# SHORTEST DISJOINT PATHS 

## Notes for our seminar

Lex Schrijver

## 1. The shortest disjoint paths problem

We show that the shortest disjoint paths problem:
given: a directed graph $D=(V, A)$, vertices $s_{1}, t_{1}, \ldots, s_{k}, t_{k} \in V$, and a 'length' function $\ell: A \rightarrow \mathbb{Z}_{+}$,
find: disjoint directed paths $P_{1}, \ldots, P_{k}$, where $P_{i}$ runs from $s_{i}$ to $t_{i}$ (for $i=$ $1, \ldots, k)$, with $\ell\left(P_{1}\right)+\cdots \ell\left(P_{k}\right)$ minimum,
is solvable in polynomial time if $D$ is planar and there exist faces $S$ and $T$ such that each $s_{i}$ is incident with $S$ and each $t_{i}$ is incident with $T$ ([1]).

## 2. We may assume

We may assume that:
(2) (i) $T$ is the unbounded face,
(ii) $t_{1}, \ldots, t_{k}$ are distinct and occur clockwise around the boundary of $T$,
(iii) $s_{1}, \ldots, s_{k}$ are distinct and occur clockwise around the boundary of $S$,
(iv) each $s_{i}$ and $t_{i}$ has total degree 1 , and each other vertex of $D$ has total degree 3 ,
(v) there is an undirected $S-T$ path $Q$ in the dual graph $D^{*}$ such that there exist disjoint curves $K_{1}, \ldots, K_{k}$ in $\mathbb{R}^{2} \backslash(S \cup T \cup Q)$, where $K_{i}$ runs from $s_{i}$ to $t_{i}$ $(i=1, \ldots, k)$.

Let $q: A \rightarrow\{-1,0,1\}$ be defined by, for $a \in A$,

$$
q(a)= \begin{cases}1 & \text { if } a \text { crosses } Q \text { from left to right }  \tag{3}\\ -1 & \text { if } a \text { crosses } Q \text { from right to left } \\ 0 & \text { otherwise }\end{cases}
$$

For $x \in \mathbb{Z}^{A}$, we call $q^{\top} x$ the winding number of $x$ and $\ell^{\top} x$ the length of $x$.

## 3. Flows

A flow is a function $f: A \rightarrow\{0,1\}$ such that for each $v \in V$ :

$$
f\left(\delta^{\text {in }}(v)\right)-f\left(\delta^{\text {out }}(v)\right)= \begin{cases}-1 & \text { if } v=s_{i} \text { for some } i  \tag{4}\\ 1 & \text { if } v=t_{i} \text { for some } i \\ 0 & \text { otherwise }\end{cases}
$$

We say that $f$ is a $w$-flow if $q^{\top} f=w$. Then, taking indices $\bmod k$ :
Proposition 1. A function $f: A \rightarrow \mathbb{Z}$ is a $w$-flow if and only if there exist directed paths $P_{1}, \ldots, P_{k}$ and directed circuits $C_{1}, \ldots, C_{m}$ such that $P_{1}, \ldots, P_{k}, C_{1}, \ldots, C_{m}$ are pairwise disjoint, such that $P_{i}$ runs from $s_{i}$ to $t_{i+w}$ (for $i=1, \ldots, k$ ), and such that

$$
\begin{equation*}
f=\chi^{P_{1}}+\cdots+\chi^{P_{k}}+\chi^{C_{1}}+\cdots+\chi^{C_{m}} . \tag{5}
\end{equation*}
$$

It follows that a flow that is shortest among all flows with winding number being a multiple of $k$, yields a solution to (1).

## 4. Circulations

A circulation is a function $c: A \rightarrow \mathbb{Z}$ such that $c\left(\delta^{\text {out }}(v)\right)=c\left(\delta^{\text {in }}(v)\right)$ for each $v \in V$. A $w$-circulation is a circulation with winding number $w$. Note that if $f$ and $g$ are flows with winding numbers $v$ and $w$, then $g-f$ is a $(w-v)$-circulation. This gives:

Proposition 2. Let $f$ be a v-flow and let $g$ be a $w$-flow. Then $g$ is a shortest $w$-flow if and only if $g-f$ is shortest among all $(w-v)$-circulations c satisfying $\mathbf{0} \leq f+c \leq \mathbf{1}$.

Let $\mathcal{F}$ be the collection of faces of $D$. Let $N$ be the $A \times \mathcal{F}$ matrix with, for any $a \in A$ and $F \in \mathcal{F}$ :

$$
N_{a, F}:= \begin{cases}1 & \text { if } F \text { is the face at the left-hand side of } a,  \tag{6}\\ -1 & \text { if } F \text { is the face at the right-hand side of } a, \\ 0 & \text { otherwise. }\end{cases}
$$

Then:
Proposition 3. A function $c: A \rightarrow \mathbb{Z}$ is a w-circulation if and only if there exists a $y \in \mathbb{Z}^{\mathcal{F}}$ with $c=N y, y_{S}=0$, and $y_{T}=w$.

## 5. Convexity

For each $w \in \mathbb{Z}$, let $\lambda_{w}$ be the minimum length of a $w$-flow.
Proposition 4. Let $w^{\prime} \leq w \leq w^{\prime \prime} \in \mathbb{Z}$ with $\lambda_{w^{\prime}}$ and $\lambda_{w^{\prime \prime}}$ finite, and $w=\alpha w^{\prime}+(1-\alpha) w^{\prime \prime}$ for $0 \leq \alpha \leq 1$. Then

$$
\begin{equation*}
\lambda_{w} \leq \alpha \lambda_{w^{\prime}}+(1-\alpha) \lambda_{w^{\prime \prime}} . \tag{7}
\end{equation*}
$$

Proof. We can assume that $w=w^{\prime}+1$. Let $r:=w^{\prime \prime}-w^{\prime}$. Let $f$ and $g$ be a shortest $w^{\prime}$-flow and a shortest $w^{\prime \prime}$-flow, respectively. Then $c:=g-f$ is a circulation with winding number $v:=w^{\prime \prime}-w^{\prime}$. As each vertex of $D$ has total degree 3 and as $c$ has values in $\{-1,0,1\}$, there exist pairwise disjoint undirected circuits $C_{1}, \ldots, C_{m}$ such that

$$
\begin{equation*}
c=\chi^{C_{1}}+\cdots+\chi^{C_{m}} . \tag{8}
\end{equation*}
$$

(For an undirected circuit $C, \chi^{C}(a)=1$ if $a$ is traversed in forward direction, $\chi^{C}(a)=-1$ if $a$ is traversed in backward direction, and $\chi^{C}(a)=0$ otherwise.) Each $\chi^{C_{j}}$ has winding number in $\{-1,0,1\}$, adding up to $v$.

This implies (by appropriately combining the $C_{j}$ ) that $c$ can be decomposed as $c=$ $c_{1}+\cdots+c_{v}$, where each $c_{i}$ is a circulation with winding number 1 . Then for each $i=$ $1, \ldots, v, f+c_{i}$ is a $w$-flow. (It is a 0,1 function since if $f(a)=0$, then $c(a)=g(a)$, so $0 \leq c(a) \leq 1$, hence $0 \leq c_{i}(a) \leq 1$, implying $0 \leq f(a)+c_{i}(a) \leq 1$. Similarly, if $f(a)=1$, then $c(a)=g(a)-1$, so $-1 \leq c(a) \leq 0$, hence $-1 \leq c_{i}(a) \leq 0$, implying again $0 \leq f(a)+c_{i}(a) \leq 1$.)

So $\lambda_{w} \leq \ell^{\top} f+\ell^{\top} c_{i}$. Hence

$$
\begin{align*}
& v \lambda_{w} \leq \sum_{i=1}^{v}\left(\ell^{\top} f+\ell^{\top} c_{i}\right)=v \ell^{\top} f+\sum_{i=1}^{v} \ell^{\top} c_{i}=(v-1) \ell^{\top} f+\left(\ell^{\top} f+\sum_{i=1}^{v} \ell^{\top} c_{i}\right)=  \tag{9}\\
& (v-1) \ell^{\top} f+\ell^{\top} g=(v-1) \lambda_{w^{\prime}}+\lambda_{w^{\prime \prime}} .
\end{align*}
$$

## 6. Shortest flow

Let $f$ be a flow of minimum length (over all winding numbers), which is a solution of the LP problem

$$
\begin{equation*}
\min \left\{\ell^{\top} f \mid 0 \leq f \leq 1, M f=\sum_{i=1}^{k}\left(e_{t_{i}}-e_{s_{i}}\right)\right\} \tag{10}
\end{equation*}
$$

where $M$ is the $V \times A$ incidence matrix of $D$ (with $M_{v, a}=1$ if $a$ enters $v, M_{v, a}=-1$ if $a$ leaves $v$, and $M_{v, a}=0$ otherwise). Note that $M$ is totally unimodular, so that 10) has an integer optimum solution.

## 7. Shortest $W$-flow

Let $f$ have winding number $v$. Given $w \in \mathbb{Z}$, we can derive from $f$ a shortest $w$-flow, by finding a shortest $(w-v)$-circulation $c$ such that $f+c$ is a 0,1 function (by Proposition 22).

By Proposition 3, such a circulation is equal to $N y$, where $y$ is an optimum solution of the LP problem

$$
\begin{equation*}
\min \left\{\ell^{\top} N y \mid-f \leq N y \leq \mathbf{1}-f, y_{S}=0, y_{T}=w-v\right\} . \tag{11}
\end{equation*}
$$

Note that again $N$ is totally unimodular, so that (11) has an integer optimum solution.

## 8. Solving (1)

Let $w^{\prime}:=k\lfloor v / k\rfloor$ and $w^{\prime \prime}:=w^{\prime}+k$. By (7), a shortest flow of winding number $w^{\prime}$ or $w^{\prime \prime}$ gives a solution of problem (11). By Section 7, we can find such a flow in polynomial time.

## References

[1] É. Colin de Verdière, A. Schrijver, Shortest vertex-disjoint two-face paths in planar graphs, ACM Transactions on Algorithms 7 (2011) no. 2, Art. 19.

