THE HALES-JEWETT THEOREM AND SHELAH'S CLAIM

Notes for our seminar — Lex Schrijver

For any $n \in \mathbb{N}$, let $[n] := \{1, \ldots, n\}$. Let $n, k, t \in \mathbb{N}$. Call a function $\varphi : [n]^t \to [n]^k$ an *embedding* if φ is injective and for each $i \in [k]$ either $\varphi(x)_i$ is constant over $x \in [n]^t$ or there exists $j \in [t]$ such that $\varphi(x)_i = x_j$ for all $x \in [n]^t$. Call a function χ on $[n]^t$ reducible if $\chi(x) = \chi(y)$ whenever y arises from x by replacing each value n by n - 1.

Shelah [2] showed:

Theorem (Shelah's Claim). For $r, n, t \in \mathbb{N}$ with $n \geq 2$ there exists k = k(r, n, t) such that for each $\chi : [n]^k \to [r]$ there exists an embedding $\varphi : [n]^t \to [n]^k$ with $\chi \circ \varphi$ reducible.

Proof. By induction on t, with k(r, n, 0) = 0 being trivial. If $t \ge 1$, define

(1)
$$k(r,n,t) := k(r^n,n,t-1) + r^{n^{k(r^n,n,t-1)}}$$

Set $k := k(n, r, t), \ \ell := k(r^n, n, t-1)$, and $m := r^{n^{k(r^n, n, t-1)}}$, and choose $\chi : [n]^k \to [r]$.

For each i = 0, ..., m, let u_i be the word $(n-1)^{m-i}n^i$ in $[n]^m$, and define $\chi_i : [n]^\ell \to [r]$ by $\chi_i(w) := \chi(wu_i)$ for $w \in [n]^\ell$. Since $m+1 > m = r^{n^\ell}$, there exist $i < j \in \{0, ..., m\}$ such that $\chi_i = \chi_j$. Define the embedding $\psi : [n]^1 \to [n]^m$ by $\psi(a) = (n-1)^{m-j}a^{j-i}n^i$ for $a \in [n]$. So $\chi(w\psi(n-1)) = \chi(w\psi(n))$ for all $w \in [n]^\ell$.

Next define $\chi' : [n]^{\ell} \to [r]^n$ by

(2)
$$\chi'(w) := (\chi(w\psi(1)), \dots, \chi(w\psi(n)))$$

for $w \in [n]^{\ell}$. By the induction hypothesis and by definition of ℓ , there exists an embedding $\varphi' : [n]^{t-1} \to [n]^{\ell}$ with $\chi' \circ \varphi'$ reducible. Then $\varphi := \varphi' \times \psi$ is an embedding $[n]^{t-1} \times [n]^1 \to [n]^{\ell} \times [n]^m$ with $\chi \circ \varphi$ reducible.

This implies the theorem of Hales and Jewett [1]:

Corollary (Hales-Jewett theorem). For each $r, n \in \mathbb{N}$ there exists m = m(r, n) such that for each $\chi : [n]^m \to [r]$ there exists an embedding $\varphi : [n]^1 \to [n]^m$ with $\chi \circ \varphi$ constant.

Proof. By induction on n, the case n = 1 being trivial. Assume $n \ge 2$. Let t := m(r, n-1) and m := m(r, n) := k(r, n, t). By Shelah's Claim, there exists an embedding $\varphi : [n]^t \to [n]^m$ with $\chi \circ \varphi$ reducible. By the induction hypothesis, there exists an embedding $\psi : [n-1]^1 \to [n-1]^t$ with $\chi \circ \varphi \circ \psi$ constant. We can extend ψ to an embedding $\psi' : [n]^1 \to [n]^t$. Then $\varphi \circ \psi'$ is an embedding $[n]^1 \to [n]^m$ with $\chi \circ \varphi \circ \psi'$ constant.

References

- A. W. Hales, R. I. Jewett, Regularity and positional games, Transactions of the American Mathematical Society 106 (1963) 222–229.
- [2] S. Shelah, Primitive recursive bounds for van der Waerden numbers, Journal of the American Mathematical Society 1 (1988) 683–697.