**Theorem** (Vizing's theorem for simple graphs)**.** $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$ *for any simple graph* $G$*.*

**Proof.** The inequality $\Delta(G) \leq \chi'(G)$ being trivial, we show $\chi'(G) \leq \Delta(G)+1$. To prove this inductively, it suffices to show for any simple graph $G$:

(0.1) Let $v$ be a vertex such that $v$ and all its neighbours have degree at most $k$, while at most one neighbour has degree precisely $k$. Then if $G-v$ is $k$-edge-colourable, also $G$ is $k$-edge-colourable.

We prove (1) by induction on $k$. We can assume that each neighbour $u$ of $v$ has degree $k-1$, except for one of degree $k$, since otherwise we can add a new vertex $w$ and an edge $uw$ without violating the conditions in (1). We can do this till all neighbours of $v$ have degree $k-1$, except for one having degree $k$.

Consider any $k$-edge-colouring of $G-v$. For $i=1,\ldots,k$, let $X_i$ be the set of neighbours of $v$ that are missed by colour $i$. So all but one neighbour of $v$ is in precisely two of the $X_i$, and one neighbour is in precisely one $X_i$. Hence

(0.2) $$\sum_{i=1}^{k} |X_i| = 2\deg(v) - 1 < 2k.$$

We can assume that we have chosen the colouring such that $\sum_{i=1}^{k} |X_i|^2$ is minimized. Then for all $i,j=1,\ldots,k$:

(0.3) $$||X_i| - |X_j|| \leq 2.$$

For if, say, $|X_1| > |X_2| + 2$, consider the subgraph $H$ made by all edges of colours 1 and 2. Each component of $H$ is a path or circuit. At least one component of $H$ contains more vertices in $X_1$ than in $X_2$. This component is a path $P$ starting in $X_1$ and not ending in $X_2$. Exchanging colours 1 and 2 on $P$ reduces $|X_1|^2 + |X_2|^2$, contradicting our minimality assumption. This proves (3).

This implies that there exists an $i$ with $|X_i| = 1$, since otherwise by (2) and (3) each $|X_i|$ is 0 or 2, while their sum is odd, a contradiction.

So we can assume $|X_k| = 1$, say $X_k := \{u\}$. Let $G'$ be the graph obtained from $G$ by deleting edge $vu$ and deleting all edges of colour $k$. So $G'-v$ is $(k-1)$-edge-coloured. Moreover, in $G'$, vertex $v$ and all its neighbours have degree at most $k-1$, and at most one neighbour has degree $k-1$. So by the induction hypothesis, $G'$ is $(k-1)$-edge-colourable. Restoring colour $k$, and giving edge $vu$ colour $k$, gives a $k$-edge-colouring of $G$. ∎