SHORTEST DISJOINT PATHS

Notes for our seminar Lex Schrijver

1. The shortest disjoint paths problem

We show that the *shortest disjoint paths problem*:

(1) given: a directed graph D = (V, A), vertices $s_1, t_1, \ldots, s_k, t_k \in V$, and a 'length' function $\ell : A \to \mathbb{Z}_+$, find: disjoint directed paths P_1, \ldots, P_k , where P_i runs from s_i to t_i (for $i = 1, \ldots, k$), with $\ell(P_1) + \cdots + \ell(P_k)$ minimum,

is solvable in polynomial time if D is planar and there exist faces S and T such that each s_i is incident with S and each t_i is incident with T ([1]).

2. We may assume

We may assume that:

(2) (i) T is the unbounded face,

- (ii) t_1, \ldots, t_k are distinct and occur clockwise around the boundary of T,
- (iii) s_1, \ldots, s_k are distinct and occur clockwise around the boundary of S,
- (iv) each s_i and t_i has total degree 1, and each other vertex of D has total degree 3,
- (v) there is an undirected S T path Q in the dual graph D^* such that there exist disjoint curves K_1, \ldots, K_k in $\mathbb{R}^2 \setminus (S \cup T \cup Q)$, where K_i runs from s_i to t_i $(i = 1, \ldots, k)$.

Let $q: A \to \{-1, 0, 1\}$ be defined by, for $a \in A$,

(3)
$$q(a) = \begin{cases} 1 & \text{if } a \text{ crosses } Q \text{ from left to right,} \\ -1 & \text{if } a \text{ crosses } Q \text{ from right to left,} \\ 0 & \text{otherwise.} \end{cases}$$

For $x \in \mathbb{Z}^A$, we call $q^{\mathsf{T}}x$ the *winding number* of x and $\ell^{\mathsf{T}}x$ the *length* of x.

3. Flows

A flow is a function $f : A \to \{0, 1\}$ such that for each $v \in V$:

(4)
$$f(\delta^{\text{in}}(v)) - f(\delta^{\text{out}}(v)) = \begin{cases} -1 & \text{if } v = s_i \text{ for some } i, \\ 1 & \text{if } v = t_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

We say that f is a w-flow if $q^{\mathsf{T}} f = w$. Then, taking indices mod k:

Proposition 1. A function $f: A \to \mathbb{Z}$ is a w-flow if and only if there exist directed paths P_1, \ldots, P_k and directed circuits C_1, \ldots, C_m such that $P_1, \ldots, P_k, C_1, \ldots, C_m$ are pairwise disjoint, such that P_i runs from s_i to t_{i+w} (for $i = 1, \ldots, k$), and such that

(5)
$$f = \chi^{P_1} + \dots + \chi^{P_k} + \chi^{C_1} + \dots + \chi^{C_m}.$$

It follows that a flow that is shortest among all flows with winding number being a multiple of k, yields a solution to (1).

4. Circulations

A circulation is a function $c : A \to \mathbb{Z}$ such that $c(\delta^{\text{out}}(v)) = c(\delta^{\text{in}}(v))$ for each $v \in V$. A *w*-circulation is a circulation with winding number w. Note that if f and g are flows with winding numbers v and w, then g - f is a (w - v)-circulation. This gives:

Proposition 2. Let f be a v-flow and let g be a w-flow. Then g is a shortest w-flow if and only if g - f is shortest among all (w - v)-circulations c satisfying $0 \le f + c \le 1$.

Let \mathcal{F} be the collection of faces of D. Let N be the $A \times \mathcal{F}$ matrix with, for any $a \in A$ and $F \in \mathcal{F}$:

(6)
$$N_{a,F} := \begin{cases} 1 & \text{if } F \text{ is the face at the left-hand side of } a, \\ -1 & \text{if } F \text{ is the face at the right-hand side of } a, \\ 0 & \text{otherwise.} \end{cases}$$

Then:

Proposition 3. A function $c : A \to \mathbb{Z}$ is a w-circulation if and only if there exists a $y \in \mathbb{Z}^{\mathcal{F}}$ with c = Ny, $y_S = 0$, and $y_T = w$.

5. Convexity

For each $w \in \mathbb{Z}$, let λ_w be the minimum length of a *w*-flow.

Proposition 4. Let $w' \leq w \leq w'' \in \mathbb{Z}$ with $\lambda_{w'}$ and $\lambda_{w''}$ finite, and $w = \alpha w' + (1 - \alpha)w''$ for $0 \leq \alpha \leq 1$. Then

(7) $\lambda_w \le \alpha \lambda_{w'} + (1 - \alpha) \lambda_{w''}.$

Proof. We can assume that w = w' + 1. Let r := w'' - w'. Let f and g be a shortest w'-flow and a shortest w''-flow, respectively. Then c := g - f is a circulation with winding number v := w'' - w'. As each vertex of D has total degree 3 and as c has values in $\{-1, 0, 1\}$, there exist pairwise disjoint undirected circuits C_1, \ldots, C_m such that

(8)
$$c = \chi^{C_1} + \dots + \chi^{C_m}.$$

(For an undirected circuit C, $\chi^{C}(a) = 1$ if a is traversed in forward direction, $\chi^{C}(a) = -1$ if a is traversed in backward direction, and $\chi^{C}(a) = 0$ otherwise.) Each $\chi^{C_{j}}$ has winding number in $\{-1, 0, 1\}$, adding up to v.

This implies (by appropriately combining the C_j) that c can be decomposed as $c = c_1 + \cdots + c_v$, where each c_i is a circulation with winding number 1. Then for each $i = 1, \ldots, v$, $f + c_i$ is a w-flow. (It is a 0,1 function since if f(a) = 0, then c(a) = g(a), so $0 \le c(a) \le 1$, hence $0 \le c_i(a) \le 1$, implying $0 \le f(a) + c_i(a) \le 1$. Similarly, if f(a) = 1, then c(a) = g(a) - 1, so $-1 \le c(a) \le 0$, hence $-1 \le c_i(a) \le 0$, implying again $0 \le f(a) + c_i(a) \le 1$.)

So $\lambda_w \leq \ell^{\mathsf{T}} f + \ell^{\mathsf{T}} c_i$. Hence

(9)
$$v\lambda_{w} \leq \sum_{i=1}^{v} (\ell^{\mathsf{T}} f + \ell^{\mathsf{T}} c_{i}) = v\ell^{\mathsf{T}} f + \sum_{i=1}^{v} \ell^{\mathsf{T}} c_{i} = (v-1)\ell^{\mathsf{T}} f + (\ell^{\mathsf{T}} f + \sum_{i=1}^{v} \ell^{\mathsf{T}} c_{i}) = (v-1)\ell^{\mathsf{T}} f + \ell^{\mathsf{T}} g = (v-1)\lambda_{w'} + \lambda_{w''}.$$

6. Shortest flow

Let f be a flow of minimum length (over all winding numbers), which is a solution of the LP problem

(10)
$$\min\{\ell^{\mathsf{T}} f \mid 0 \le f \le 1, Mf = \sum_{i=1}^{k} (e_{t_i} - e_{s_i})\},\$$

where M is the $V \times A$ incidence matrix of D (with $M_{v,a} = 1$ if a enters v, $M_{v,a} = -1$ if a leaves v, and $M_{v,a} = 0$ otherwise). Note that M is totally unimodular, so that (10) has an integer optimum solution.

7. Shortest W-flow

Let f have winding number v. Given $w \in \mathbb{Z}$, we can derive from f a shortest w-flow, by finding a shortest (w - v)-circulation c such that f + c is a 0, 1 function (by Proposition 2).

By Proposition 3, such a circulation is equal to Ny, where y is an optimum solution of the LP problem

(11)
$$\min\{\ell^{\mathsf{T}} Ny \mid -f \leq Ny \leq 1 - f, y_S = 0, y_T = w - v\}.$$

Note that again N is totally unimodular, so that (11) has an integer optimum solution.

8. Solving (1)

Let $w' := k \lfloor v/k \rfloor$ and w'' := w' + k. By (7), a shortest flow of winding number w' or w'' gives a solution of problem (1). By Section 7, we can find such a flow in polynomial time.

References

[1] É. Colin de Verdière, A. Schrijver, Shortest vertex-disjoint two-face paths in planar graphs, ACM Transactions on Algorithms 7 (2011) no. 2, Art. 19.