V. Szemerédi's regularity lemma

1. Szemerédi's regularity lemma

The 'regularity lemma' of Endre Szemerédi [5] roughly asserts that, for each $\varepsilon > 0$, there exists a number k such that the vertex set V of any graph $G = (V, E)$ can be partitioned into at most k *almost* equal-sized classes so that between *almost* any two classes, the edges are distributed *almost* homogeneously. Here *almost* depends on ε. The important issue is that k only depends on ε, and not on the size of the graph.

Let $G = (V, E)$ be a directed graph. For nonempty $I, J \subseteq V$, let $e(I, J) := |E \cap (I \times J)|$ and $d(I, J) := e(I, J)/|I||J|$. Call the pair (I, J) *ε-regular* if for all $X \subseteq I, Y \subseteq J$:

$$\text{(1)} \qquad \text{if } |X| > \varepsilon|I| \text{ and } |Y| > \varepsilon|J| \text{ then } |d(X, Y) - d(I, J)| \le \varepsilon.$$

A partition P of V is called *ε-regular* if

$$\text{(2)} \qquad \sum_{\substack{I,J \in P \\ (I,J)\ \varepsilon\text{-irregular}}} |I||J| \le \varepsilon|V|^2.$$

Moreover, P is called *ε-balanced* if P contains a subcollection C such that all sets in C have the same size and such that $|V \setminus \bigcup C| \le \varepsilon|V|$.

For $I, J \subseteq V$, let $L_{I,J}$ be the linear subspace of $\mathbb{R}^{V\times V}$ consisting of all scalar multiples of the incidence matrix of $I \times J$ in $\mathbb{R}^{V\times V}$. For any $M \in \mathbb{R}^{V\times V}$, let $M_{I,J}$ be the orthogonal projection of M onto $L_{I,J}$ (with respect to the inner product $\mathrm{Tr}(MN^\mathsf{T})$ for matrices $M, N \in \mathbb{R}^{V\times V}$). So the entries of $M_{I,J}$ on $I \times J$ are all equal to the average value of M on $I \times J$.

If P is a partition of V, let L_P be the sum of the spaces $L_{I,J}$ with $I, J \in P$, and let M_P be the orthogonal projection of M onto L_P. So $M_P = \sum_{I,J\in P} M_{I,J}$.

Define $f_\varepsilon(x) := (1 + \varepsilon^{-1})x4^x$ for $x \in \mathbb{R}$.

Lemma 1. *Let $\varepsilon > 0$ and $G = (V, E)$ be a directed graph, with adjacency matrix A. Then each ε-irregular partition P has an ε-balanced refinement Q with $|Q| \le f_\varepsilon(|P|)$ and $\|A_Q\|^2 > \|A_P\|^2 + \varepsilon^5|V|^2$.*

Proof. Let $(I_1, J_1), \ldots, (I_n, J_n)$ be the ε-irregular pairs in P^2. For each $i = 1, \ldots, n$, we can choose (by definition (1)) subsets $X_i \subseteq I_i$ and $Y_i \subseteq J_i$ with $|X_i| > \varepsilon|I_i|$, $|Y_i| > \varepsilon|J_i|$ and $|d(X_i, Y_i) - d(I_i, J_i)| > \varepsilon$. For any fixed $K \in P$, there exists a partition R_K of K such that each X_i with $I_i = K$ and each Y_i with $J_i = K$ is a union of classes of R_K and such that $|R_K| \le 2^{2|P|} = 4^{|P|}$. Let $R := \bigcup_{K\in P} R_K$. Then R is a refinement of P such that each X_i and each Y_i is a union of classes of R. Moreover, $|R| \le |P|4^{|P|}$.

Now note that for each i, since $(A_R)_{X_i,Y_i} = A_{X_i,Y_i}$ (as $L_{X_i,Y_i} \subseteq L_R$) and since A_{X_i,Y_i} and A_P are constant on $X_i \times Y_i$, with values $d(X_i, Y_i)$ and $d(I_i, J_i)$, respectively:

$$\text{(3)} \qquad \|(A_R - A_P)_{X_i,Y_i}\|^2 = \|A_{X_i,Y_i} - (A_P)_{X_i,Y_i}\|^2 = |X_i||Y_i|(d(X_i, Y_i) - d(I_i, J_i))^2 > \varepsilon^4|I_i||J_i|.$$

Then negating (2) gives with Pythagoras, as A_P is orthogonal to $A_R - A_P$ (as $L_P \subseteq L_R$), and as the spaces L_{X_i,Y_i} are pairwise orthogonal,

$$(4) \qquad \|A_R\|^2 - \|A_P\|^2 = \|A_R - A_P\|^2 \geq \sum_{i=1}^{n} \|(A_R - A_P)_{X_i,Y_i}\|^2 \geq \sum_{i=1}^{n} \varepsilon^4 |I_i||J_i| > \varepsilon^5 |V|^2.$$

To obtain an ε-balanced partition Q, define $t := \varepsilon|V|/|R|$. Split each class of R into classes, each of size $\lceil t \rceil$, except for at most one of size less than t. This gives partition Q. Then $|Q| \leq |R| + |V|/t = (1 + \varepsilon^{-1})|R| \leq f_\varepsilon(|P|)$. Moreover, the union of the classes of Q of size less than t has size at most $|R|t = \varepsilon|V|$. So Q is ε-balanced. As $L_R \subseteq L_Q$, we have, using (4), $\|A_Q\|^2 \geq \|A_R\|^2 > \|A_P\|^2 + \varepsilon^5|V|^2$. ∎

For $n \in \mathbb{N}$, f_ε^n denotes the n-th iterate of f_ε.

Theorem 1 (Szemerédi's regularity lemma)**.** *For each $\varepsilon > 0$ and directed graph $G = (V, E)$, each partition P of V has an ε-balanced ε-regular refinement of size $\leq f_\varepsilon^{\lceil \varepsilon^{-5} \rceil}(|P|)$.*

Proof. Let A be the adjacency matrix of G. Set $P_0 = P$. For $i \geq 0$, if P_i has been set, let P_{i+1} be an ε-balanced refinement of P_i with $|P_{i+1}| \leq f_\varepsilon(|P_i|)$ and with $\|A_{P_{i+1}}\|$ maximal. As $\|A_{P_i}\|^2 \leq \|A\|^2 \leq |V|^2$ for all i, $\|A_{P_{i+1}}\|^2 \leq \|A_{P_i}\|^2 + \varepsilon^5|V|^2$ for some i with $1 \leq i \leq \lceil \varepsilon^{-5} \rceil$. Then, by Lemma 1, P_i is ε-regular. Moreover $|P_i| \leq f_\varepsilon^i(|P|) \leq f_\varepsilon^{\lceil \varepsilon^{-5} \rceil}(|P|)$. ∎

It is important to observe that the bound on $|Q|$, though generally huge, only depends on ε and $|P|$, and not on the size of the graph. Gowers [1] showed that the bound necessarily is huge (at least a tower of powers of 2's of height proportional to $\varepsilon^{-1/16}$).

Exercise

1.1. Let P be an ε-balanced ε-regular partition of V, and let $C \subseteq P$ be such that all sets in C have the same size and such that $|V \setminus \bigcup C| \leq \varepsilon|V|$. Prove that at most $(\varepsilon/(1-\varepsilon)^2)|C|^2$ pairs in C^2 are ε-irregular.

2. Arithmetic progressions

An *arithmetic progression* of *length* k is a sequence of numbers $a_1, \ldots, a_k$ with $a_i - a_{i-1} = a_2 - a_1 \neq 0$ for $i = 2, \ldots, k$. For any k and n, let $\alpha_k(n)$ be the maximum size of a subset of $[n]$ containing no arithmetic progression of length k. (Here $[n] := \{1, \ldots, n\}$.)

We can now derive the theorem of Roth [3], which implies that any set X of natural numbers with $\limsup_{n\to\infty} |X \cap [n]|/n > 0$ contains an arithmetic progression of length 3. ($f(n) = o(g(n))$ means $\lim_{n\to\infty} f(n)/g(n) = 0$.)

Corollary 1a. $\alpha_3(n) = o(n)$.

Proof. Choose $\varepsilon > 0$, define $K := f_\varepsilon^{\lceil \varepsilon^{-5} \rceil}(1)$, and let $n > \varepsilon^{-3}K$. It suffices to show that $\alpha_3(n) \leq 30\varepsilon n$, so suppose $\alpha_3(n) > 30\varepsilon n$. Let S be a subset of $[n]$ of size $\alpha_3(n)$ containing no arithmetic progressions of length 3. Define the directed graph $G = (V, E)$ by $V := [2n]$ and $E := \{(u, v) \mid u, v \in V, v - u \in S\}$. So $|E| \geq |S|n > 30\varepsilon n^2$.

By Theorem 1, there exists an ε-regular partition P of V of size at most K. Let $\mathcal{Q}$ be the set of ε-regular pairs $(I,J) \in P^2$ with $d(I,J) > 2\varepsilon$ and $|I| > \varepsilon^{-2}$. Then

$$(5) \qquad \sum_{(I,J)\in\mathcal{Q}} e(I,J) > 16\varepsilon n^2.$$

Indeed, as P is ε-regular and as $e(I,J) \leq |I||J|$, (2) implies that the sum of $e(I,J)$ over all ε-irregular pairs (I,J) is at most $\varepsilon|V|^2 = 4\varepsilon n^2$. Moreover, the sum of $e(I,J)$ over all pairs $(I,J) \in P^2$ with $d(I,J) \leq 2\varepsilon$ is at most $2\varepsilon|V|^2 = 8\varepsilon n^2$. Finally, the sum of $e(I,J)$ over all $(I,J) \in P^2$ with $|I| \leq \varepsilon^{-2}$ is at most $|P|\varepsilon^{-2}|V| \leq K\varepsilon^{-2}|V| = 2K\varepsilon^{-2}n \leq 2\varepsilon n^2$. As $\sum_{I,J\in P} e(I,J) = |E| > 30\varepsilon n^2$, we obtain (5).

Now let $A := [4n]$. For each $a \in A$, define $E_a := \{(u,v) \in E \mid u+v = a\}$, and let T_a and H_a be the sets of tails and of heads, respectively, of the edges in E_a. Then

(6) there exist $a \in A$ and $(I,J) \in \mathcal{Q}$ such that $|T_a \cap I| > \varepsilon|I|$ and $|H_a \cap J| > \varepsilon|J|$.

Suppose such a, I, J do not exist. For $a \in A$, $I, J \in P$, let $e_a(I,J)$ be the number of pairs in $I \times J$ that are adjacent in (V, E_a). So $e(I,J) = \sum_{a\in A} e_a(I,J)$ for all $I, J \in P$. Now the sum of $e_a(I,J)$ over all a, I, J with $|T_a \cap I| \leq \varepsilon|I|$ is equal to the sum of $|T_a \cap I|$ over all a, I with $|T_a \cap I| \leq \varepsilon|I|$, which is at most $\sum_{a,I} \varepsilon|I| = \varepsilon|A||V| = 8\varepsilon n^2$. Similarly, the sum of $e_a(I,J)$ over all a, I, J with $|H_a \cap J| < \varepsilon|J|$ is at most $8\varepsilon n^2$. Hence, with (5) we obtain (6).

Set $X := T_a \cap I$ and $Y := H_a \cap J$. So $|X| > \varepsilon|I|$ and $|Y| > \varepsilon|J|$. As (I,J) is ε-regular, $d(I,J) > 2\varepsilon$, and $|I| > \varepsilon^{-2}$, we have $d(X,Y) \geq d(I,J) - \varepsilon > \varepsilon > \varepsilon^{-1}|I|^{-1} > |X|^{-1}$. So $e(X,Y) = d(X,Y)|X||Y| > |Y|$. Hence there is an edge (u,v) in $X \times Y$ with $u+v = b \neq a$ (as E_a is a matching). By definition of T_a and H_a, there exist $v', u' \in V$ with $(u,v'), (u',v) \in E_a$. Then $v'-u, v-u, v-u'$ is an arithmetic progression in S of length 3, since $v' \neq v$ and $v - v' = u - u'$, as $u + v' = a = u' + v$. ∎

(Note that ε-balancedness of partition P of V is not used in this proof.) This was extended to $\alpha_k(n) = o(n)$ for any k by Szemerédi [4]. Recently, Green and Tao [2] proved that there exist arbitrarily long arithmetic progressions of primes.

References

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