On the graphon space

Notes for our seminar

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Let $\mathcal{W}$ be the set of measurable functions $[0,1]^2 \to \mathbb{R}$, and let $\mathcal{W}_0$ be the set of those $W \in \mathcal{W}$ with image in $[0,1]$ (the *graphons*). Let G be the group of measure preserving bijections on $[0,1]$, as acting on $\mathcal{W}$. For any $W \in \mathcal{W}_0$ and any simple graph F, let

$$(1) \qquad t(F,W) := \int_{[0,1]^{VF}} \prod_{ij \in EF} w(x_i,x_j)dx.$$

The following was proved by Borgs, Chayes, Lovász, Sós, and Vesztergombi [1]:

Theorem 1. *For $U,W \in \mathcal{W}_0$, if $t(F,U) = t(F,W)$ for each simple graph F, then $\delta_\square(U,W) = 0$.*

Proof. I. Let $\delta_1 := d_1/G$. We first show that for each $W \in \mathcal{W}$:

$$(2) \qquad \lim_{k\to\infty} \mathbf{E}[\delta_1(W, \mathbb{H}(k,W))] = 0.$$

Suppose to the contrary that for some $\varepsilon > 0$ there are infinitely many k with $\mathbf{E}[\delta_1(W,\mathbb{H}(k,W))] > \varepsilon$. Choose a step function U with $d_1(U,W) < \varepsilon/3$. Then (where, for $x \in [0,1]^2$, W_x denotes the weighted graph with vertex set $[k]$ and weight $W(x_i,x_j)$ on edge ij with $i \neq j$):

$$(3) \qquad \mathbf{E}[d_1(\mathbb{H}(k,U),\mathbb{H}(k,W))] = \int_{[0,1]^k} d_1(U_x,W_x)dx$$
$$= \int_{[0,1]^k} k^{-2} \sum_{\substack{i,j=1\\ i\neq j}}^{k} |U(x_i,x_j) - W(x_i,x_j)|dx \leq \int_{[0,1]^2} |U(y_1,y_2) - W(y_1,y_2)|dy =$$
$$d_1(U,W).$$

Hence $\mathbf{E}[\delta_1(U,\mathbb{H}(k,U))] > \varepsilon/3$ for infinitely many k. So to prove (2), we can assume that W is a step function, with intervals as steps. Let $J_k := \{x \in [0,1]^k \mid x_1 \leq x_2 \leq \cdots \leq x_k\}$. Then it suffices to show

$$(4) \qquad \lim_{k\to\infty} k! \int_{J_k} d_1(W,W_x)dx = 0.$$

To prove (4), we can assume that $W = \mathbf{1}_{[\alpha,\alpha+\beta]^2}$ for some $\alpha,\beta \in [0,1]$ (by the sublinearity of $d_1(W,W_x)$ in W). Setting $\gamma := 1-\alpha-\beta$ we have, if $i+j+l = k$,

$$(5) \qquad d_1(W, \mathbf{1}_{[\frac{i}{k},\frac{i+j}{k}]^2}) \leq 2(|\tfrac{i}{k} - \alpha| + |\tfrac{l}{k} - \gamma|).$$

This gives the following, where the term $\frac{1}{k}$ corrects for the zeros on the diagonal of W_x:

$$k!\int_{J_k} d_1(W,W_x)dx \le \tfrac{1}{k} + 2\sum_{i+j+l=k}\tbinom{k}{i,j,l}\alpha^i\beta^j\gamma^l\big(\big|\tfrac{i}{k}-\alpha\big| + \big|\tfrac{l}{k}-\gamma\big|\big). \tag{6}$$

With Cauchy-Schwarz we have

$$\sum_{i+j+l=k}\tbinom{k}{i,j,l}\alpha^i\beta^j\gamma^l\big|\tfrac{i}{k}-\alpha\big| = \sum_{i=0}^{k}\tbinom{k}{i}\alpha^i(1-\alpha)^{k-i}\big|\tfrac{i}{k}-\alpha\big| \le \tag{7}$$
$$\big(\sum_{i=0}^{k}\tbinom{k}{i}\alpha^i(1-\alpha)^{k-i}(\tfrac{i}{k}-\alpha)^2\big)^{1/2} = \big(\tfrac{\alpha-\alpha^2}{k}\big)^{1/2},$$

which tends to 0 as $k\to\infty$. By symmetry, we have a similar estimate for the other part in the summation in (6), and we have (4), and hence (2).

II. We next show that for each $W\in\mathcal{W}_0$:

$$\lim_{k\to\infty}\mathbf{E}[d_\Box(\mathbb{H}(k,W),\mathbb{G}(k,W))] = 0. \tag{8}$$

For any weighted graph (H,w), let $\mathbf{G}(H)$ be the random graph where edge ij is chosen independently with probability $w(ij)$ $(i\neq j)$. Let H have vertex set $[k]$. Then for any fixed $S\subseteq[k]$, by the Chernoff-Hoeffding inequality,

$$\mathbf{Pr}[|\sum_{i,j\in S}(e_{\mathbf{G}(H)}(ij)-w(ij)|) > 2k^{3/2}] = \mathbf{Pr}[|\sum_{\substack{i,j\in S\\ i<j}}(e_{\mathbf{G}(H)}(ij)-w(ij))| > k^{3/2}] < \tag{9}$$
$$2e^{-k^3/|S|^2} \le 2e^{-k},$$

where $e_{\mathbf{G}(H)}(ij)=1$ if $ij\in E(\mathbf{G}(H))$ and $e_{\mathbf{G}(H)}(ij)=0$ otherwise. This gives

$$\mathbf{Pr}[d_\Box(\mathbf{G}(H),H) > 4k^{-1/2}] \le \mathbf{Pr}[\exists S\subseteq[k] : |\sum_{i,j\in S}(e_{\mathbf{G}(H)}(ij)-w(ij))| > 2k^{3/2}] \tag{10}$$
$$< 2^k 2e^{-k}.$$

Since $d_\Box(\mathbf{G}(H),H)\le 1$, this implies

$$\mathbf{E}[d_\Box(\mathbf{G}(H),H)] \le 4k^{-1/2} + 2^k 2e^{-k}. \tag{11}$$

Now substitute $H := \mathbb{H}(k,W)$. As $\mathbf{G}(\mathbb{H}(k,W)) = \mathbb{G}(k,W)$ and as the right hand side of (11) tends to 0 as $k\to\infty$, we get (8).

III. We finally derive the theorem. We have for any k:

$$\delta_\Box(U,W) \le \mathbf{E}[\delta_\Box(U,\mathbb{G}(k,W))] + \mathbf{E}[\delta_\Box(\mathbb{G}(k,W),W)] = \tag{12}$$

$$\mathbf{E}[\delta_\square(U,\mathbb{G}(k,U))]+\mathbf{E}[\delta_\square(\mathbb{G}(k,W),W)].$$

The equality follows from the condition in the theorem. By (2) and (8), the last expression in (12) tends to 0 as $k\to\infty$ (using $d_\square\le d_1$). So $\delta_\square(U,W)=0$. ∎

Let $\mathcal{F}$ be the collection of all connected simple graphs. Note that the distance function

$$d(x,y):=\sup_{F\in\mathcal{F}}\frac{|x(F)-y(F)|}{|E(F)|} \tag{13}$$

for $x,y\in[0,1]^{\mathcal{F}}$ gives the Tychonoff product topology on $[0,1]^{\mathcal{F}}$, since for each m, there are only finitely many $F\in\mathcal{F}$ with $|E(F)|\le m$.

Let $\mathcal{W}_0//G$ be the space obtained from $(\mathcal{W}_0,d_\square)/G$ by identifying points at distance 0. (So its points are the closures of the G-orbits in $\mathcal{W}_0$.) Define $\tau:\mathcal{W}_0//G\to[0,1]^{\mathcal{F}}$ by

$$\tau(W)(F):=t(F,W) \tag{14}$$

for $W\in\mathcal{W}_0$ and $F\in\mathcal{F}$. Since $|t(F,U)-t(F,W)|/|E(F)|\le\delta_\square(U,W)$ for all $U,W\in\mathcal{W}_0$, τ is continuous.

Corollary 1a. *τ is injective.*

Proof. This is equivalent to Theorem 1. ∎

By (2) and (8), the graphs among the graphons span $\mathcal{W}_0$, and hence also the range of τ. The latter can be characterized by reflection positivity (Lovász and Szegedy [2]).

Corollary 1a implies a strengthening of Theorem 1:

Corollary 1b. *There exists a function $\varphi:(0,1]\to(0,1]$ such that*

$$\frac{|t(F,U)-t(F,W)|}{|E(F)|}\ge\varphi(\delta_\square(U,W)) \tag{15}$$

for all $U,W\in\mathcal{W}_0$ and $F\in\mathcal{F}$.

Proof. This follows from the fact that τ is continuous and bijective between compact metric spaces, and that hence τ^{-1} is uniformly continuous. ∎

Bound (15) is qualitative. In [1] it is proved that one can take φ of order $(\exp\exp(1/x))^{-1}$.

Appendix: The Chernoff-Hoeffding inequality

First note that for any $a\in[0,1]$ and $t\in\mathbb{R}$ we have

$$ae^{(1-a)t}+(1-a)e^{-at}\le\tfrac{1}{2}(e^t+e^{-t})\le e^{t^2/2}, \tag{16}$$

as $(0, ae^{(1-a)t} + (1-a)e^{-at}) = (1-a)(-a, e^{-at}) + a(1-a, e^{(1-a)t})$, hence it is below the line connecting $(-1, e^{-t})$ and $(1, e^{t})$, by the convexity of e^x. The second inequality in (16) follows by Taylor expansion.

Theorem 2 (Chernoff-Hoeffding inequality)**.** *Let $x_1, \ldots, x_n$ be independent random variables from $\{0,1\}$. Then for $\lambda \geq 0$:*

$$\mathbf{Pr}[\sum_{i=1}^{n}(x_i - \mathbf{E}[x_i]) > \lambda] < e^{-\lambda^2/2n}. \tag{17}$$

Proof. We have

$$e^{\lambda^2/n}\mathbf{Pr}[\sum_{i=1}^{n}(x_i - \mathbf{E}[x_i]) > \lambda] = e^{\lambda^2/n}\mathbf{Pr}[e^{\lambda(\sum_{i=1}^{n}(x_i - \mathbf{E}[x_i]))/n} > e^{\lambda^2/n}] < \mathbf{E}[e^{\lambda(\sum_{i=1}^{n}(x_i - \mathbf{E}[x_i]))/n}] = \mathbf{E}[\prod_{i=1}^{n} e^{\lambda(x_i - \mathbf{E}[x_i])/n}] = \prod_{i=1}^{n}\mathbf{E}[e^{\lambda(x_i - \mathbf{E}[x_i])/n}] \leq \prod_{i=1}^{n} e^{\lambda^2/2n^2} = e^{\lambda^2/2n}, \tag{18}$$

where the first inequality is Markov's inequality and the last inequality follows from (16). ∎

References

[1] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, K. Vesztergombi, Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing, 2007, `arXiv:math/0702004v1`

[2] L. Lovász, B. Szegedy, Limits of dense graph sequences, *Journal of Combinatorial Theory, Series B* 96 (2006) 933–957.