# THE HALES-JEWETT THEOREM AND SHELAH'S CLAIM 

Notes for our seminar - Lex Schrijver
For any $n \in \mathbb{N}$, let $[n]:=\{1, \ldots, n\}$. Let $n, k, t \in \mathbb{N}$. Call a function $\varphi:[n]^{t} \rightarrow[n]^{k}$ an embedding if $\varphi$ is injective and for each $i \in[k]$ either $\varphi(x)_{i}$ is constant over $x \in[n]^{t}$ or there exists $j \in[t]$ such that $\varphi(x)_{i}=x_{j}$ for all $x \in[n]^{t}$. Call a function $\chi$ on $[n]^{t}$ reducible if $\chi(x)=\chi(y)$ whenever $y$ arises from $x$ by replacing each value $n$ by $n-1$.

Shelah [2] showed:
Theorem (Shelah's Claim). For $r, n, t \in \mathbb{N}$ with $n \geq 2$ there exists $k=k(r, n, t)$ such that for each $\chi:[n]^{k} \rightarrow[r]$ there exists an embedding $\varphi:[n]^{t} \rightarrow[n]^{k}$ with $\chi \circ \varphi$ reducible.

Proof. By induction on $t$, with $k(r, n, 0)=0$ being trivial. If $t \geq 1$, define

$$
\begin{equation*}
k(r, n, t):=k\left(r^{n}, n, t-1\right)+r^{n^{k\left(r^{n}, n, t-1\right)}} . \tag{1}
\end{equation*}
$$

Set $k:=k(n, r, t), \ell:=k\left(r^{n}, n, t-1\right)$, and $m:=r^{n^{k\left(r^{n}, n, t-1\right)}}$, and choose $\chi:[n]^{k} \rightarrow[r]$.
For each $i=0, \ldots, m$, let $u_{i}$ be the word $(n-1)^{m-i} n^{i}$ in $[n]^{m}$, and define $\chi_{i}:[n]^{\ell} \rightarrow[r]$ by $\chi_{i}(w):=\chi\left(w u_{i}\right)$ for $w \in[n]^{\ell}$. Since $m+1>m=r^{n^{\ell}}$, there exist $i<j \in\{0, \ldots, m\}$ such that $\chi_{i}=\chi_{j}$. Define the embedding $\psi:[n]^{1} \rightarrow[n]^{m}$ by $\psi(a)=(n-1)^{m-j} a^{j-i} n^{i}$ for $a \in[n]$. So $\chi(w \psi(n-1))=\chi(w \psi(n))$ for all $w \in[n]^{\ell}$.

Next define $\chi^{\prime}:[n]^{\ell} \rightarrow[r]^{n}$ by

$$
\begin{equation*}
\chi^{\prime}(w):=(\chi(w \psi(1)), \ldots, \chi(w \psi(n))) \tag{2}
\end{equation*}
$$

for $w \in[n]^{\ell}$. By the induction hypothesis and by definition of $\ell$, there exists an embedding $\varphi^{\prime}:[n]^{t-1} \rightarrow[n]^{\ell}$ with $\chi^{\prime} \circ \varphi^{\prime}$ reducible. Then $\varphi:=\varphi^{\prime} \times \psi$ is an embedding $[n]^{t-1} \times[n]^{1} \rightarrow$ $[n]^{\ell} \times[n]^{m}$ with $\chi \circ \varphi$ reducible.

This implies the theorem of Hales and Jewett [1]:
Corollary (Hales-Jewett theorem). For each $r, n \in \mathbb{N}$ there exists $m=m(r, n)$ such that for each $\chi:[n]^{m} \rightarrow[r]$ there exists an embedding $\varphi:[n]^{1} \rightarrow[n]^{m}$ with $\chi \circ \varphi$ constant.

Proof. By induction on $n$, the case $n=1$ being trivial. Assume $n \geq 2$. Let $t:=m(r, n-1)$ and $m:=m(r, n):=k(r, n, t)$. By Shelah's Claim, there exists an embedding $\varphi:[n]^{t} \rightarrow[n]^{m}$ with $\chi \circ \varphi$ reducible. By the induction hypothesis, there exists an embedding $\psi:[n-1]^{1} \rightarrow$ $[n-1]^{t}$ with $\chi \circ \varphi \circ \psi$ constant. We can extend $\psi$ to an embedding $\psi^{\prime}:[n]^{1} \rightarrow[n]^{t}$. Then $\varphi \circ \psi^{\prime}$ is an embedding $[n]^{1} \rightarrow[n]^{m}$ with $\chi \circ \varphi \circ \psi^{\prime}$ constant.

## References

[1] A. W. Hales, R. I. Jewett, Regularity and positional games, Transactions of the American Mathematical Society 106 (1963) 222-229.
[2] S. Shelah, Primitive recursive bounds for van der Waerden numbers, Journal of the American Mathematical Society 1 (1988) 683-697.

