## RAINBOW PATHS IN EDGE-COLOURED REGULAR GRAPHS

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Abstract. We show that if  $k \leq 10$ , then each properly k-edge-coloured k-regular simple graph contains a rainbow path of length k - 1. The proof requires a computer search.

In this note, all graphs are simple and all edge-colourings are proper. A path is *rainbow* if all its edges have different colours.

For any k-edge-coloured graph G, let p(G) denote the length of the longest rainbow path in G. For  $k \in \mathbb{N}$ , let p(k) be the minimum of p(G) taken over all k-regular k-edge-coloured graphs G.

Trivially,  $p(k) \leq k$  and p(k) = k for  $k \leq 2$ . A theorem of Babu, Sunil Chandran, and Rajendraprasad [1] implies  $p(k) \geq \frac{2}{3}k$  for all k (short proof in [2]). For  $k \geq 3$  one has  $p(k) \leq k - 1$ .<sup>1</sup> We show that p(k) = k - 1 if  $3 \leq k \leq 10$ .

Call a (not necessarily regular) graph H a rainbow-bar if H is (|V(H)|+1)-edge-coloured and contains a rainbow Hamilton path, and if for each rainbow Hamilton path P in H, for each color c missing on P and for each end vertex v of P, there is an edge of color c incident with v.

**Theorem 1.** No rainbow-bar H with |V(H)| < 10 exists.

**Proof.** This follows from a computer search.

**Theorem 2.** Let G be a k-regular k-edge-coloured graph with p(G) = k - 2. Then G contains a rainbow-bar H with |V(H)| = k - 1 as induced subgraph.

**Proof.** Let P be a rainbow path of length k - 2. Let H be the subgraph of G induced by V(P). Then H is a rainbow-bar. Indeed, since P is a rainbow path of length k - 2 it is a Hamilton path in H. Moreover, consider any Hamilton path Q in H. So Q is a rainbow path of length k - 2. Consider any color c not occurring on Q and any end v of Q. As Q is a longest rainbow path, the c-colored edge of G incident with v must have its end other than v on V(P), hence it is an edge of H.

**Corollary 2a.** If  $k \leq 10$ , then each k-regular k-edge-coloured graph G contains a rainbow path of length k - 1.

**Proof.** Suppose not. Inductively (by deleting the edges of some color) we know p(G) = k-2. This contradicts Theorems 1 and 2.

**Corollary 2b.** If  $3 \le k \le 10$ , then p(k) = k - 1.

<sup>&</sup>lt;sup>1</sup> If  $k \ge 3$ , there exist distinct nonzero vectors  $a_1, \ldots, a_k$  in  $GF(2)^k$  with  $\sum_i a_i = 0$ . (For instance, take the incidence vectors of the edges of a cycle on  $\{1, \ldots, k\}$ .) Let G be the graph with vertex set  $GF(2)^k$ , and an edge of colour  $i \in \{1, \ldots, k\}$  between vertices u and v if  $u + v = a_i$ . There are no other edges. So G is k-regular and k-edge-coloured. Then any walk on which all colours occur exactly once must be a closed walk, as  $\sum_i a_i = 0$ . Hence no rainbow path of length k exists.

**Proof.** Directly from Corollary 2a.

Question. Do rainbow-bars exist?

As above, a negative answer implies p(k) = k - 1 for all  $k \ge 3$ .

With a similar method one may prove, for  $k \leq 8$ , that each k-regular k-edge-colored graph has a rainbow path or a rainbow cycle of length k.

## References

- J. Babu, L. Sunil Chandran, D. Rajendraprasad, Heterochromatic paths in edge colored graphs without small cycles and heterochromatic-triangle-free graphs, *European Journal of Combinatorics* 48 (2015) 110-126.
- [2] D. Johnston, C. Palmer, A. Sarkar, Rainbow Turán problems for paths and forests of stars, *Electronic Journal of Combinatorics* 24 (2017), no. 1, Paper 1.34, 15 pp.