.

Each player $i \in \mathcal{N}$ picks a resource $A_i \in R$, giving an **action profile** $A = (A_1, \ldots, A_n) \in \mathbb{R}^n$. The **congestion vector** $x(A) = (x(A)_r)_{r \in \mathbb{R}}$ specifies the number of players picking each resource:

$$x(A)_r = |\{i \in \mathcal{N} : r \in A_i\}|.$$

The cost player $i \in \mathcal{N}$ has to pay is given by

$$c_i(A) = d_{A_i}(x(A)_{A_i}),$$

that is, each player pays the delay / cost of the resource they chose, which is a nondecreasing function of the number of players picking the resource. The goal of each player is to minimise their own cost. One can think for example of jobs (players) choosing a machine (resource) and "paying" the total processing time of the machine.

We call $A \in \mathbb{R}^n$ a **Nash equilibrium** if no player has an incentive to change its action, given that the others don't change their action. That is, for each $r, s \in \mathbb{R}$ with $r = A_i$ for some $i \in \mathcal{N}$,

$$d_r(x(A)_r) = c_i(A) \le c_i(A_1, \dots, A_{i-1}, s, A_{i+1}, \dots, A_n) = d_s(x(A)_s + 1).$$

In a *sequential-move version* of the game, the players move one after the other and provide a *strategy* which tells you the resource they will choose given the choices of their predecessors.

Example 1.1. Consider $\mathcal{N} = \{1, 2\}$, $R = \{r, s\}$ and costs $d_r(x_r) = x_r$ and $d_s(x_s) = 2x_s$. Below the costs $(c_1(A), c_2(A))$ are given for all possible action profiles A.

	Player 2 plays r	Player 2 plays s
Player 1 plays r	(2,2)	(1,2)
Player 1 plays s	(2,1)	(4,4)

There are two possible sequential-move versions of this game: the one in which player 1 moves first and the one in which player 2 moves first. Consider the sequential-move version where player 1 moves first. A strategy S_1 for player 1 is an element in $\{r, s\}$. A strategy for player 2 is a function

$$S_2: \{r, s\} \to \{r, s\}$$

that tells us which action player 2 plays given the action of player 1. The game tree of the sequential-move version with possible strategies is given in Figure 1.

A subgame-perfect equilibrium $S = (S_i)_{i \in \mathcal{N}}$ consists of a strategy S_i for each player $i \in \mathcal{N}$ so that S is a Nash equilibrium in each subgame. For example, since $c_2(s,s) > c_2(s,r)$, player 2 is strictly better off with playing r if player 1 plays s. Hence his only best-response in the subgame given by (s) ("given by forcing player 1 to play s") is r. If player 1 plays

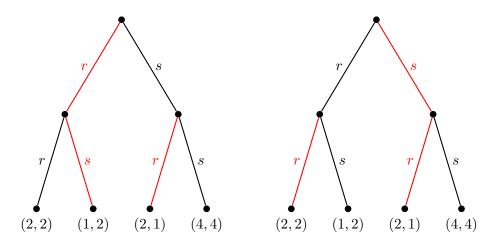


Figure 1: The game trees of sequential-move versions of the game are given. At a leaf of the tree, the costs of both players are given of the corresponding action profile. Two possible subgame-perfect equilibria are drawn red. The corresponding subgame-perfect outcomes are (r, s) and (s, r). In the tree to the right, it is also subgame-perfect for the first player to play r, which gives subgame-perfect outcome (r, r).

r, player 2 is indifferent between r and s, so either choice is subgame-perfect. Strategy $S_1 = r$ is a best-response to any strategy S_2 because

$$c_1(r, S_2(r)) = 2 \le c_1(s, S_2(s))$$

independent of how we define S_2 . However, we may also take $S_1 = s$ if $S_2(r) = r$ and $S_2(s) = r$, since then

$$c_1(s, S_2(s)) = c_1(s, r) = 2 \le 2 = c_1(r, s) = c_1(r, S_2(r)).$$

A *subgame-perfect outcome* is the action profile of the original game that we find by following the red path in the game tree from the root to a leave. In this case, the set of subgame-perfect outcomes

$$\{(r,r), (r,s), (s,r)\}$$

equals the set of Nash equilibria.

Proposition 1.2. Every sequential-move version has a subgame-perfect equilibrium.

In layer n of the tree, one connects each node to a leave node by minimising the cost of the last player. After this is fixed, you know the resulting cost vectors given the actions of the first n-1 players, so you are in your initial situation but with one player less. Hence, by "backtracking bottom-up" we will find a subgame-perfect equilibrium.

2 Each Nash equilibrium is a subgame-perfect outcome

The worst cost W = W(G) of G is defined as

$$W := \min_{A \in \mathbb{R}^n} W_A(G) = \min_{A \in \mathbb{R}^n} \max_{r \in A} d_r(x(A)_r).$$

If we order the numbers

$$d_r(x_r)$$
 for $x_r \in \{1, \ldots, n\}$ and $r \in R$

from low to high, then W will be the *n*th number. The intuition behind the worst cost is that nobody should get anything higher than W (as there exists a profile for which this is the case), yet someone will always have to get cost W (since W is minimal). Define $d_r(0) = -\infty$. Let

$$M_r = \max\{m \in \{0, \dots, n\} \mid d_r(m) \le W\} \ (r \in R)$$

denote the number of times resource r can be chosen while keeping its cost $\leq W$. A resources costs more than W for a new player if and only if it has been chosen at least M_r times.

Given the choices of the other players, you can always pick a resource costing at most W, because the action profile A achieving the minimum in the definition of W satisfies

$$\sum_{r \in R} M_r \ge \sum_{r \in R} |\{i \in \mathcal{N} : A_i = r\}| = n.$$

(We see each action profile as a tuple and perform set operations as if it were a multiset.)

Lemma 2.1. Any player has a cost of at most W in a subgame-perfect outcome.

Proof. The proof goes by induction on the number of players n. The case n = 1 is clear. Suppose the claim holds for all games with at most k players for some $k \ge 1$ and consider n = k + 1. Suppose A is a subgame-perfect outcome; renumber the players so that the corresponding order is the identity. Assume towards contradiction that player j picks a resource costing > W. At the point players $1, \ldots, j - 1$ have played, there must still be a resource s costing at most W at that moment. If player j picks s instead, then the cost of s will be > W in the outcome of the game (since A_j is subgame-perfect for j), hence one of the successors of j must pick s. However, the subgame induced by $(A_1, \ldots, A_{j-1}, s)$ has strictly less than n players, the same or lower worst cost (because $\sum_{r \in R} M_r \ge n$) and is still a singleton symmetric congestion game, hence all players in this game have cost at most W by the induction hypothesis. This is a contradiction: at least one of the successors of j picks s as well and hence must receive a cost > W as well.

Theorem 2.2. Each Nash equilibrium is a subgame-perfect outcome.

Proof. Let $A = (A_1, \ldots, A_n)$ be a Nash equilibrium. Note that by symmetry in the players, it suffices to show that a permutation of A is a subgame-perfect outcome corresponding to the identity.

Consider the game tree of the sequential-move version corresponding to the identity. We are going to draw in a subgame-perfect equilibrium (with a permutation of A on the equilibrium path) in red.

Consider any player $1 \leq j \leq n$ any any sequence of actions B_1, \ldots, B_{j-1} of his predecessors. If we are allowed to colour a resource from the multiset $M := \{A_1, \ldots, A_n\} \setminus \{B_1, \ldots, B_{j-1}\}$ red, then we are done: if we keep making such choices, we will get a red path from top to bottom which is a permutation of A (and this suffices by symmetry in the players). We will prove the following.

- (1) Any $r \in R \setminus M$ costs at least W if player j picks r.
- (2) Let $s \in M$. Then $(B_1, \ldots, B_{j-1}, s)$ has only subgame-perfect equilibria in which s costs at most W.

We need to pick an arc for player j that leads to a lowest possible cost. If we show the statements above, we are sure that at least one of those "perfect responses" will be in M, which completes our proof.

Proof of (1). Let $r \in R \setminus M$. Then resource r has been chosen at least $x(A)_r$ times already at the point j makes his choice. Assume j picks r as well. If (1) were false, we would have $d_r(x(A) + 1) < W$ (since the cost functions are non-decreasing), hence Awould not be a Nash equilibrium, as the player paying at least W (which must exist by minimality of W) would prefer to switch to resource r.

Proof of (2). At the point B_1, \ldots, B_{j-1} , s are played, there has to be a choice for the other n - j - 1 players in which all of these get $\text{cost} \leq W$. Hence the new W of this subgame is lower or equal to the original W, and hence we get from Lemma 2.1 that $(B_1, \ldots, B_{j-1}, s)$ has only subgame-perfect equilibria in which each successor of j has cost at most W. Since $s \in M$, it costs at most W before the other players make their move, and because of the fact above, no subgame-perfect move of a successor will make it cost more than W.

In fact, it follows from Lemma 2.1 that the set of Nash equilibria coincides with the set of subgame-perfect outcomes, which was already proved by different methods as well [2, 1].

References

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