# RAINBOW PATHS IN EDGE-COLOURED REGULAR GRAPHS 

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#### Abstract

We show that if $k \leq 10$, then each properly $k$-edge-coloured $k$-regular simple graph contains a rainbow path of length $k-1$. The proof requires a computer search.


In this note, all graphs are simple and all edge-colourings are proper. A path is rainbow if all its edges have different colours.

For any $k$-edge-coloured graph $G$, let $p(G)$ denote the length of the longest rainbow path in $G$. For $k \in \mathbb{N}$, let $p(k)$ be the minimum of $p(G)$ taken over all $k$-regular $k$-edge-coloured graphs $G$.

Trivially, $p(k) \leq k$ and $p(k)=k$ for $k \leq 2$. A theorem of Babu, Sunil Chandran, and Rajendraprasad [1] implies $p(k) \geq \frac{2}{3} k$ for all $k$ (short proof in [2]). For $k \geq 3$ one has $p(k) \leq k-1,1$ We show that $p(k)=k-1$ if $3 \leq k \leq 10$.

Call a (not necessarily regular) graph $H$ a rainbow-bar if $H$ is $(|V(H)|+1)$-edge-coloured and contains a rainbow Hamilton path, and if for each rainbow Hamilton path $P$ in $H$, for each color $c$ missing on $P$ and for each end vertex $v$ of $P$, there is an edge of color $c$ incident with $v$.

Theorem 1. No rainbow-bar $H$ with $|V(H)|<10$ exists.
Proof. This follows from a computer search.

Theorem 2. Let $G$ be a $k$-regular $k$-edge-coloured graph with $p(G)=k-2$. Then $G$ contains a rainbow-bar $H$ with $|V(H)|=k-1$ as induced subgraph.

Proof. Let $P$ be a rainbow path of length $k-2$. Let $H$ be the subgraph of $G$ induced by $V(P)$. Then $H$ is a rainbow-bar. Indeed, since $P$ is a rainbow path of length $k-2$ it is a Hamilton path in $H$. Moreover, consider any Hamilton path $Q$ in $H$. So $Q$ is a rainbow path of length $k-2$. Consider any color $c$ not occurring on $Q$ and any end $v$ of $Q$. As $Q$ is a longest rainbow path, the $c$-colored edge of $G$ incident with $v$ must have its end other than $v$ on $V(P)$, hence it is an edge of $H$.

Corollary 2a. If $k \leq 10$, then each $k$-regular $k$-edge-coloured graph $G$ contains a rainbow path of length $k-1$.

Proof. Suppose not. Inductively (by deleting the edges of some color) we know $p(G)=k-2$. This contradicts Theorems 1 and 2 .

Corollary 2b. If $3 \leq k \leq 10$, then $p(k)=k-1$.

[^0]Proof. Directly from Corollary 2a,

Question. Do rainbow-bars exist?
As above, a negative answer implies $p(k)=k-1$ for all $k \geq 3$.
With a similar method one may prove, for $k \leq 8$, that each $k$-regular $k$-edge-colored graph has a rainbow path or a rainbow cycle of length $k$.

## References

[1] J. Babu, L. Sunil Chandran, D. Rajendraprasad, Heterochromatic paths in edge colored graphs without small cycles and heterochromatic-triangle-free graphs, European Journal of Combinatorics 48 (2015) 110-126.
[2] D. Johnston, C. Palmer, A. Sarkar, Rainbow Turán problems for paths and forests of stars, Electronic Journal of Combinatorics 24 (2017), no. 1, Paper 1.34, 15 pp.


[^0]:    ${ }^{1}$ If $k \geq 3$, there exist distinct nonzero vectors $a_{1}, \ldots, a_{k}$ in $\operatorname{GF}(2)^{k}$ with $\sum_{i} a_{i}=0$. (For instance, take the incidence vectors of the edges of a cycle on $\{1, \ldots, k\}$.) Let $G$ be the graph with vertex set $\operatorname{GF}(2)^{k}$, and an edge of colour $i \in\{1, \ldots, k\}$ between vertices $u$ and $v$ if $u+v=a_{i}$. There are no other edges. So $G$ is $k$-regular and $k$-edge-coloured. Then any walk on which all colours occur exactly once must be a closed walk, as $\sum_{i} a_{i}=0$. Hence no rainbow path of length $k$ exists.

