# Characterizing partition functions of the vertex model

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**Abstract.** We characterize which graph parameters are partition functions of a vertex model over an algebraically closed field of characteristic 0 (in the sense of de la Harpe and Jones, Graph invariants related to statistical mechanical models: examples and problems, *Journal of Combinatorial Theory, Series B* 57 (1993) 207–227).

We moreover characterize when the vertex model can be taken so that its moment matrix has finite rank. Basic instruments are the Nullstellensatz and the First and Second Fundamental Theorems of Invariant theory for the orthogonal group.

## 1. Introduction and survey of results

Let $\mathcal{G}$ denote the collection of all undirected graphs, two of them being the same if they are isomorphic. In this paper, all graphs are finite and may have loops and multiple edges. We denote by $\delta(v)$ the set of edges incident with a vertex $v$. An edge connecting $u$ and $v$ is denoted by $uv$. The vertex set and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. Moreover, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and for $k \in \mathbb{N}$:

(1) $$[k] := \{1, \ldots, k\}.$$

Let $k \in \mathbb{N}$ and let $\mathbb{F}$ be a commutative ring. Following de la Harpe and Jones [5], call any function $y : \mathbb{N}^k \to \mathbb{F}$ a ($k$-*color*) *vertex model* (*over* $\mathbb{F}$).[6] The *partition function* of $y$ is the function $p_y : \mathcal{G} \to \mathbb{F}$ defined for any graph $G = (V, E)$ by

(2) $$p_y(G) := \sum_{\kappa: E \to [k]} \prod_{v \in V} y_{\kappa(\delta(v))}.$$

Here $\kappa(\delta(v))$ is a multisubset of $[k]$, which we identify with its incidence vector in $\mathbb{N}^k$.

We can visualize $\kappa$ as a coloring of the edges of $G$ and $\kappa(\delta(v))$ as the multiset of colors 'seen' from $v$. The vertex model was considered by de la Harpe and Jones [5] as a physical model, where vertices serve as particles, edges as interactions between particles, and colors as states or energy levels. They also introduced the 'spin model', where the role of vertices and edges is interchanged. The partition function of any spin model is also the partition

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[6] In [10] it is called an *edge coloring model*. Colors are also called *states*.

function of some vertex model, as was shown by Szegedy [10][7]. Hence it includes the Ising-Potts model (cf. Section 2 below). Also several graph parameters (like the number of matchings) are partition functions of some vertex model. There are real-valued graph parameters that are partition functions of a vertex model over $\mathbb{C}$, but not over $\mathbb{R}$. (A simple example is $(-1)^{|E(G)|}$.)

In this paper, we characterize which functions $f : \mathcal{G} \to \mathbb{F}$ are the partition function of a vertex model over $\mathbb{F}$, when $\mathbb{F}$ is an algebraically closed field of characteristic 0.

To describe the characterization, let $GH$ denote the disjoint union of graphs $G$ and $H$. Call a function $f : \mathcal{G} \to \mathbb{F}$ *multiplicative* if $f(\emptyset) = 1$ and $f(GH) = f(G)f(H)$ for all $G, H \in \mathcal{G}$.

Moreover, for any graph $G = (V, E)$, any $U \subseteq V$, and any $s : U \to V$, define

$$E_s := \{us(u) \mid u \in U\} \text{ and } G_s := (V, E \cup E_s) \tag{3}$$

(adding multiple edges if $E_s$ intersects $E$). Let $S_U$ be the group of permutations of $U$.

**Theorem 1.** *Let $\mathbb{F}$ be an algebraically closed field of characteristic 0. A function $f : \mathcal{G} \to \mathbb{F}$ is the partition function of some $k$-color vertex model over $\mathbb{F}$ if and only if $f$ is multiplicative and for each graph $G = (V, E)$, each $U \subseteq V$ with $|U| = k + 1$, and each $s : U \to V$:*

$$\sum_{\pi \in S_U} \operatorname{sgn}(\pi) f(G_{s \circ \pi}) = 0. \tag{4}$$

Let $y : \mathbb{N}^k \to \mathbb{F}$. The corresponding *moment matrix* is

$$M_y := (y_{\alpha+\beta})_{\alpha,\beta \in \mathbb{N}^k}. \tag{5}$$

Abusing language we say that $y$ has *rank* $r$ if $M_y$ has rank $r$. For any graph $G = (V, E)$, $U \subseteq V$, and $s : U \to V$, let $G/s$ be the graph obtained from $G_s$ by contracting all edges in $E_s$.

**Theorem 2.** *Let $f$ be the partition function of a $k$-color vertex model over an algebraically closed field $\mathbb{F}$ of characteristic 0. Then $f$ is the partition function of a $k$-color vertex model over $\mathbb{F}$ of rank at most $r$ if and only if for each graph $G = (V, E)$, each $U \subseteq V$ with $|U| = r + 1$, and each $s : U \to V \setminus U$:*

$$\sum_{\pi \in S_U} \operatorname{sgn}(\pi) f(G/s \circ \pi) = 0. \tag{6}$$

It is easy to see that the conditions in Theorem 2 imply those in Theorem 1 for $k := r$, since for each $u \in U$ we can add to $G$ a new vertex $u'$ and a new edge $uu'$, thus obtaining graph $G'$. Then (6) for $G'$, $U'$, and $s'(u') := s(u)$ gives (4). This implies that if $f$ is the

[7] The construction given in [5] only extends the spin model for line graphs.

partition function of a vertex model of rank $r$, it is also the partition function of an $r$-color vertex model of rank $r$.

It is also direct to see that in both theorems we may restrict $s$ to injective functions. However, in Theorem 1, $s(U)$ should be allowed to intersect $U$ (otherwise $f(G) := 2^{\#\text{ of loops}}$ would satisfy the condition for $k = 1$, but is not the partition function of some 1-color vertex model). Moreover, in Theorem 2, $s(U)$ may not intersect $U$ (otherwise $f(G) := 2^{|V(G)|}$ would not satisfy the condition for $k = r = 1$, while it is the partition function of some 1-color vertex model of rank 1).

## 2. Background

In this section, we give some background to the results described in this paper. The definitions and results given in this section will not be used in the remainder of this paper.

As mentioned, the vertex model has its roots in mathematical physics, see de la Harpe and Jones [5], and for more background on the relations between graph theory and models in statistical mechanics, [1], [9], and [12]. De la Harpe en Jones also gave the dual 'spin model', where the roles of vertices and edges are interchanged. Partition functions of spin models were characterized by Freedman, Lovász, and Schrijver [3] and Schrijver [8]. Szegedy [10] showed that the partition function of any spin model is also the partition function of some vertex model (it extends a result of [5]).

Let us illustrate these results by applying them to the Ising model. The Ising model (a spin model) has the following partition function:

$$f(G) := \sum_{\sigma:V(G)\to\{+1,-1\}} \prod_{uv\in E(G)} \exp(\sigma(u)\sigma(v)L/kT), \tag{7}$$

where $L$ is a positive constant, $k$ is the Boltzmann constant and $T$ is the temperature. Now for each $U \subseteq V(G)$ with $|U| = 3$ and each $s : U \to V(G)$, condition (4) is satisfied, that is, equivalently,

$$\sum_{\sigma:V(G)\to\{+1,-1\}} \sum_{\pi\in S_U} \operatorname{sgn}(\pi) \prod_{uv\in E(G_{s\circ\pi})} \exp(\sigma(u)\sigma(v)L/kT) = 0. \tag{8}$$

This follows from the fact that for each fixed $\sigma : V(G) \to \{+1, -1\}$ there exist distinct $u_1, u_2 \in U$ with $\sigma(u_1) = \sigma(u_2)$. Let $\rho$ be the permutation in $S_U$ that exchanges $u_1$ and $u_2$. Then the terms in (8) for $\pi$ and $\pi \circ \rho$ cancel.

So by Theorem 1, $f$ is the partition function $p_y$ of some 2-color vertex model $y$. With Theorem 2 one may similarly show that one can take $y$ of rank 2. Indeed, one may check that one has $f = p_y$ by taking $y : \mathbb{N}^2 \to \mathbb{R}$ with

$$y(k,l) := \gamma^k\delta^l + \gamma^l\delta^k, \tag{9}$$

where $\gamma, \delta$ are real numbers satisfying $\gamma^2 + \delta^2 = \exp(L/kT)$ and $2\gamma\delta = \exp(-L/kT)$.

We next describe some results of Szegedy [8,9] concerning the vertex model that are related to, and have motivated, our results. They require the notions of $l$-labeled graphs and $l$-fragments.

For $l \in \mathbb{N}$, an *$l$-labeled graph* is an undirected graph $G = (V, E)$ together with an injective 'label' function $\lambda : [l] \to V$. (So unlike in the usual meaning of labeled graph, in an $l$-labeled graph only $l$ of the vertices are labeled, while the remaining vertices are unlabeled.)

If $G$ and $H$ are two $l$-labeled graphs, let $GH$ be the graph obtained from the disjoint union of $G$ and $H$ by identifying equally labeled vertices. (We can identify (unlabeled) graphs with 0-labeled graphs, and then this notation extends consistently the notation $GH$ given in Section 1.)

An *$l$-fragment* is an $l$-labeled graph where each labeled vertex has degree 1. (If you like, you may alternatively view the degree-1 vertices as ends of 'half-edges'.) If $G$ and $H$ are $l$-fragments, the graph $G \cdot H$ is obtained from $GH$ by ignoring each of the $l$ identified points as vertex, joining its two incident edges into one edge. (A good way to imagine this is to see a graph as a topological 1-complex.) Note that it requires that we also should consider the 'vertexless loop' as possible edge of a graph, as we may create it in $G \cdot H$.

Let $\mathcal{G}_l$ and $\mathcal{G}'_l$ denote the collections of $l$-labeled graphs and of $l$-fragments, respectively. For any $f : \mathcal{G} \to \mathbb{F}$ and $l \in \mathbb{N}$, the *connection matrices* $C_{f,l}$ and $C'_{f,l}$ are the $\mathcal{G}_l \times \mathcal{G}_l$ and $\mathcal{G}'_l \times \mathcal{G}'_l$ matrices defined by

$$(10) \qquad C_{f,l} := (f(GH))_{G,H \in \mathcal{G}_l} \quad \text{and} \quad C'_{f,l} := (f(G \cdot H))_{G,H \in \mathcal{G}'_l}.$$

Now we can formulate Szegedy's theorem ([10]):

(11) A function $f : \mathcal{G} \to \mathbb{R}$ is the partition function of a vertex model over $\mathbb{R}$ if and only if $f$ is multiplicative and $C'_{f,l}$ is positive semidefinite for each $l$.

Note that the number of colors is equal to the $f$-value of the vertexless loop. The proof of (11) is based on the First Fundamental Theorem for the orthogonal group and on the Real Nullstellensatz.

Next consider the complex case. Szegedy [11] observed that if $y$ is a vertex model of rank $r$, then $\text{rank}(C_{p_y,l}) \leq r^l$ for each $l$. It made him ask whether, conversely, for each function $f : \mathcal{G} \to \mathbb{C}$ with $f(\emptyset) = 1$ such that there exists a number $r$ for which $\text{rank}(C_{f,l}) \leq r^l$ for each $l$, there exists a finite rank vertex model $y$ over $\mathbb{C}$ with $f = p_y$. The answer is negative however: the function $f$ defined by

$$(12) \qquad f(G) := \begin{cases} (-2)^{\#\text{ of components}} & \text{if } G \text{ is 2-regular,} \\ 0 & \text{otherwise,} \end{cases}$$

has $f(\emptyset) = 1$ and can be shown to satisfy $\text{rank}(C_{f,l}) \leq 4^l$ for each $l$. However, $f$ is not the partition function of a vertex model (as for no $k$ it satisfies condition (4) of Theorem 1). The characterizations given in the present paper may serve as alternatives to Szegedy's question.

## 3. A useful framework

In the proofs of both Theorem 1 and 2 we will use the following framework and results.

Let $k \in \mathbb{N}$. Introduce a variable $y_\alpha$ for each $\alpha \in \mathbb{N}^k$ and define the ring $R$ of polynomials in these (infinitely many) variables:

$$R := \mathbb{F}[y_\alpha \mid \alpha \in \mathbb{N}^k]. \tag{13}$$

There is a bijection between the variables $y_\alpha$ in $R$ and the monomials $x^\alpha = \prod_{i=1}^k x_i^{\alpha_i}$ in $\mathbb{F}[x_1, \ldots, x_k]$. (Note that $x^\alpha x^\beta$ does not correspond to $y_\alpha y_\beta$, but with $y_{\alpha+\beta}$.) In this way, functions $y : \mathbb{N}^k \to \mathbb{F}$ correspond to elements of $\mathbb{F}[x_1, \ldots, x_k]^*$.

Define $p : \mathcal{G} \to R$ by $p(G)(y) := p_y(G)$ for any graph $G = (V, E)$ and $y : \mathbb{N}^k \to \mathbb{F}$. Let $\mathbb{F}\mathcal{G}$ denote the set of formal $\mathbb{F}$-linear combinations of elements of $\mathcal{G}$. The elements of $\mathbb{F}\mathcal{G}$ are called *quantum graphs*. We can extend $p$ linearly to $\mathbb{F}\mathcal{G}$. Taking disjoint union of graphs $G$ and $H$ as product $GH$, makes $\mathbb{F}\mathcal{G}$ to an algebra. Then $p$ is an algebra homomorphism.

The main ingredients of the proof are two basic facts about $p$: a characterization of the image $\operatorname{Im} p$ of $p$ and a characterization of the kernel $\operatorname{Ker} p$ of $p$. The characterization of $\operatorname{Im} p$ is similar to that given by Szegedy [10].

To characterize $\operatorname{Im} p$, let $O_k$ be the group of orthogonal matrices over $\mathbb{F}$ of order $k$. Observe that $O_k$ acts on $\mathbb{F}[x_1, \ldots, x_k]$, and hence on $R$, through the bijection $y_\alpha \leftrightarrow x^\alpha$ mentioned above. As usual, $Z^{O_k}$ denotes the set of $O_k$-invariant elements of $Z$, if $O_k$ acts on a set $Z$.

To characterize $\operatorname{Ker} p$, let $I$ be the subspace of $\mathbb{F}\mathcal{G}$ spanned by the quantum graphs

$$\sum_{\pi \in S_U} \operatorname{sgn}(\pi) G_{s\circ\pi}, \tag{14}$$

where $G = (V, E)$ is a graph, $U \subseteq V$ with $|U| = k + 1$, and $s : U \to V$.

**Proposition 1.** $\operatorname{Im} p = R^{O_k}$ *and* $\operatorname{Ker} p = I$.

**Proof.** For $n \in \mathbb{N}$, let $\mathcal{G}_n$ be the collection of graphs with $n$ vertices, again two of them being the same if they are isomorphic. Let $S\mathbb{F}^{n\times n}$ be the set of symmetric matrices in $\mathbb{F}^{n\times n}$. For any linear space $X$, let $\mathcal{O}(X)$ denote the space of regular functions on $X$ (the algebra generated by the linear functions on $X$). Then $\mathcal{O}(S\mathbb{F}^{n\times n})$ is spanned by the monomials $\prod_{ij\in E} x_{i,j}$ in the variables $x_{i,j}$, where $([n], E)$ is a graph. Here $x_{i,j} = x_{j,i}$ are the standard coordinate functions on $S\mathbb{F}^{n\times n}$, while taking $ij$ as unordered pair.

Let $\mathbb{F}\mathcal{G}_n$ be the linear space of formal $\mathbb{F}$-linear combinations of elements of $\mathcal{G}_n$, and $R_n$ be the set of homogeneous polynomials in $R$ of degree $n$. We set $p_n := p|\mathbb{F}\mathcal{G}_n$. So $p_n : \mathbb{F}\mathcal{G}_n \to R_n$. Hence it suffices to show, for each $n$,

$$\operatorname{Im} p_n = R_n^{O_k} \text{ and } \operatorname{Ker} p_n = I \cap \mathbb{F}\mathcal{G}_n. \tag{15}$$

To show (15), we define linear functions $\mu$, $\sigma$, and $\tau$ so that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{F}\mathcal{G}_n & \xrightarrow{p_n} & R_n \\
\uparrow \mu & & \uparrow \sigma \\
\mathcal{O}(S\mathbb{F}^{n\times n}) & \xrightarrow{\tau} & \mathcal{O}(\mathbb{F}^{k\times n})
\end{array} . \tag{16}
$$

Define $\mu$ by

$$
\mu(\textstyle\prod_{ij\in E} x_{i,j}) := G \tag{17}
$$

for any graph $G = ([n], E)$. Define $\sigma$ by

$$
\sigma(\prod_{j=1}^{n}\prod_{i=1}^{k} z_{i,j}^{\alpha(i,j)}) := \prod_{j=1}^{n} y_{\alpha_j} \tag{18}
$$

for $\alpha \in \mathbb{N}^{k\times n}$, where $z_{i,j}$ are the standard coordinate functions on $\mathbb{F}^{k\times n}$ and where $\alpha_j = (\alpha(1,j),\ldots,\alpha(k,j)) \in \mathbb{N}^k$. Then $\sigma$ is $O_k$-equivariant, for the natural action of $O_k$ on $\mathcal{O}(\mathbb{F}^{k\times n})$.

Finally, define $\tau$ by

$$
\tau(q)(z) := q(z^T z) \tag{19}
$$

for $q \in \mathcal{O}(S\mathbb{F}^{n\times n})$ and $z \in \mathbb{F}^{k\times n}$.

Now (16) commutes; in other words,

$$
p_n \circ \mu = \sigma \circ \tau. \tag{20}
$$

To prove it, consider any monomial $q := \prod_{ij\in E} x_{i,j}$ in $\mathcal{O}(S\mathbb{F}^{n\times n})$, where $G = ([n], E)$ is a graph. Then for $z \in \mathbb{F}^{k\times n}$,

$$
\tau(q)(z) = q(z^T z) = \prod_{ij\in E}\sum_{h=1}^{k} z_{h,i}z_{h,j} = \sum_{\kappa:E\to[k]}\prod_{i\in[n]}\prod_{e\in\delta(i)} z_{\kappa(e),i}. \tag{21}
$$

So, by definition (18) of $\sigma$ and (17) of $\mu$,

$$
\sigma(\tau(q)) = \sum_{\kappa:E\to[k]}\prod_{i\in[n]} y_{\kappa(\delta(i))} = p_n(G) = p_n(\mu(q)). \tag{22}
$$

This proves (20).

Note that $\tau$ is an algebra homomorphism, but $\mu$ and $\sigma$ generally are not. ($\mathbb{F}\mathcal{G}_n$ and $R_n$ are not algebras.) The latter two functions are surjective, and their restrictions to the $S_n$-invariant part of their respective domains are bijective.

The First Fundamental Theorem (FFT) for $O_k$ (cf. [4] Theorem 5.2.2) says that Im $\tau = (\mathcal{O}(\mathbb{F}^{k\times n}))^{O_k}$. Hence, as $\mu$ and $\sigma$ are surjective, and as $\sigma$ is $O_k$-equivariant,

(23) $\operatorname{Im} p_n = p_n(\mathbb{F}\mathcal{G}_n) = p_n(\mu(\mathcal{O}(S\mathbb{F}^{n\times n}))) = \sigma(\tau(\mathcal{O}(S\mathbb{F}^{n\times n}))) = \sigma(\mathcal{O}(\mathbb{F}^{k\times n})^{O_k}) = R_n^{O_k}$.

(The last equality follows from the fact that $\sigma$ is $O_k$-equivariant, so that we have $\subseteq$. To see $\supseteq$, take $q \in R_n^{O_k}$. As $\sigma$ is surjective, $q = \sigma(r)$ for some $r \in \mathcal{O}(\mathbb{F}^{k\times n})$. Then $q = \sigma(\rho_{O_k}(r))$, where $\rho_{O_k}$ is the Reynolds operator.) This is the first statement in (15).

To see $I \cap \mathbb{F}\mathcal{G}_n \subseteq \operatorname{Ker} p_n$, let $G = ([n], E)$ be a graph, $U \subseteq [n]$ with $|U| = k+1$, and $s : U \to [n]$. Then $\sum_{\pi\in S_U} \operatorname{sgn}(\pi) G_{s\circ\pi}$ belongs to $\operatorname{Ker} p_n$, as

$$p(\sum_{\pi\in S_U} \operatorname{sgn}(\pi) G_{s\circ\pi}) = \sum_{\kappa: E\cup E_s \to [k]} \sum_{\pi\in S_U} \operatorname{sgn}(\pi) \prod_{v\in V} y_{\kappa(\delta_{G_{s\circ\pi}}(v))}. \tag{24}$$

For fixed $\kappa$, there exist distinct $u_1, u_2 \in U$ with $\kappa(u_1 s(u_1)) = \kappa(u_2 s(u_2))$. So if $\rho$ is the permutation of $U$ interchanging $u_1$ and $u_2$, we have that the terms corresponding to $\pi$ and $\pi \circ \rho$ cancel. Hence (24) is zero.

We finally show $\operatorname{Ker} p_n \subseteq I$. The Second Fundamental Theorem (SFT) for $O_k$ (cf. [4] Theorem 12.2.14) says that $\operatorname{Ker} \tau = K$, where $K$ is the ideal in $\mathcal{O}(S\mathbb{F}^{n\times n})$ generated by the $(k+1)\times(k+1)$ minors of $S\mathbb{F}^{n\times n}$. Then

$$\mu(K) \subseteq I. \tag{25}$$

It suffices to show that for any $(k+1)\times(k+1)$ submatrix $N$ of $\mathbb{F}^{n\times n}$ and any graph $G = ([n], E)$ one has

$$\mu(\det N \prod_{ij\in E} x_{i,j}) \in I. \tag{26}$$

There is a subset $U$ of $[n]$ with $|U| = k+1$, and an injective function $s : U \to [n]$ such that $\{(u, s(u)) \mid u \in U\}$ forms the diagonal of $N$. So

$$\det N = \sum_{\pi\in S_U} \operatorname{sgn}(\pi) \prod_{u\in U} x_{u,s\circ\pi(u)}. \tag{27}$$

Then

$$\mu(\det N \prod_{ij\in E} x_{i,j}) = \sum_{\pi\in S_U} \operatorname{sgn}(\pi) \mu( \prod_{u\in U} x_{u,s\circ\pi(u)} \cdot \prod_{ij\in E} x_{i,j}) = \sum_{\pi\in S_U} \operatorname{sgn}(\pi) G_{s\circ\pi} \in I, \tag{28}$$

by definition of $I$. This proves (26).

To prove $\operatorname{Ker} p_n \subseteq I$, let $\gamma \in \mathbb{F}\mathcal{G}_n$ with $p_n(\gamma) = 0$. Then $\gamma = \mu(q)$ for some $q \in (\mathcal{O}(S\mathbb{F}^{n\times n}))^{S_n}$. Hence $\sigma(\tau(q)) = p(\mu(q)) = p(\gamma) = 0$. As $\tau(q)$ is $S_n$-invariant, this implies $\tau(q) = 0$ (as $\sigma$ is bijective on $\mathcal{O}(\mathbb{F}^{k\times n})^{S_n}$). So $q \in K$, hence $\gamma = \mu(q) \in \mu(K) \subseteq I$. ∎

## 4. Proof of Theorem 1

We fix $k$. Necessity of the conditions is direct. Condition (4) follows from the fact that Ker $p = I$ (Proposition 1).

To prove sufficiency, we must show that the polynomials $p(G) - f(G)$ have a common zero. Here $f(G)$ denotes the constant polynomial with value $f(G)$. So a common zero means an element $y : \mathbb{N}^k \to \mathbb{F}$ with for all $G \in \mathcal{G}$, $(p(G) - f(G))(y) = 0$, equivalently $p_y(G) = f(G)$, as required.

As $f$ is multiplicative, $f$ extends linearly to an algebra homomorphism $f : \mathbb{F}\mathcal{G} \to \mathbb{F}$. By the condition in Theorem 1, $f(I) = 0$. So by Proposition 1, Ker $p \subseteq$ Ker $f$. Hence there exists an algebra homomorphism $\hat{f} : p(\mathbb{F}\mathcal{G}) \to \mathbb{F}$ such that $\hat{f} \circ p = f$.

Let $\mathcal{I}$ be the ideal in $R$ generated by the polynomials $p(G) - f(G)$ for graphs $G$. Let $\rho_{O_k}$ denote the Reynolds operator on $R$. By Proposition 1, $\rho_{O_k}(\mathcal{I})$ is equal to the ideal in $p(\mathbb{F}\mathcal{G}) = R^{O_k}$ generated by the polynomials $p(G) - f(G)$. (This follows essentially from the fact that if $q \in R^{O_k}$ and $r \in R$, then $\rho_{O_k}(qr) = q\rho_{O_k}(r)$.) This implies, as $\hat{f}(p(G) - f(G)) = 0$, that

$$(29) \qquad \hat{f}(\rho_{O_k}(\mathcal{I})) = 0,$$

hence $1 \notin \mathcal{I}$.

If $|\mathbb{F}|$ is uncountable (e.g. if $\mathbb{F} = \mathbb{C}$), the Nullstellensatz for countably many variables (Lang [7]) yields the existence of a common zero $y$.

To prove the existence of a common zero $y$ for general algebraically closed fields $\mathbb{F}$ of characteristic 0, let, for each $d \in \mathbb{N}$, $A_d := \{\alpha \in \mathbb{N}^k \mid |\alpha| \leq d\}$ and

$$(30) \qquad Y_d := \{z|A_d \mid z : \mathbb{N}^k \to \mathbb{F}, q(z) = \hat{f}(q) \text{ for each } q \in \mathbb{F}[y_\alpha \mid \alpha \in A_d]^{O_k}\}.$$

(Since $\mathbb{F}[y_\alpha \mid \alpha \in A_d]$ is a subset of $\mathbb{F}[y_\alpha \mid \alpha : \mathbb{N}^k \to \mathbb{F}]$, $\hat{f}(q)$ is defined.) So $Y_d$ consists of the common zeros of the polynomials $p(G) - f(G)$ where $G$ ranges over the graphs of maximum degree at most $d$.

By the Nullstellensatz, since $|A_d|$ is finite, $Y_d \neq \emptyset$. Note that $Y_d$ is $O_k$-stable. This implies that $Y_d$ contains a unique $O_k$-orbit $C_d$ of minimal (Krull) dimension (cf. [6] Satz 2, page 101 or [2] 1.11 and 1.24).

Let $\pi_d$ be the projection $z \mapsto z|A_d$ for $z : A_{d'} \to \mathbb{F}$ $(d' \geq d)$. Note that if $d' \geq d$ then $\pi_d(C_{d'})$ is an $O_k$-orbit contained in $Y_d$. Hence

$$(31) \qquad \dim C_d \leq \dim \pi_d(C_{d'}) \leq \dim C_{d'}.$$

As $\dim C_d \leq \dim O_k$ for all $d$, there is a $d_0$ such that for each $d \geq d_0$, $\dim C_d = \dim C_{d_0}$. Hence we have equality throughout in (31) for all $d' \geq d \geq d_0$.

By the uniqueness of the orbit of smallest dimension, this implies that, for all $d' \geq d \geq d_0$, $C_d = \pi_d(C_{d'})$. Hence there exists $y : \mathbb{N}^k \to \mathbb{F}$ such that $y|A_d \in C_d$ for each $d \geq d_0$. This $y$ is as required.

## 5. Proof of Theorem 2

Necessity can be seen as follows. Choose $y : \mathbb{N}^k \to \mathbb{F}$ with $\text{rank}(M_y) \leq r$ and choose $\kappa : E \to [k]$, $U \subseteq V$ with $|U| = r + 1$, and $s : U \to V \setminus U$. Then

$$\sum_{\pi \in S_U} \text{sgn}(\pi) p_y(G/s \circ \pi) = \sum_{\kappa : E \to [k]} \sum_{\pi \in S_U} \text{sgn}(\pi) \prod_{u \in U} y_{\kappa(\delta(u) \cup \delta(s(\pi(u))))} \cdot \prod_{v \in V \setminus (U \cup s(U))} y_{\kappa(\delta(v))} = \sum_{\kappa : E \to [k]} \det(y_{\kappa(\delta(u) \cup \delta(s(v)))})_{u,v \in U} \prod_{v \in V \setminus (U \cup s(U))} y_{\kappa(\delta(v))} = 0. \tag{32}$$

To see sufficiency, let $J$ be the ideal in $\mathbb{F}\mathcal{G}$ spanned by the quantum graphs

$$\sum_{\pi \in S_U} \text{sgn}(\pi) G/s \circ \pi, \tag{33}$$

where $G = (V, E)$ is a graph, $U \subseteq V$ with $|U| = r + 1$, and $s : U \to V \setminus U$. Let $\mathcal{J}$ be the ideal in $R$ generated by the polynomials $\det N$ where $N$ is an $(r + 1) \times (r + 1)$ submatrix of $M_y$.

**Proposition 2.** *$\rho_{O_k}(\mathcal{J}) \subseteq p(J)$.*

**Proof.** It suffices to show that for any $(r+1)\times(r+1)$ submatrix $N$ of $M_y$ and any monomial $a$ in $R$, $\rho_{O_k}(a \det N)$ belongs to $p(J)$. Let $a$ have degree $d$, and let $n := 2(r + 1) + d$. Let $U := [r + 1]$ and let $s : U \to [n] \setminus U$ be defined by $s(i) := r + 1 + i$ for $i \in [r + 1]$.

We use the framework of Proposition 1, with $\tau$ as in (19). For each $\pi \in S_{r+1}$ we define linear function $\mu_\pi$ and $\sigma_\pi$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{F}\mathcal{G}_m & \xrightarrow{p} & R_m \\ \uparrow{\scriptstyle \mu_\pi} & & \uparrow{\scriptstyle \sigma_\pi} \\ \mathcal{O}(S\mathbb{F}^{n\times n}) & \xrightarrow{\tau} & \mathcal{O}(\mathbb{F}^{k\times n}) \end{array}, \tag{34}$$

where $m := r + 1 + d$.

The function $\mu_\pi$ is defined by

$$\mu_\pi(\prod_{ij \in E} x_{i,j}) := G/s \circ \pi \tag{35}$$

for any graph $G = ([n], E)$. It implies that for each $q \in \mathcal{O}(S\mathbb{F}^{n\times n})$,

$$\sum_{\pi \in S_{r+1}} \text{sgn}(\pi) \mu_\pi(q) \in J, \tag{36}$$

by definition of $J$.

Next $\sigma_\pi$ is defined by

(37) $$\sigma_\pi(\prod_{j=1}^{n}\prod_{i=1}^{k} z_{i,j}^{\alpha_{i,j}}) := \prod_{j=1}^{r+1} y_{\alpha_j+\alpha_{r+1+\pi(j)}} \cdot \prod_{j=2r+3}^{n} y_{\alpha_i}$$

for any $\alpha \in \mathbb{N}^{k\times n}$. So

(38) $$a \det N = \sum_{\pi\in S_{r+1}} \operatorname{sgn}(\pi)\sigma_\pi(u)$$

for some monomial $u \in \mathcal{O}(\mathbb{F}^{k\times n})$. Note that $\sigma_\pi$ is $O_k$-equivariant.

Now one directly checks that diagram (34) commutes, that is,

(39) $$p \circ \mu_\pi = \sigma_\pi \circ \tau.$$

By the FFT, $\rho_{O_k}(u) = \tau(q)$ for some $q \in \mathcal{O}(S\mathbb{F}^{n\times n})$. Hence $\sigma_\pi(\rho_{O_k}(u)) = \sigma_\pi(\tau(q)) = p(\mu_\pi(q))$. Therefore, using (38) and (36),

(40) $$\rho_{O_k}(a \det N) = \sum_{\pi\in S_{r+1}} \operatorname{sgn}(\pi)\sigma_\pi(\rho_{O_k}(u)) = \sum_{\pi\in S_{r+1}} \operatorname{sgn}(\pi)p(\mu_\pi(q)) \in p(J),$$

as required. ▮

(In fact equality holds in this proposition, but we do not need it.)

Since $f$ is the partition function of a $k$-color vertex model, there exists $\hat{f} : R \to \mathbb{F}$ with $\hat{f} \circ p = f$. If the condition in Theorem 2 is satisfied, then $f(J) = 0$, and hence with Proposition 2

(41) $$\hat{f}(\rho_{O_k}(\mathcal{J})) \subseteq \hat{f}(p(J)) = f(J) = 0.$$

With (29) this implies that $1 \notin \mathcal{I} + \mathcal{J}$, where $\mathcal{I}$ again is the ideal generated by the polynomials $p(G) - f(G)$ $(G \in \mathcal{G})$. Hence $\mathcal{I} + \mathcal{J}$ has a common zero, as required.

## 6. Analogues for directed graphs

Similar results hold for directed graphs, with similar proofs, now by applying the FFT and SFT for $\mathrm{GL}(k, \mathbb{F})$. The corresponding models were also considered by de la Harpe and Jones [5]. We state the results.

Let $\mathcal{D}$ denote the collection of all directed graphs, two of them being the same if they are isomorphic. Directed graphs are finite and may have loops and multiple edges.

The *directed partition function* of a $2k$-color vertex model $y$ is the function $p_y : \mathcal{D} \to \mathbb{F}$ defined for any directed graph $G = (V, E)$ by

(42) $$p_y(G) := \sum_{\kappa:E\to[k]} \prod_{v\in V} y_{\kappa(\delta^-(v)),\kappa(\delta^+(v))}.$$

Here $\delta^-(v)$ and $\delta^+(v)$ denote the sets of arcs entering $v$ and leaving $v$, respectively. Moreover, $\kappa(\delta^-(v)), \kappa(\delta^+(v))$ stands for the concatenation of the vectors $\kappa(\delta^-(v))$ and $\kappa(\delta^+(v))$ in $\mathbb{N}^k$, so as to obtain a vector in $\mathbb{N}^{2k}$.

Call a function $f : \mathcal{D} \to \mathbb{F}$ *multiplicative* if $f(\emptyset) = 1$ and $f(GH) = f(G)f(H)$ for all $G, H \in \mathcal{D}$. Again, $GH$ denotes the disjoint union of $G$ and $H$.

Moreover, for any directed graph $G = (V, E)$, any $U \subseteq V$, and any $s : U \to V$, define

$$A_s := \{(u, s(u)) \mid u \in U\} \text{ and } G_s := (V, E \cup A_s). \tag{43}$$

**Theorem 3.** *Let $\mathbb{F}$ be an algebraically closed field of characteristic $0$. A function $f : \mathcal{D} \to \mathbb{F}$ is the directed partition function of some $2k$-color vertex model over $\mathbb{F}$ if and only if $f$ is multiplicative and for each directed graph $G = (V, E)$, each $U \subseteq V$ with $|U| = k + 1$, and each $s : U \to V$:*

$$\sum_{\pi \in S_U} \operatorname{sgn}(\pi) f(G_{s\circ\pi}) = 0. \tag{44}$$

For any directed graph $G = (V, E)$, $U \subseteq V$, and $s : U \to V$, let $G/s$ be the directed graph obtained from $G_s$ by contracting all edges in $A_s$.

**Theorem 4.** *Let $f$ be the directed partition function of a $2k$-color vertex model over an algebraically closed field $\mathbb{F}$ of characteristic $0$. Then $f$ is the directed partition function of a $2k$-color vertex model over $\mathbb{F}$ of rank at most $r$ if and only if for each directed graph $G = (V, E)$, each $U \subseteq V$ with $|U| = r + 1$, and each $s : U \to V \setminus U$:*

$$\sum_{\pi \in S_U} \operatorname{sgn}(\pi) f(G/s \circ \pi) = 0. \tag{45}$$

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