RISK-RETURN TRADE-OFF
in
OPTIMIZATION

Marco C. Campi

thanks to :

Simone Garatti

Bernardo Pagnoncelli

Giuseppe Calafiore

Algo Care’

Maria Prandini

Daniel Reich

PART I: Principles

PART II: Algorithms

PART I: Principles

Optimization

- management
- controller synthesis
- portfolio selection

  ⋮

→ optimization program

Optimization

- management
- controller synthesis
- portfolio selection

  ⋮

→ optimization program

Uncertain environment

- exercise caution

Uncertain Optimization Program

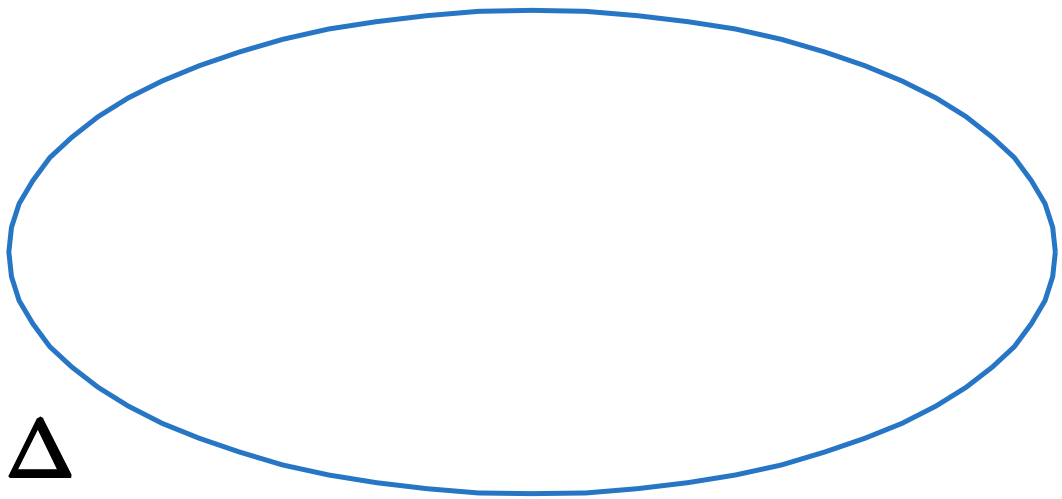
U-OP: $$\min_{\theta} \ell(\theta, \delta), \qquad \delta \in \Delta$$

Uncertain Optimization Program

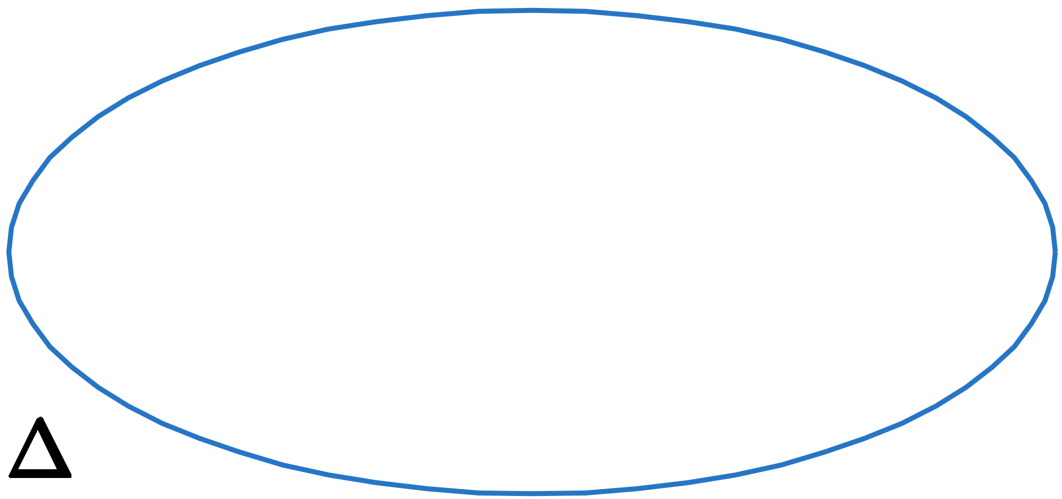
U-OP: $\min_{\theta} \ell(\theta, \delta), \qquad \delta \in \Delta$

not well-defined

Uncertainty

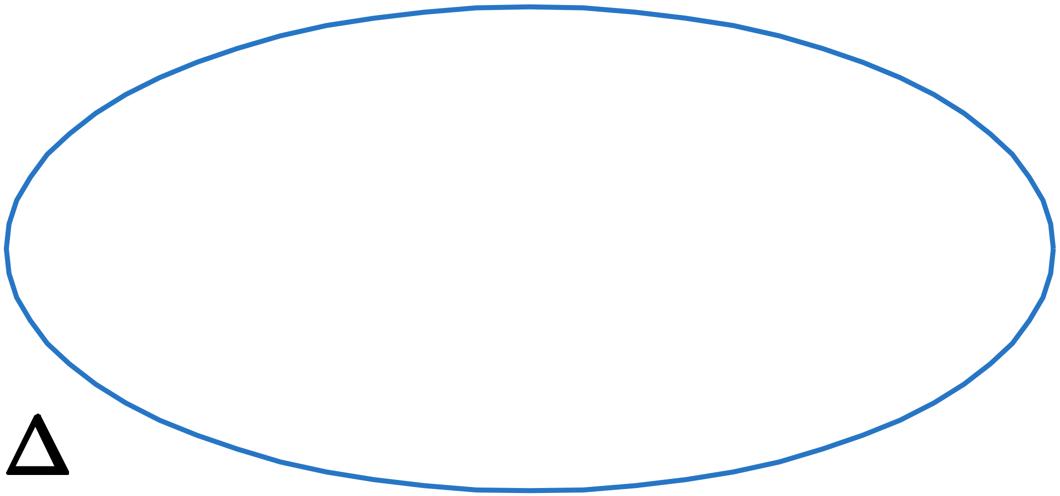


Uncertainty



$$\min_{\theta}\left[\max_{\delta\in\Delta}\ell(\theta,\delta)\right] \qquad \text{(worst-case approach)}$$

Uncertainty



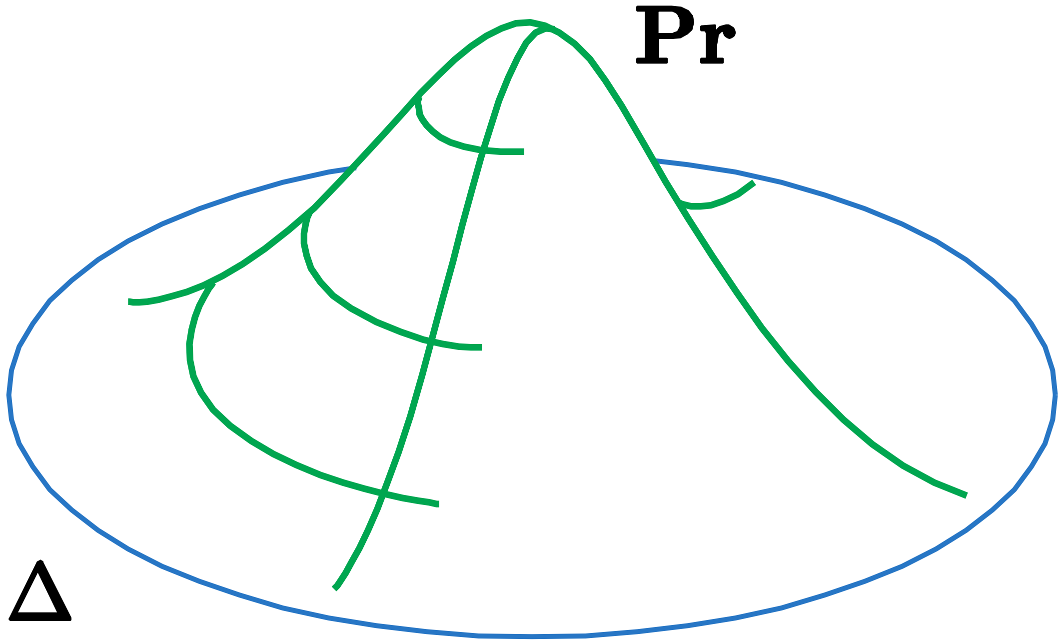
$\boldsymbol{\Delta}$

$$\min_{\theta} \left[\max_{\delta \in \Delta} \ell(\theta, \delta) \right] \qquad \text{(worst-case approach)}$$

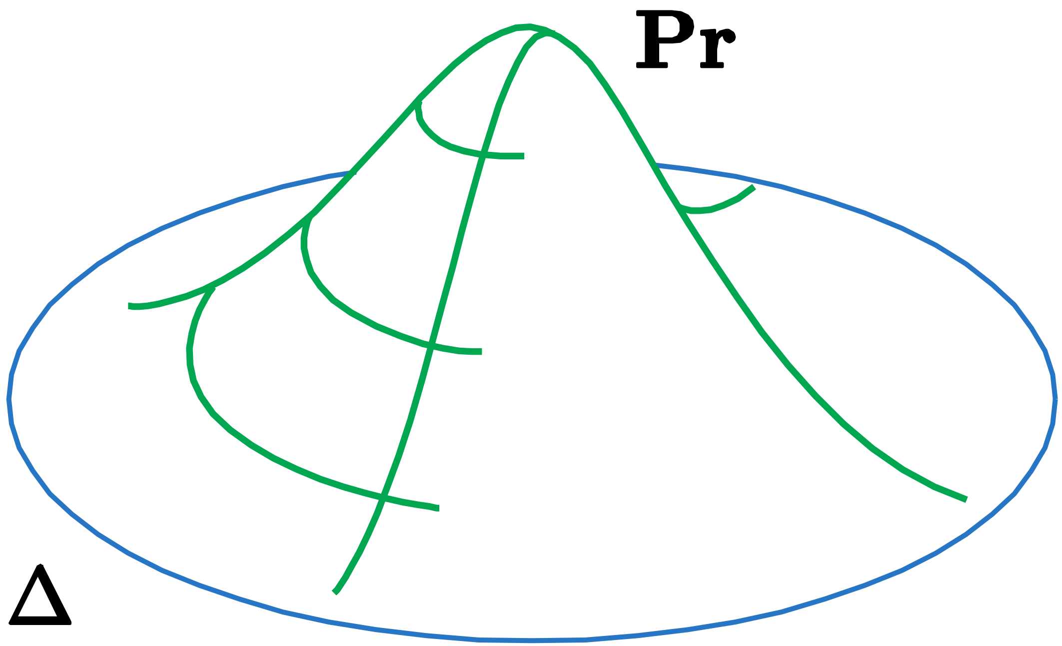
H_∞ control theory [G. Zames, 1981]

optimization [A. Ben-Tal & A. Nemirovski, 2002]

Probabilistic uncertainty

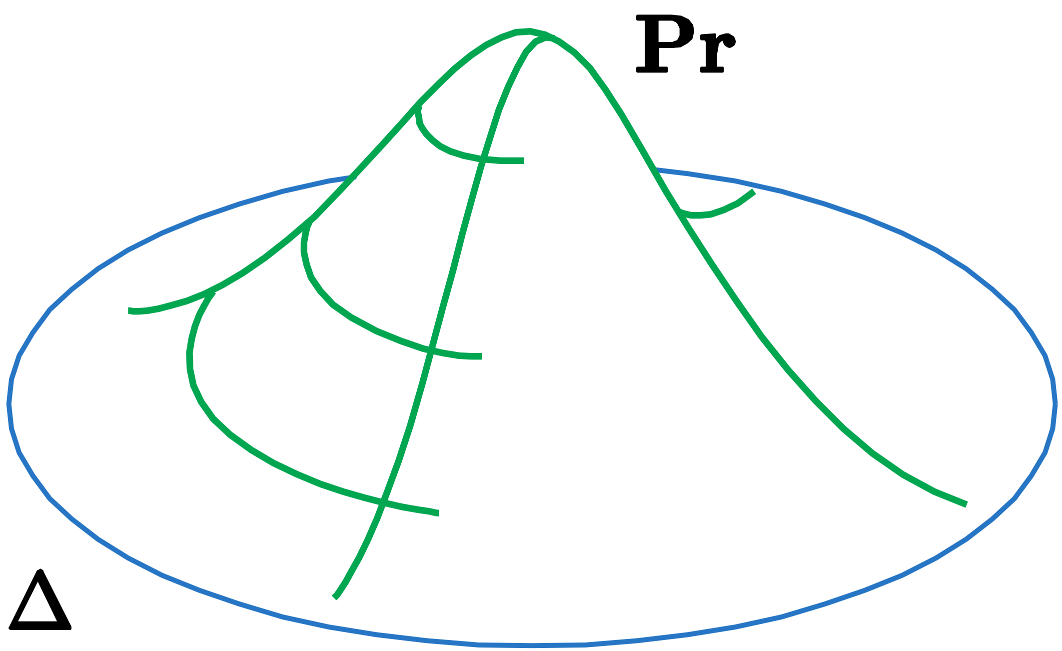


Probabilistic uncertainty



$$\min_{\theta} E_{\Delta}\left[\ell(\theta, \delta)\right]$$ (average approach)

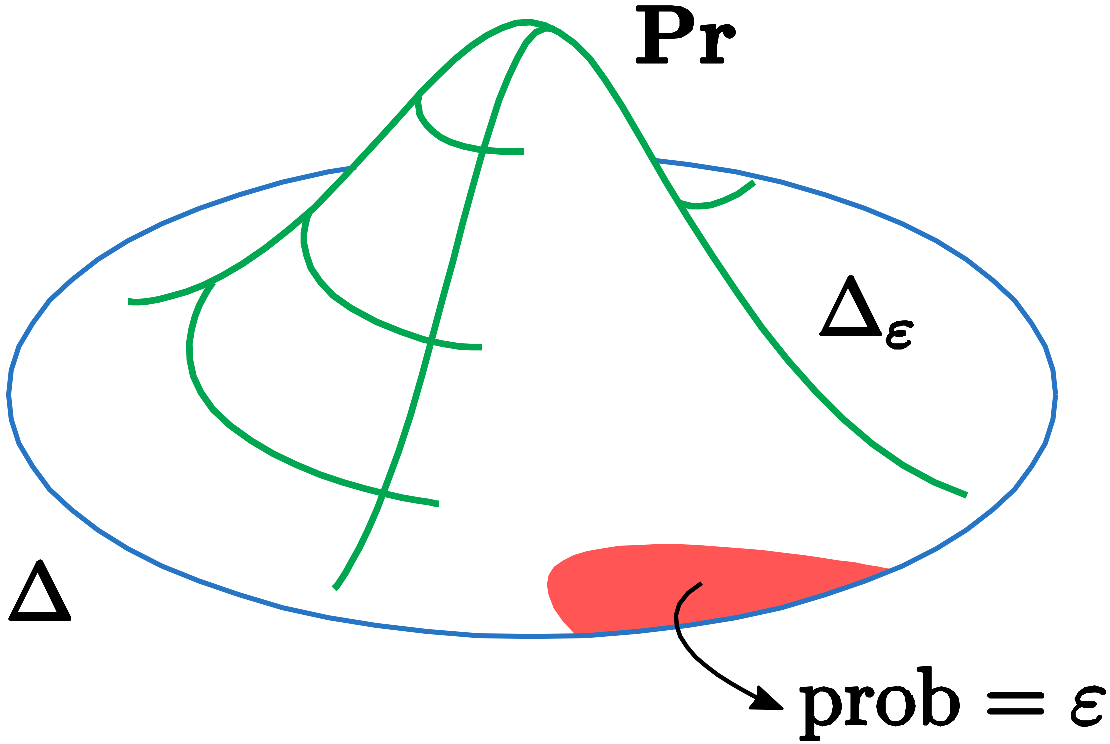
Probabilistic uncertainty



$\min_{\theta} E_{\Delta} [\ell(\theta, \delta)]$ (average approach)

stochastic control: $E_{\Delta}[\sum_t x_t^T Q x_t + u_t^T R u_t]$

Probabilistic uncertainty



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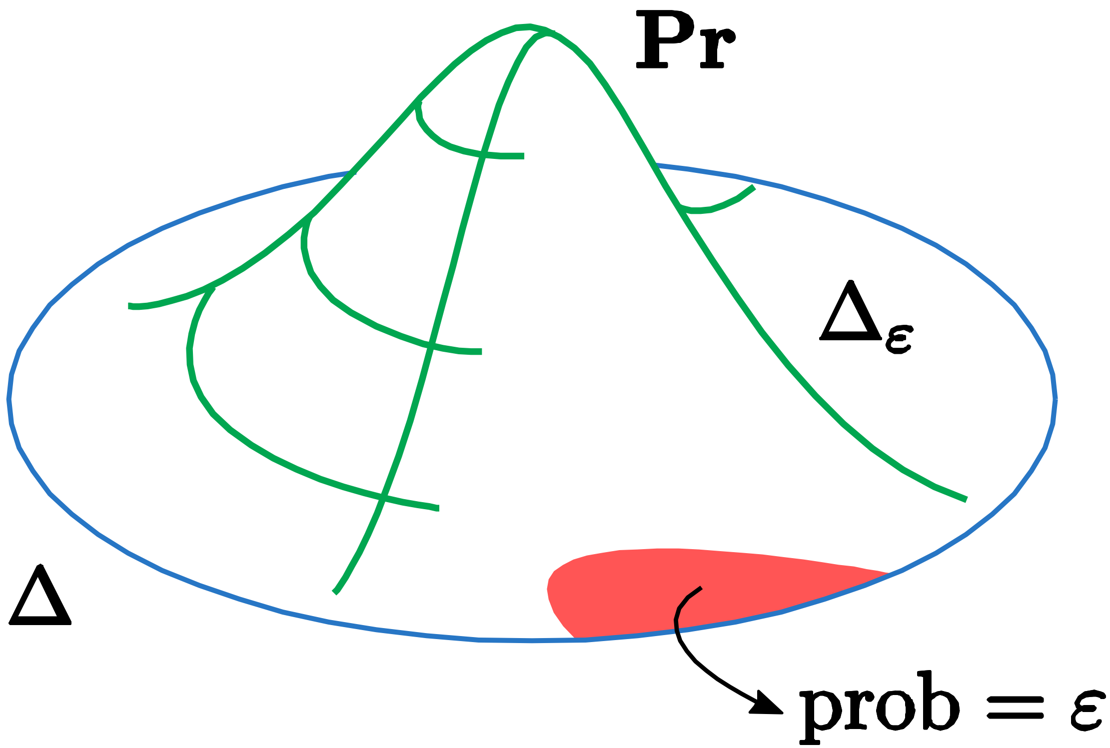

$$\min_{\theta}\left[\max_{\delta\in\Delta_\epsilon}\ell(\theta,\delta)\right]$$

$Pr(\Delta_\epsilon) = 1-\epsilon$ (chance-constrained approach)

Probabilistic uncertainty

chance-constrained approach:

[A. Charnes, W.W. Cooper, and G.H. Symonds, 1958]

Probabilistic uncertainty

chance-constrained approach:

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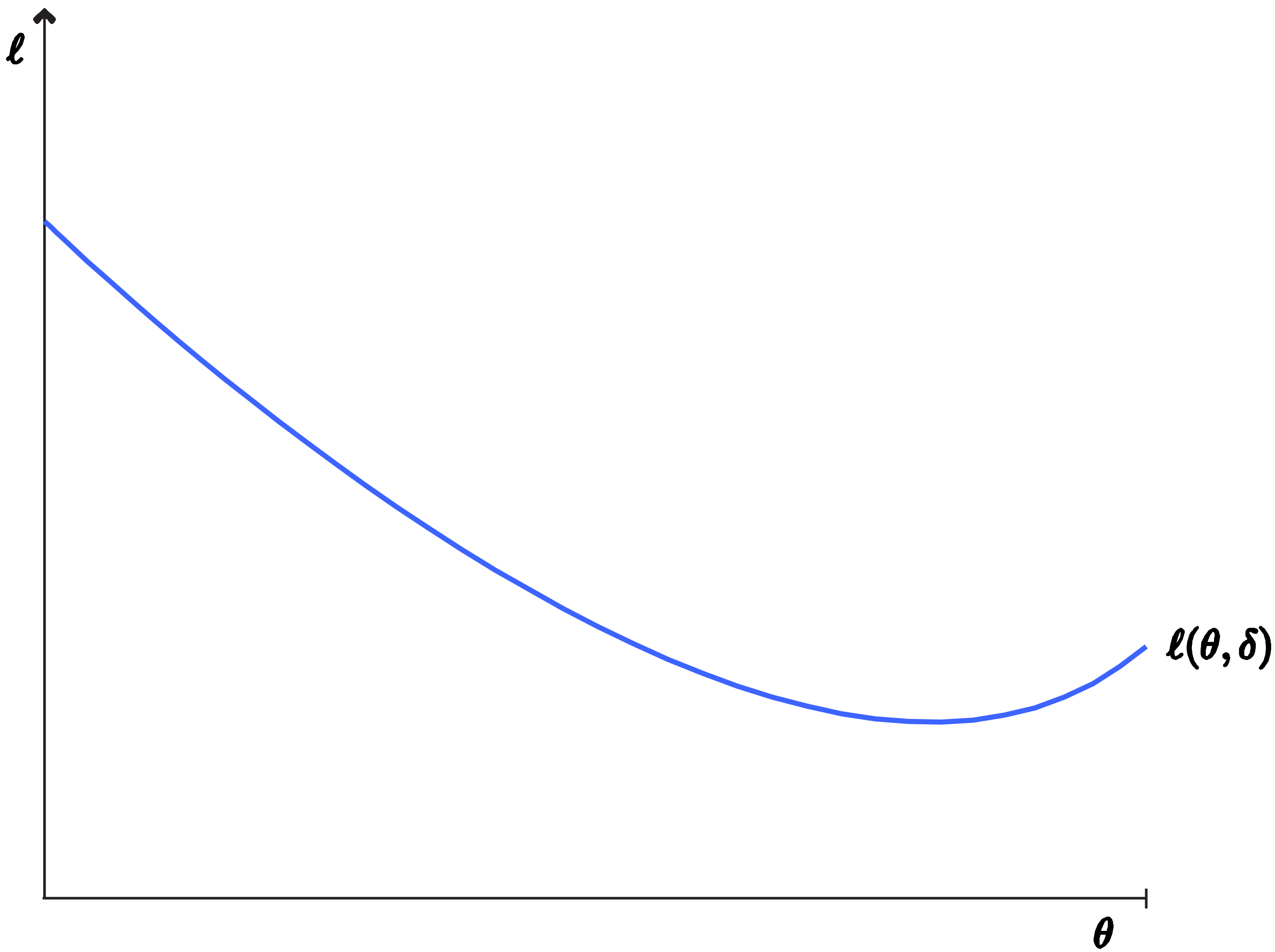
very difficult to solve, … with exceptions

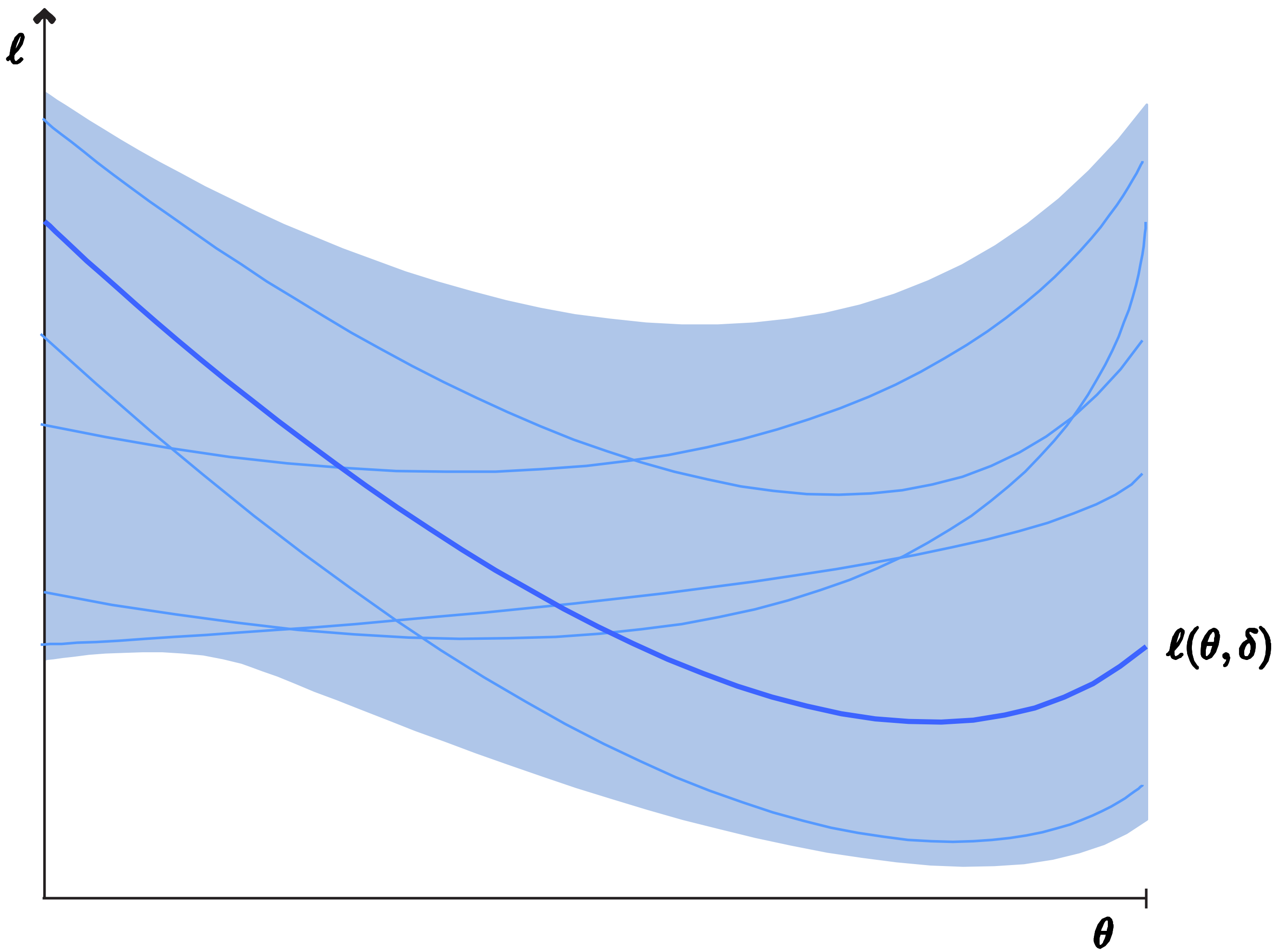
[A. Prékopa, 1995]

GOAL: **provide algorithmic tools**

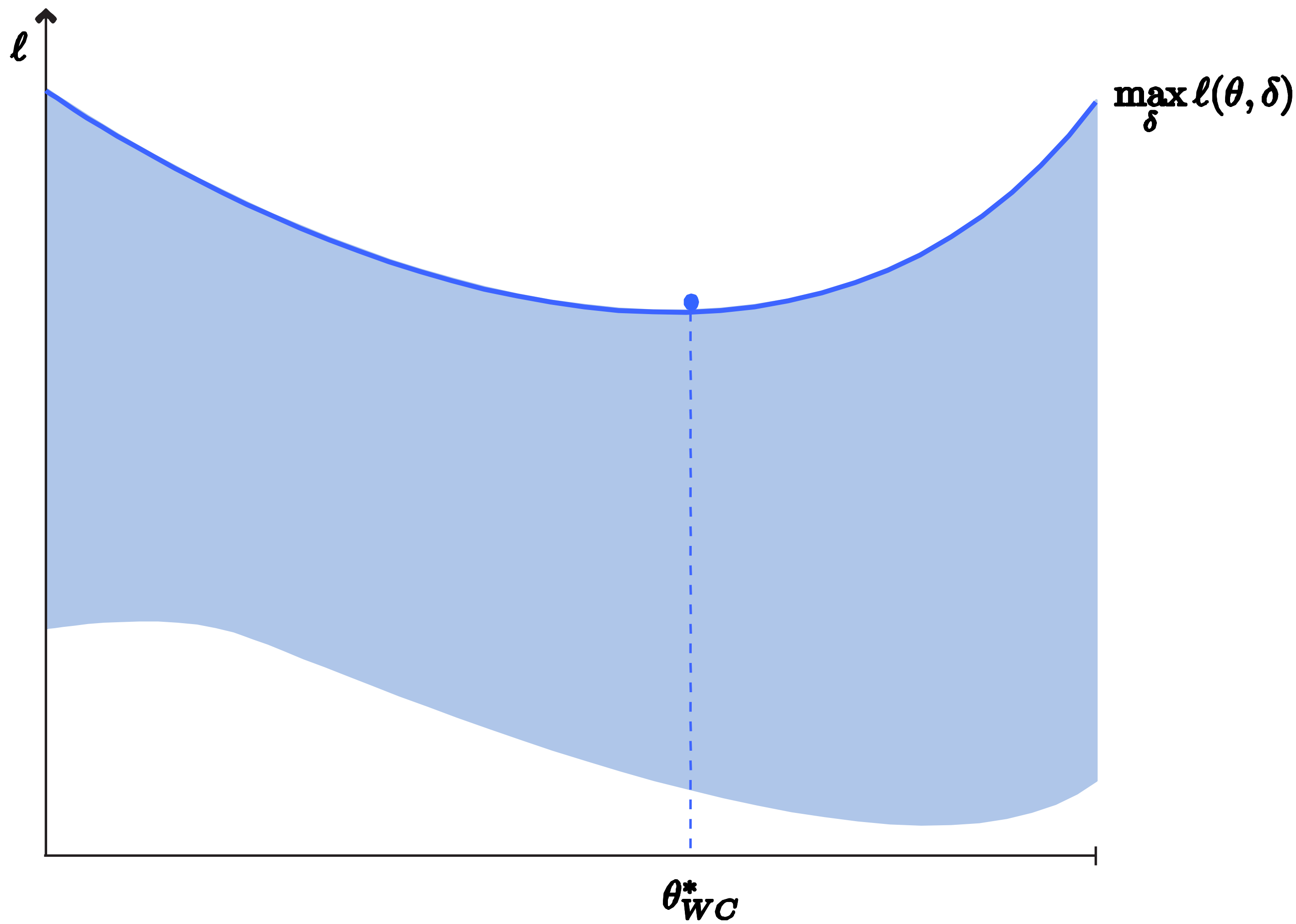
a look at optimization in the $\theta - \ell$ space



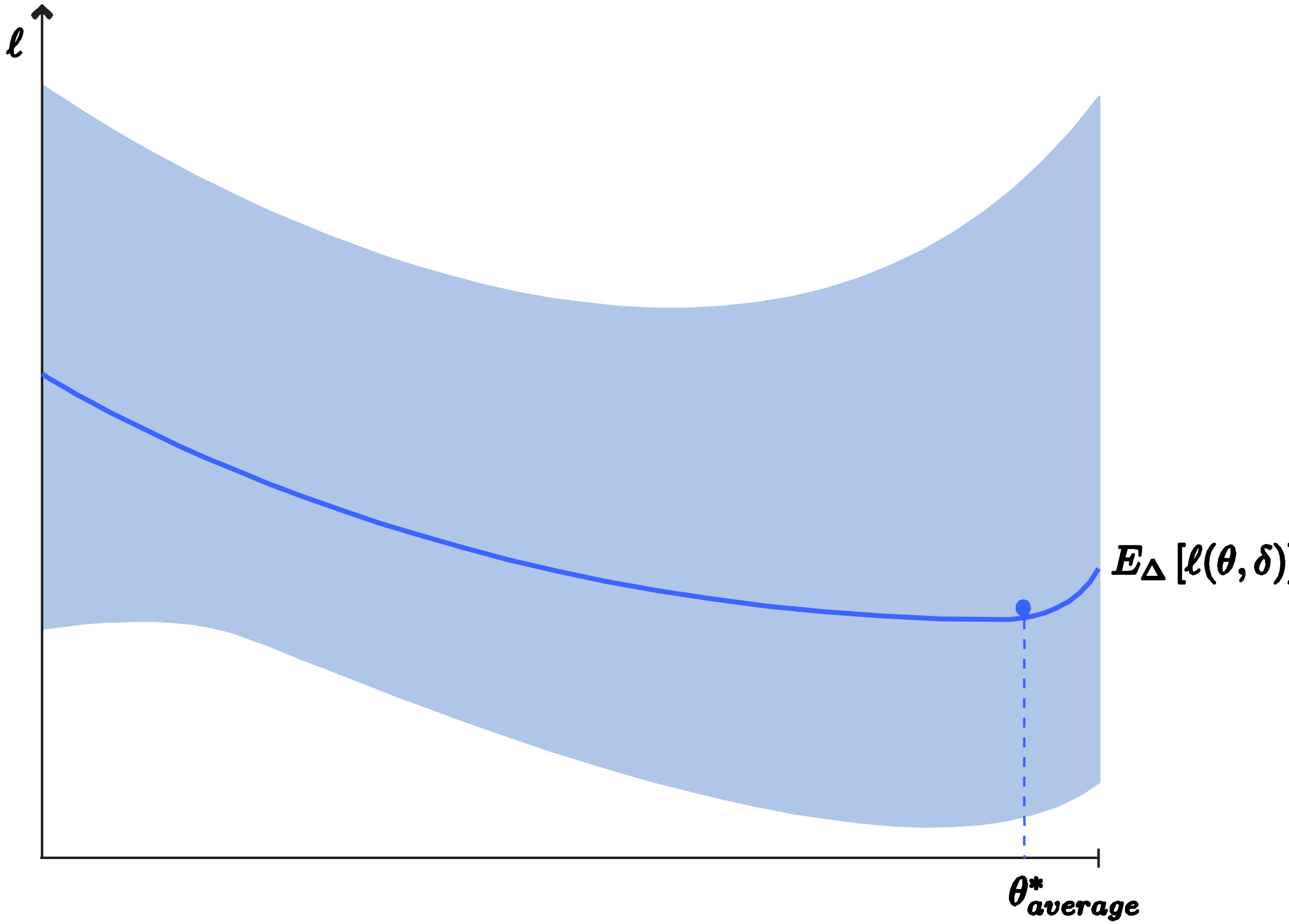
performance cloud



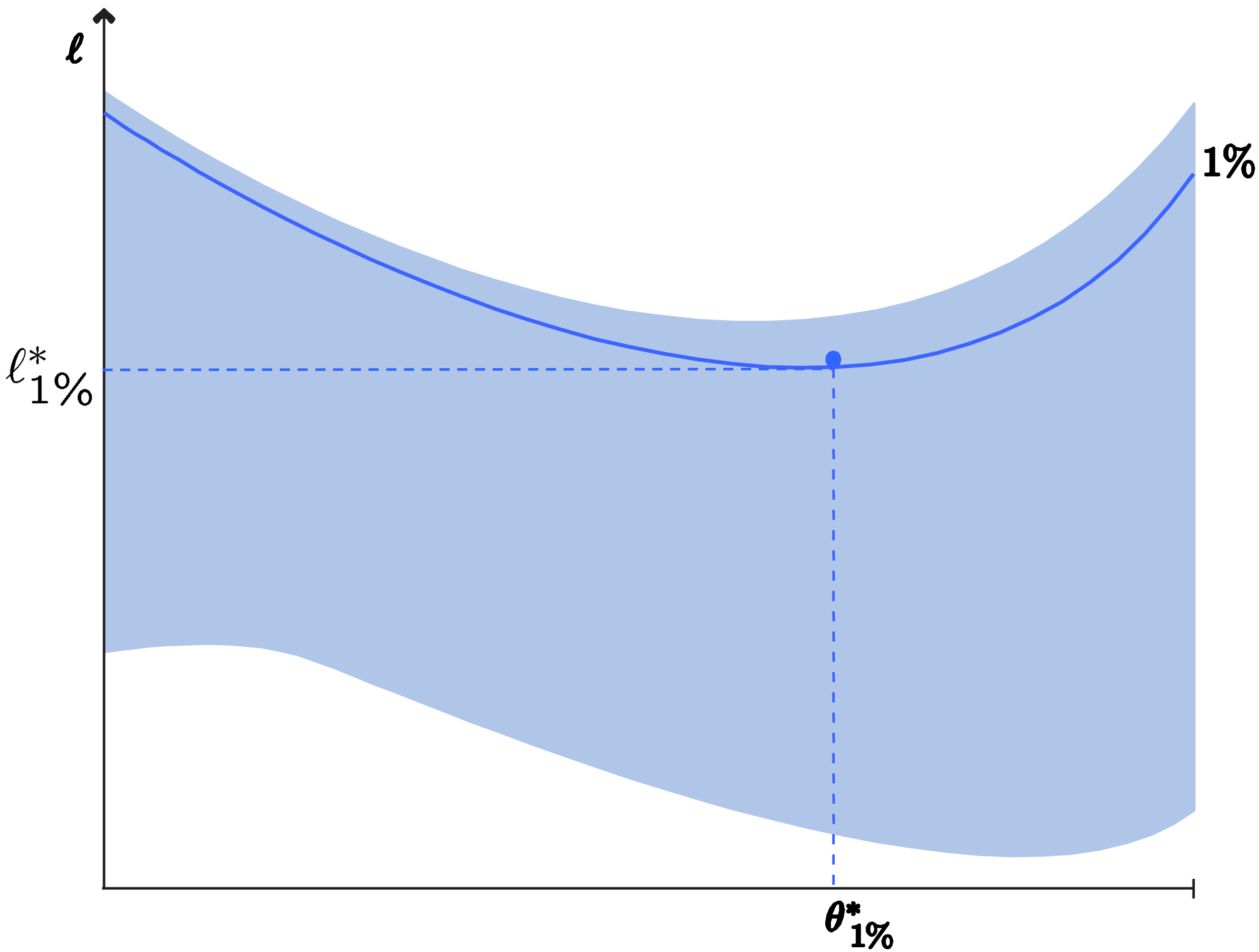
worst-case



average



chance-constrained approach



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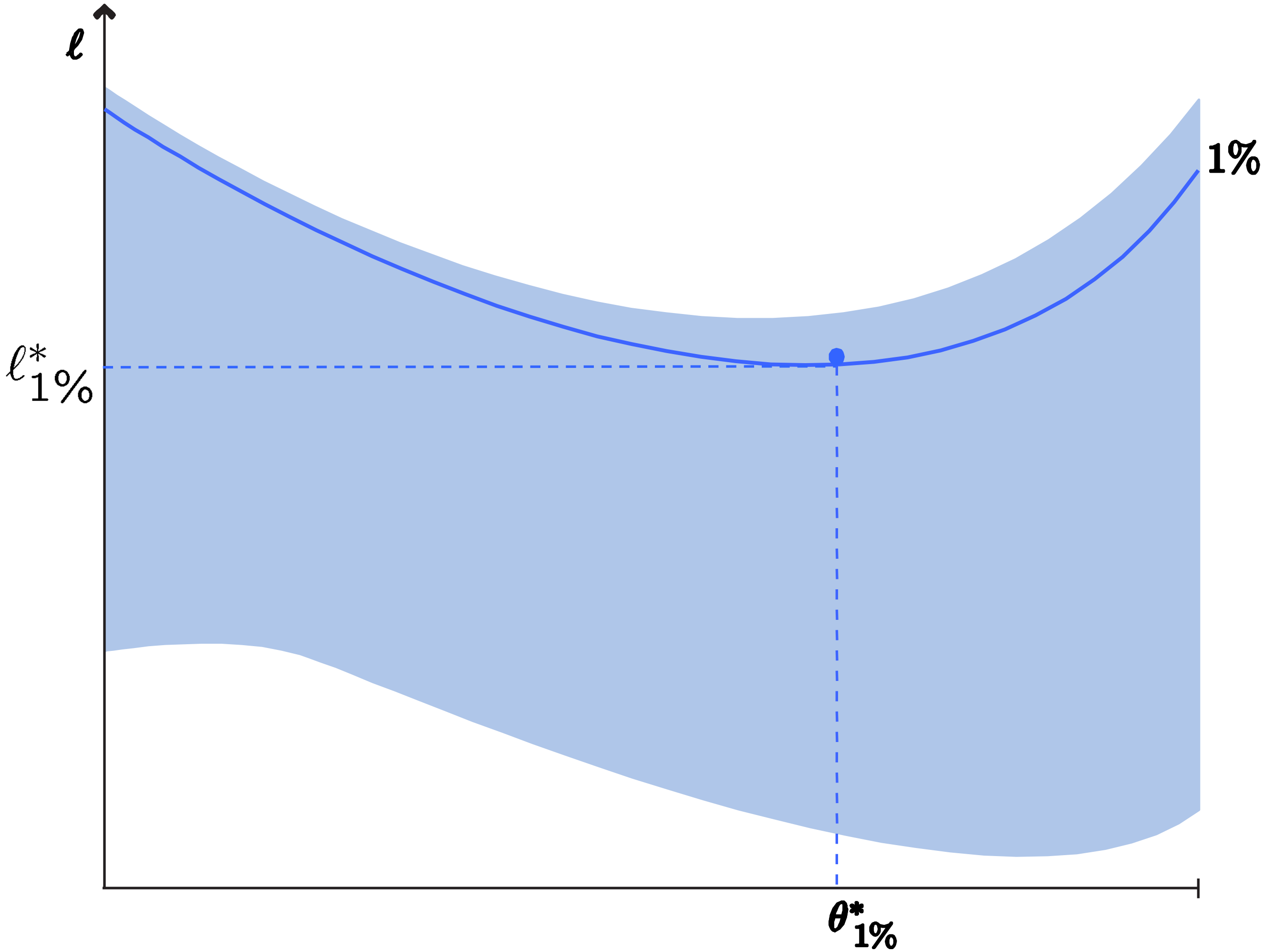
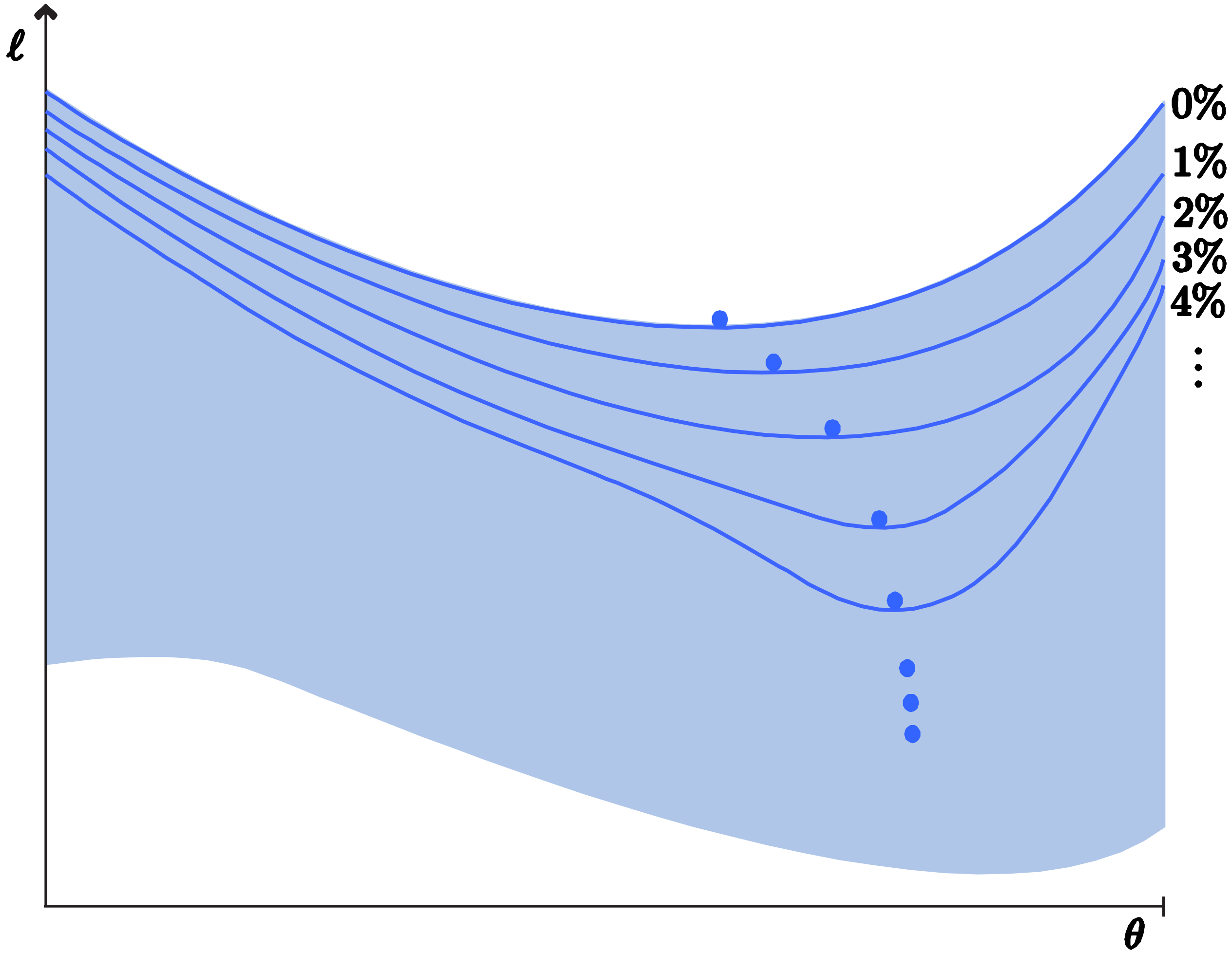
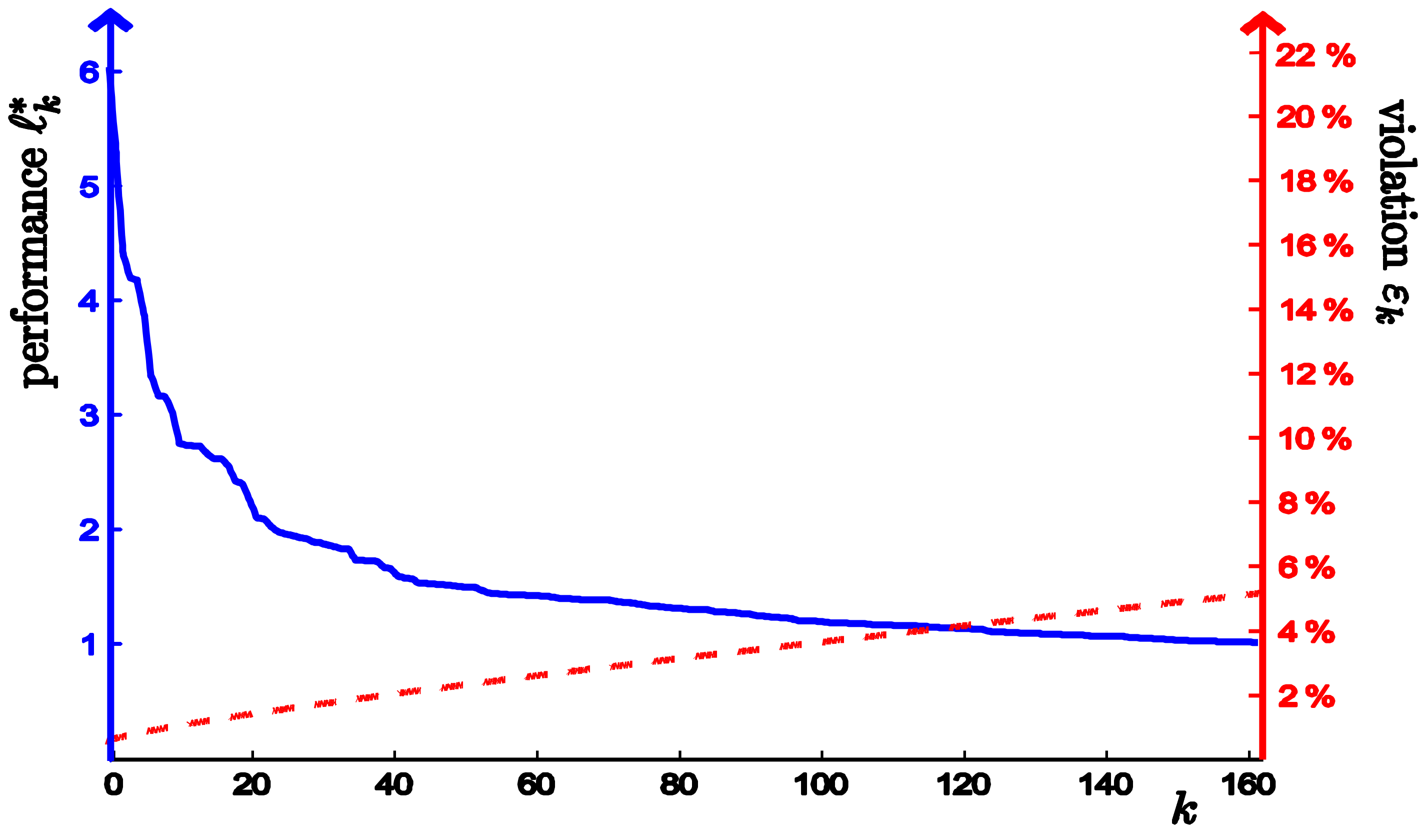


very hard to solve!

chance-constrained approach



performance - violation plot



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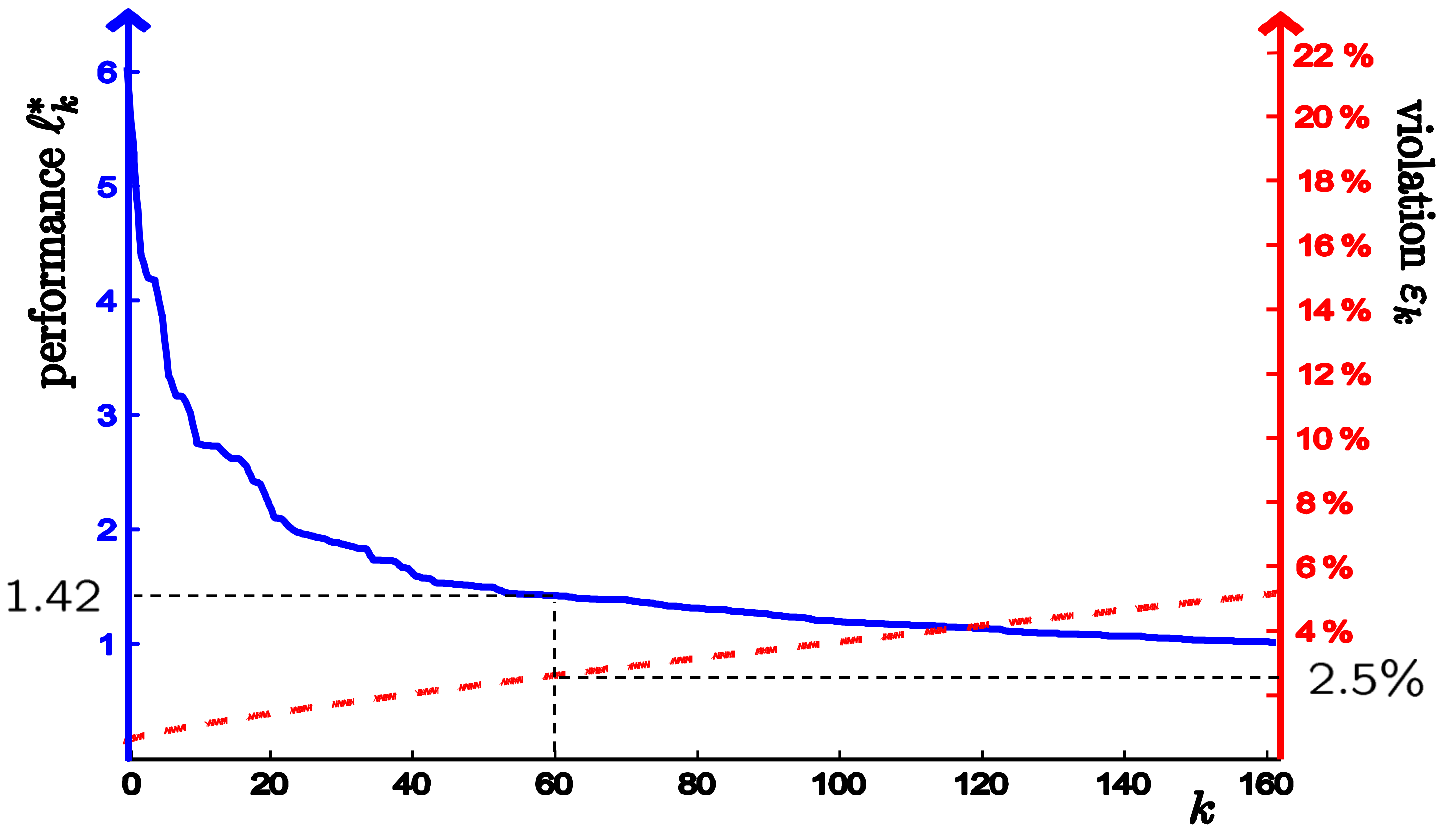
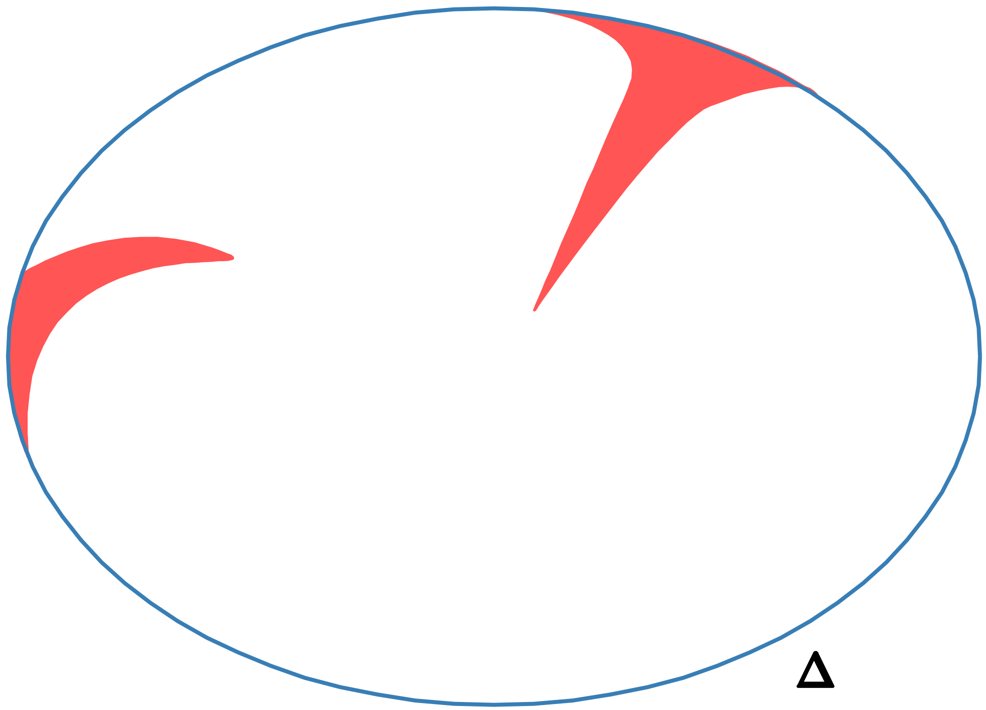


icicle geometry [C.M. Lagoa & B.R. Barmish, 2002]



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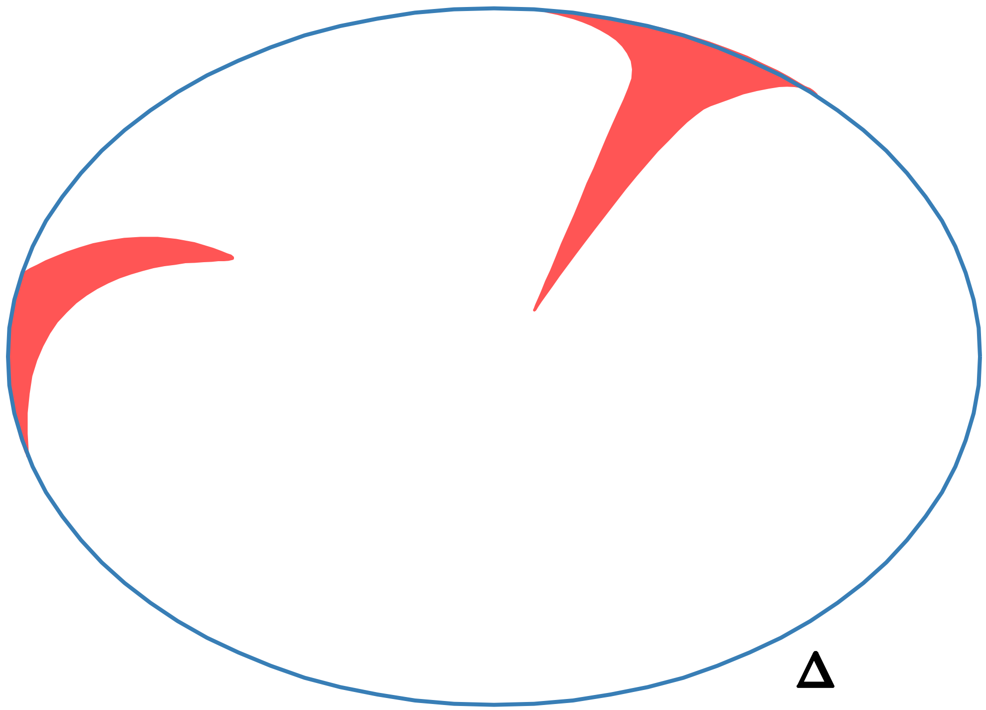
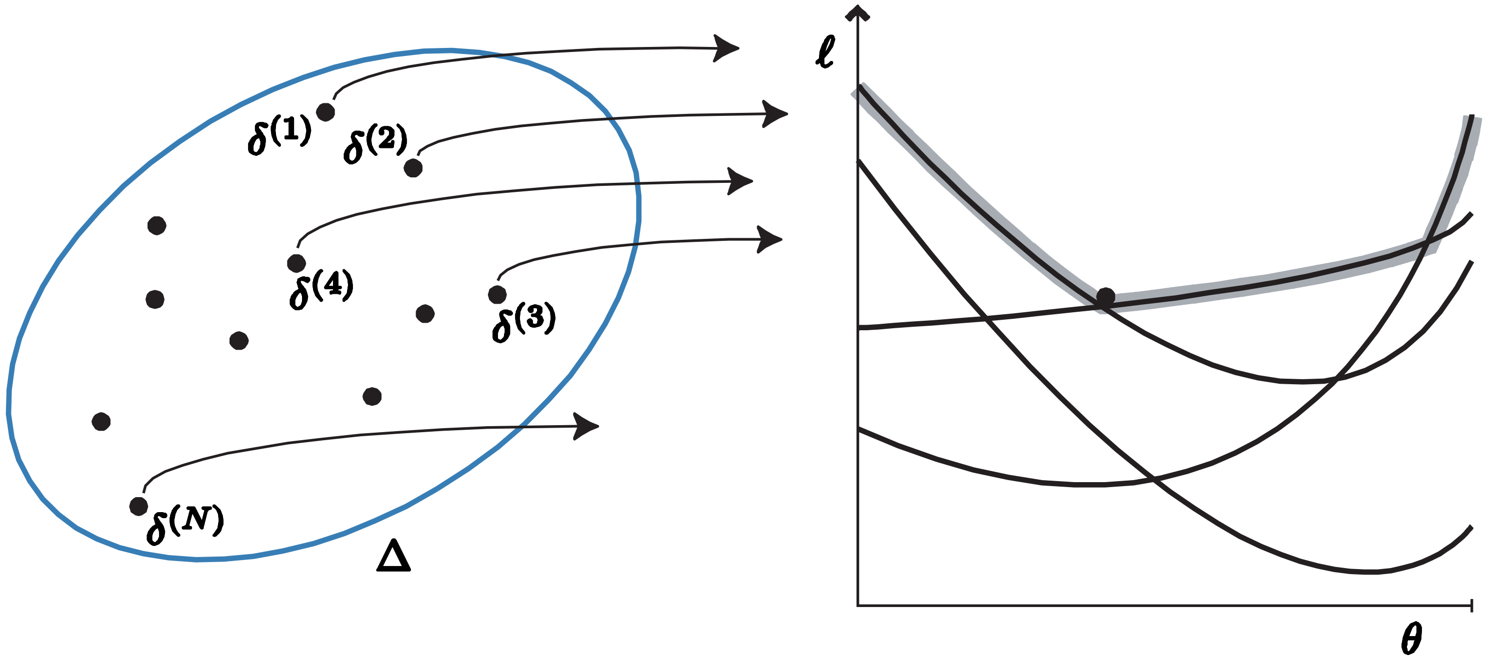


… let the problem speak

PART II: Algorithms

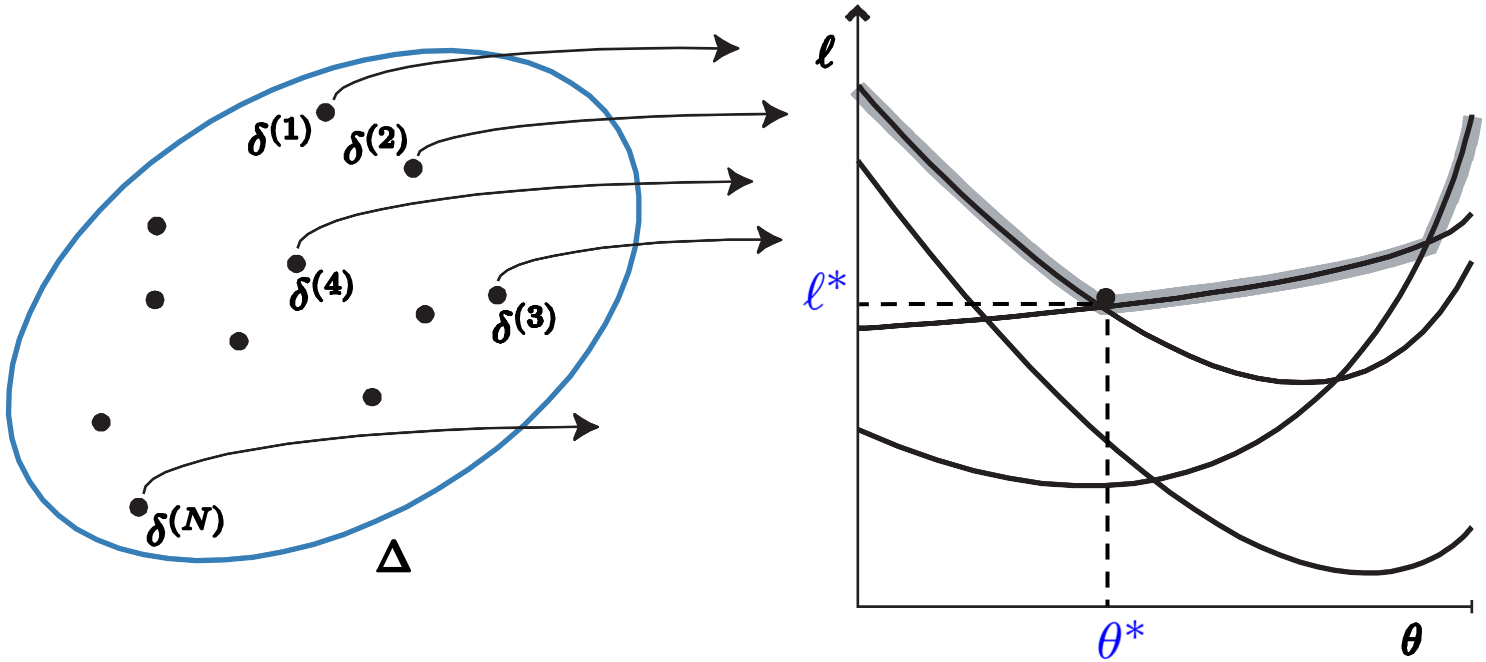
(convex case)

The “scenario” paradigm



[G. Calafiore & M. Campi, 2005, 2006]

The “scenario” paradigm



SP_N = scenario program

- SP_N is a standard finite convex optimization problem

[G. Calafiore & M. Campi, 2005, 2006]

Fundamental question:

what's the risk of ℓ^*?

Example: feedforward noise compensation

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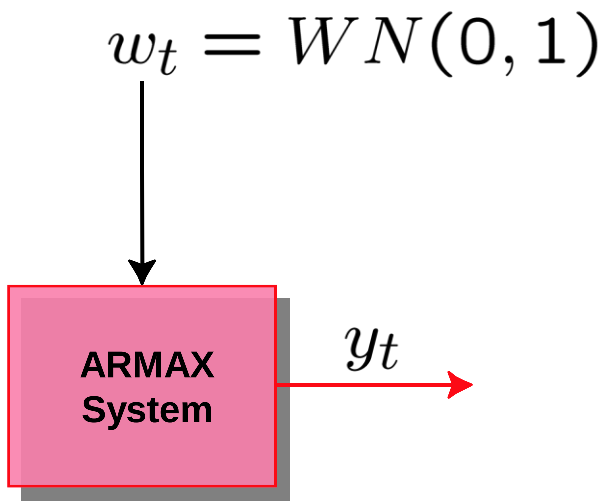

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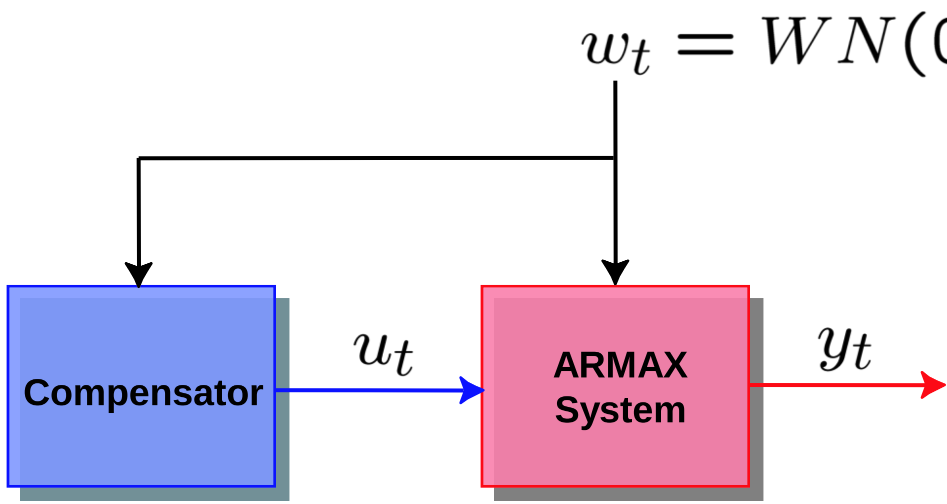

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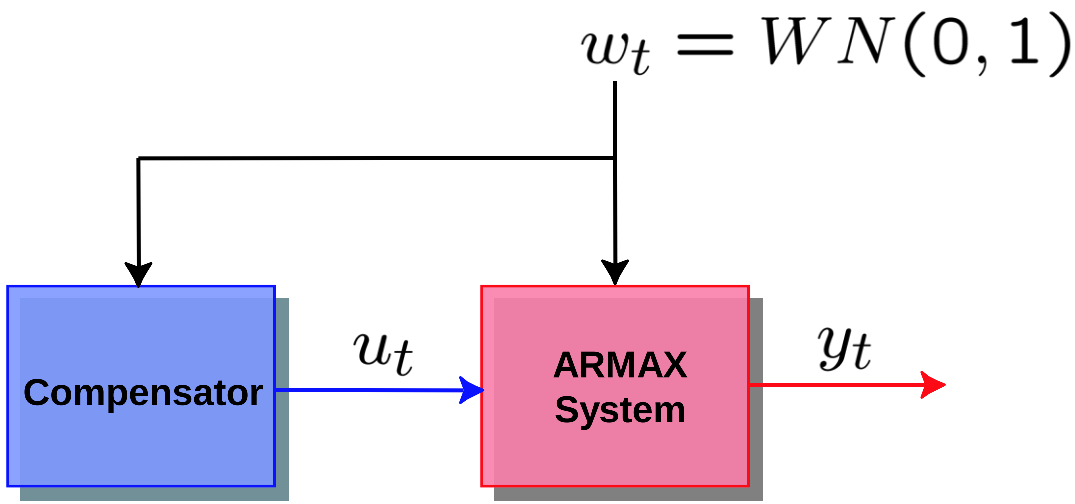
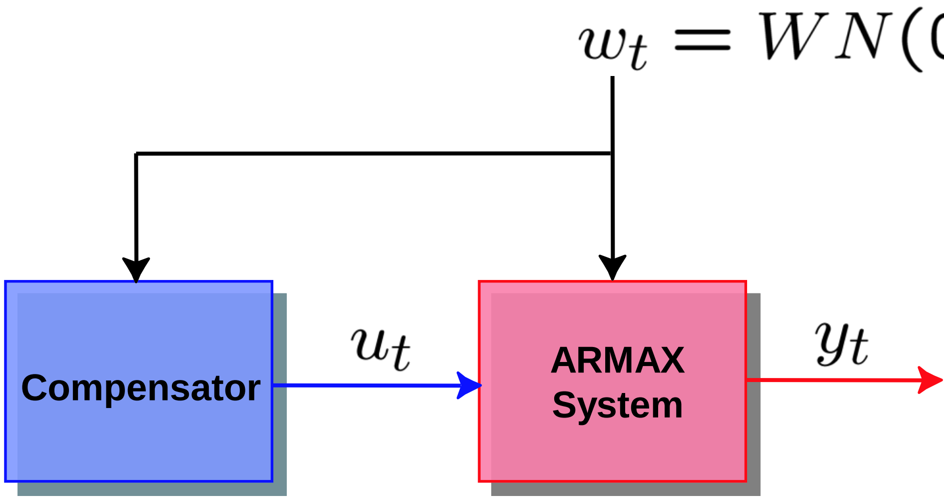


Objective: reduce the effect of noise

Example: feedforward noise compensation



ARMAX System:

$$y_{t+1} = ay_t + bu_t + cw_t + dw_{t-1}$$

Compensator:

$$u_t = k_1 w_t + k_2 w_{t-1}$$

Goal:

$$\min var[y_t]$$

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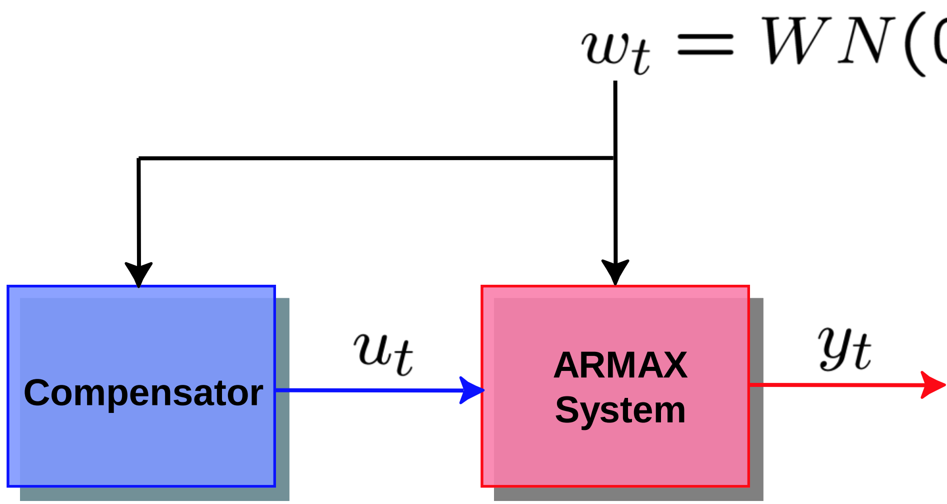

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system parameters unknown: $a, b, c, d \in \Delta$

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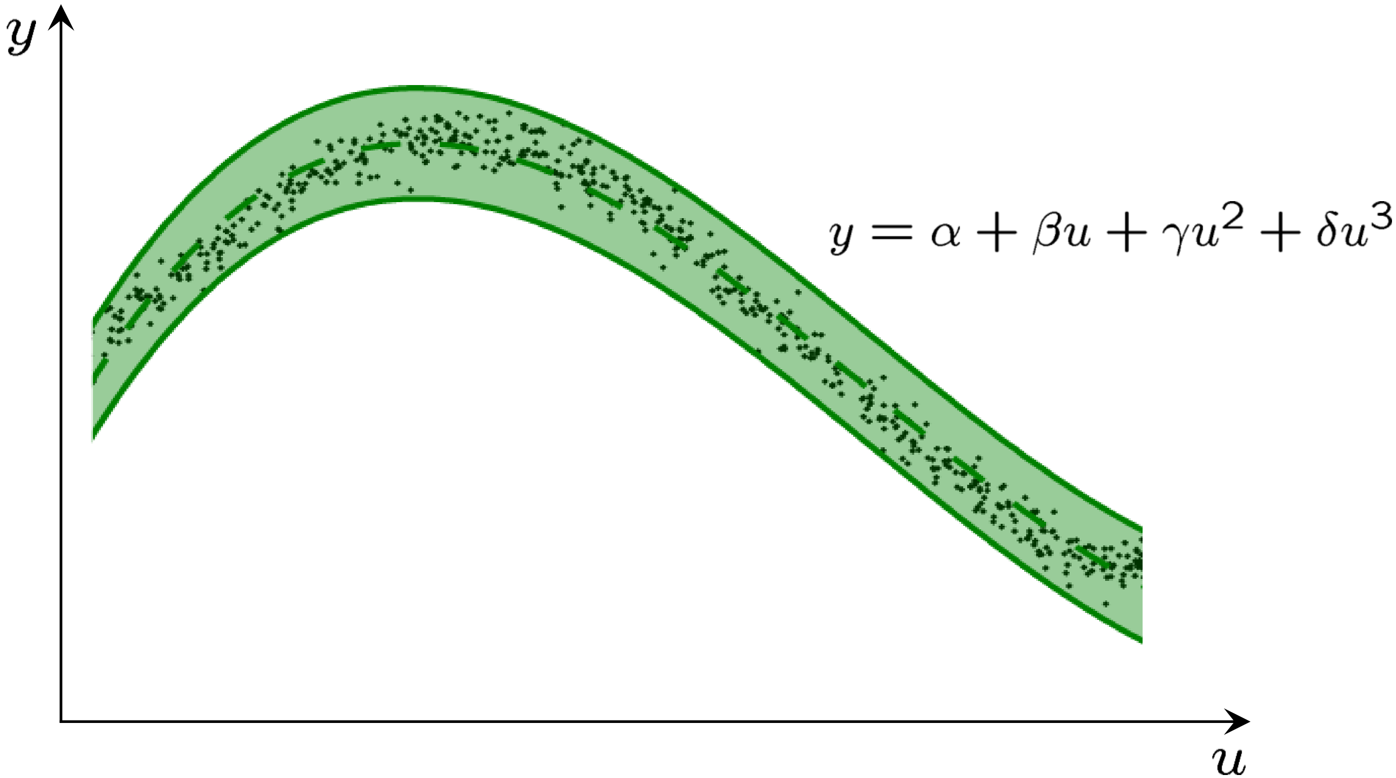
scenario approach:

sample: $a_i, b_i, c_i, d_i \in \Delta, \quad i = 1, 2, \ldots, N;$

solve:

$$\min_{k_1, k_2} \left[\max_i \frac{(c_i + b_i k_1)^2 + (d_i + b_i k_2)^2 + 2a_i(c_i + b_i k_1)(d_i + b_i k_2)}{1 - a_i^2} \right]$$

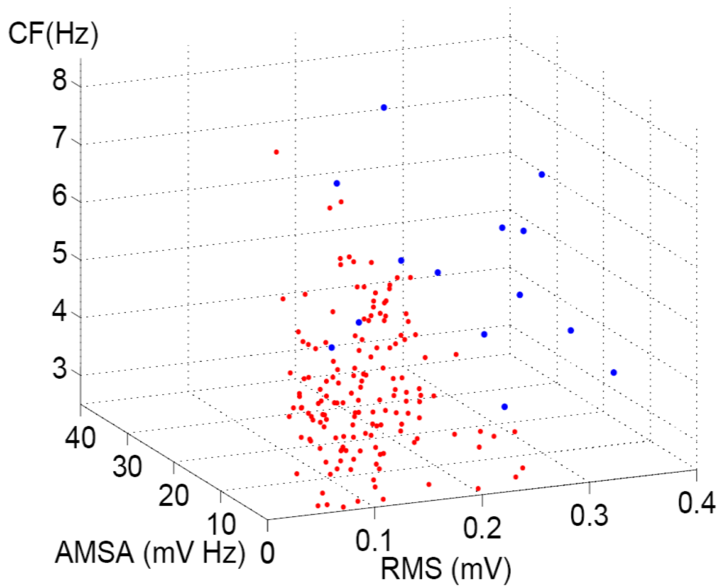
more examples: minimax prediction

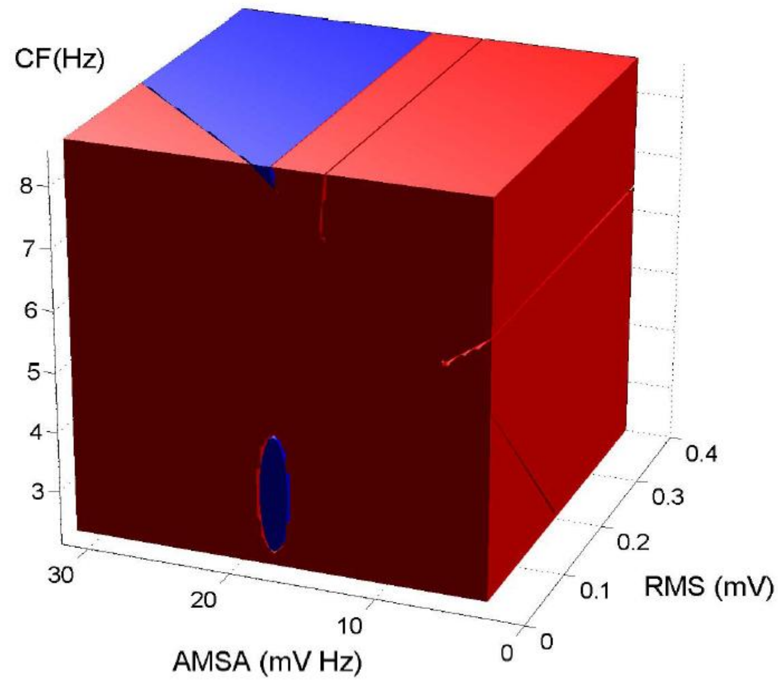


$$\min_{\alpha,\beta,\gamma,\delta} \left[\max_i |y_i - [\alpha + \beta u_i + \gamma u_i^2 + \delta u_i^3]| \right]$$

[M. Campi, G. Calafiore & S. Garatti, 2009]

more examples: machine learning





[M. Campi, 2010]

more examples: portfolio optimization

$$\min_{w_1,\ldots,w_p} \left[\max_i \left(-\sum_{k=1}^{p} w_k R_k(i) \right) \right]$$

$R_k(i)$ = return of asset k, i = instance in the record

with B. Pagnoncelli & D. Reich

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from the "visible" to the "invisible"

Theorem (with S. Garatti - G. Calafiore)

Fix $\epsilon \in (0,1)$ (risk parameter)
$\beta \in (0,1)$ (confidence parameter)

If $N \geq N(\epsilon,\beta) \doteq \frac{2}{\epsilon}\left(\ln\frac{1}{\beta} + n_\theta\right)$,
then,
with probability $\geq 1-\beta$,
ℓ^* is ϵ-level guaranteed.

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Theorem (with S. Garatti - G. Calafiore)

Fix $\epsilon \in (0,1)$ (risk parameter)

If $N \geq N(\epsilon) \doteq \frac{2}{\epsilon}(7 \ln 10 + n_\theta)$,
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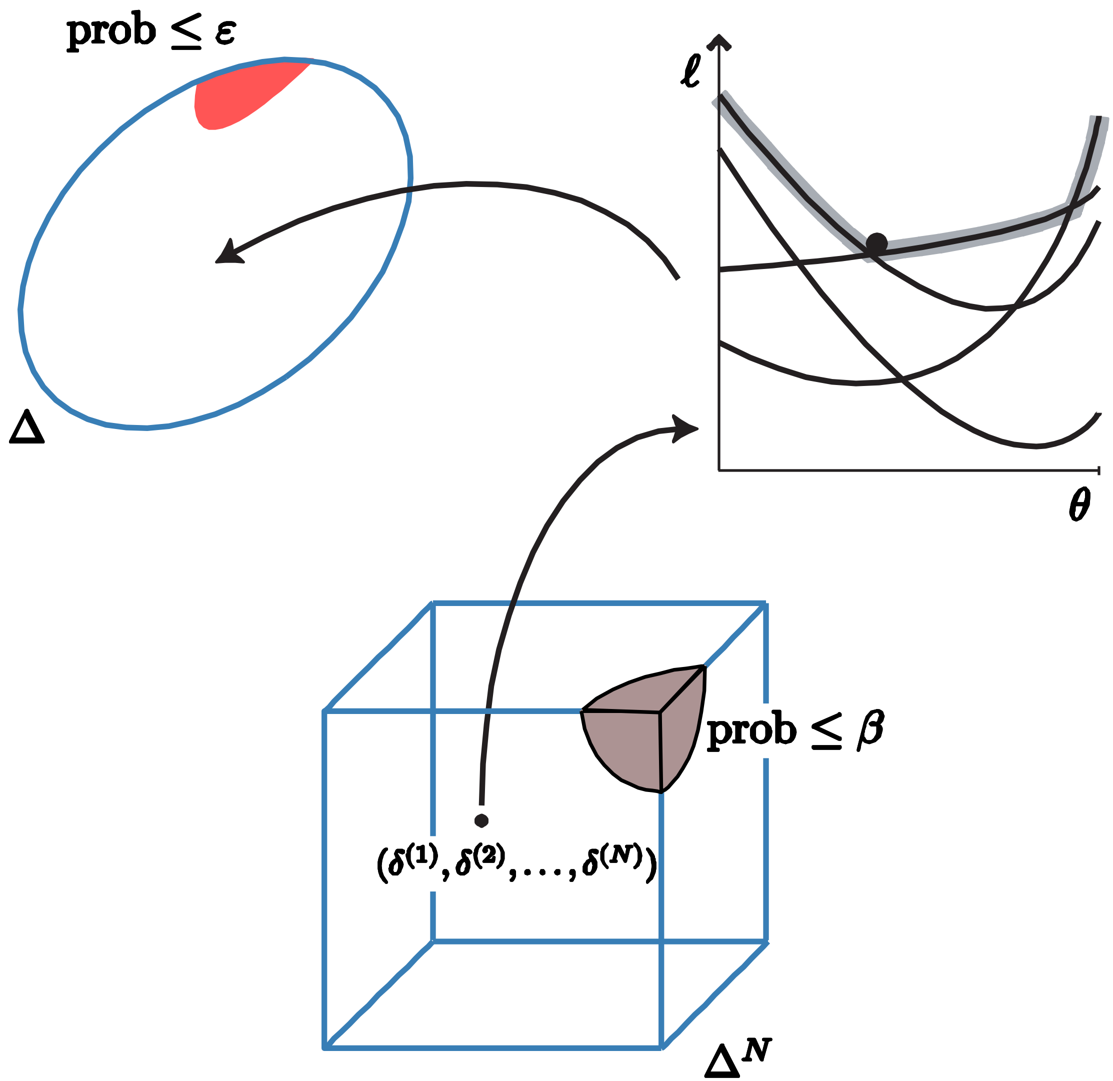

prob $\leq \varepsilon$
Δ
ℓ
θ
prob $\leq \beta$
$(\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(N)})$
Δ^N

Comments

generalization $\longrightarrow$ need for structure

Good news: the structure we need
is only convexity

… more comments

$$N = \frac{2}{\epsilon}\left(\ln\frac{1}{\beta} + n_\theta\right)$$

- *N* often tractable by standard solvers
- *N* easy to compute
- *N* independent of *Pr*
- permits to address problems otherwise intractable

Ex: feedforward noise compensation

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$$\begin{aligned}
\Delta = \{a, b, c, d \quad : \quad & a = \frac{3.5\sigma_1^2 - 0.2}{3\sigma_1^2 + 0.3} \cdot (0.32\sigma_1 + 0.6), \\
& b = 1 + \frac{\sigma_1\sigma_2^2}{10}, \\
& c = \frac{-0.01 + (\sigma_1 + \sigma_2^2)^2}{0.02 + (\sigma_1 + \sigma_2^2)^2} \cdot \left(1 - \frac{(\sigma_1 - 1)(\sigma_2 - 1)}{2}\right), \\
& d = \frac{0.05}{0.025 + (\sigma_1 + \sigma_2 - 2)^2}, \\
& \\
& (\sigma_1, \sigma_2) \in [-1, 1]^2\}.
\end{aligned}$$

Example: feedforward noise compensation

$\varepsilon = 0.005 \quad \beta = 10^{-7}$ ⇨ $N = 5427$

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$$\varepsilon = 0.005 \quad \beta = 10^{-7} \quad \Rightarrow \quad N = 5427$$

sample: $a_i, b_i, c_i, d_i \in \Delta, \quad i = 1, 2, \ldots, 5427;$

solve:

$$\min_{k_1, k_2} \left[\max_i \frac{(c_i + b_i k_1)^2 + (d_i + b_i k_2)^2 + 2a_i (c_i + b_i k_1)(d_i + b_i k_2)}{1 - a_i^2} \right]$$

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$$\Rightarrow \quad k_1^* = -0.9022, \quad k_2^* = -0.9028, \quad \ell^* = 5.8$$

Example: feedforward noise compensation

$\ell^* = 5.8$ ⇨ **Output variance below 5.8 for all plants but a small fraction (ε = 0.5%)**

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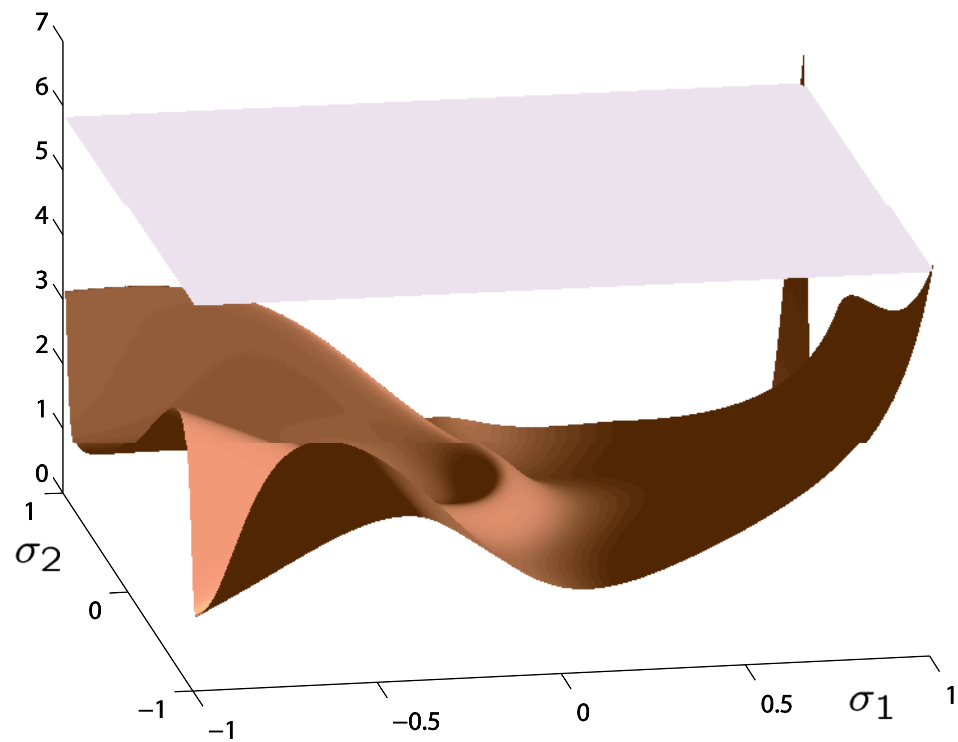
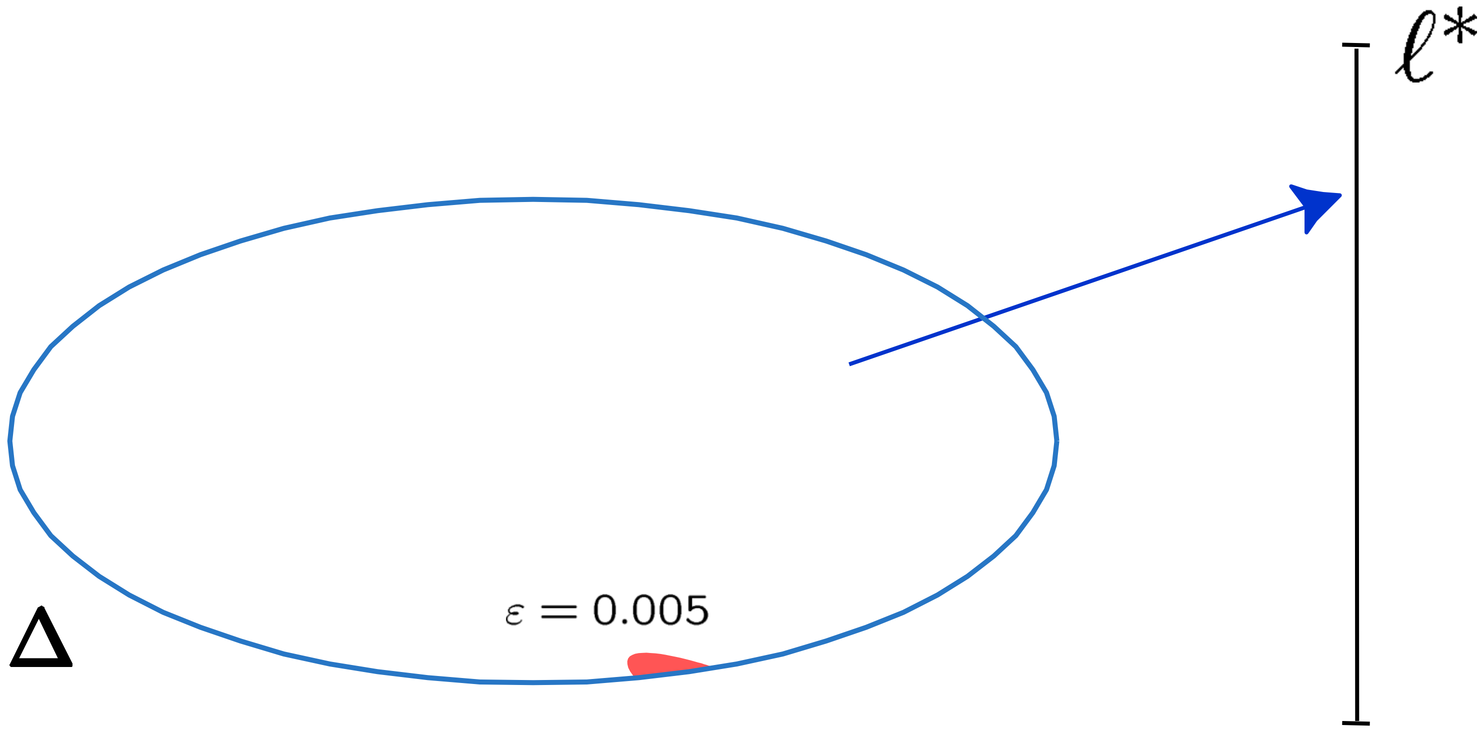


performance profile

Risk-Return Tradeoff



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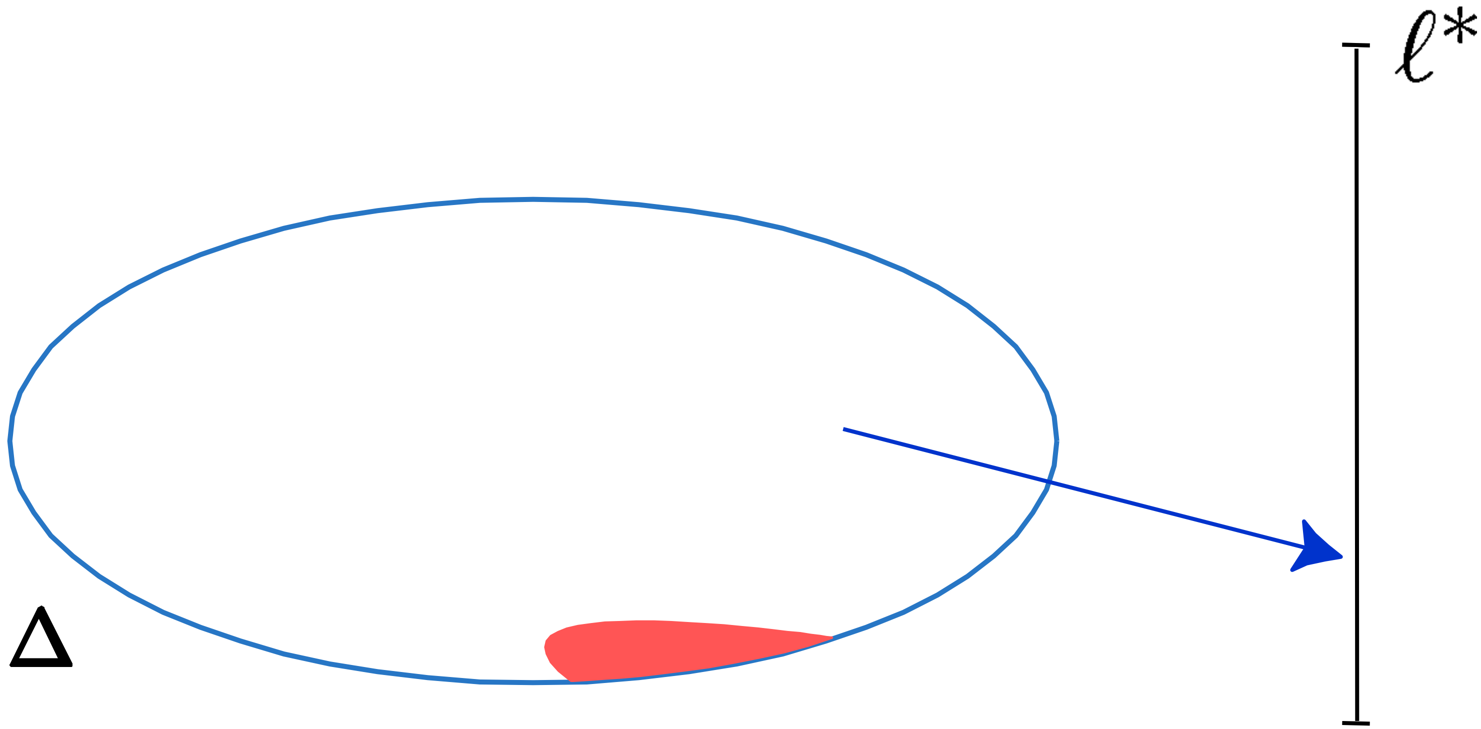

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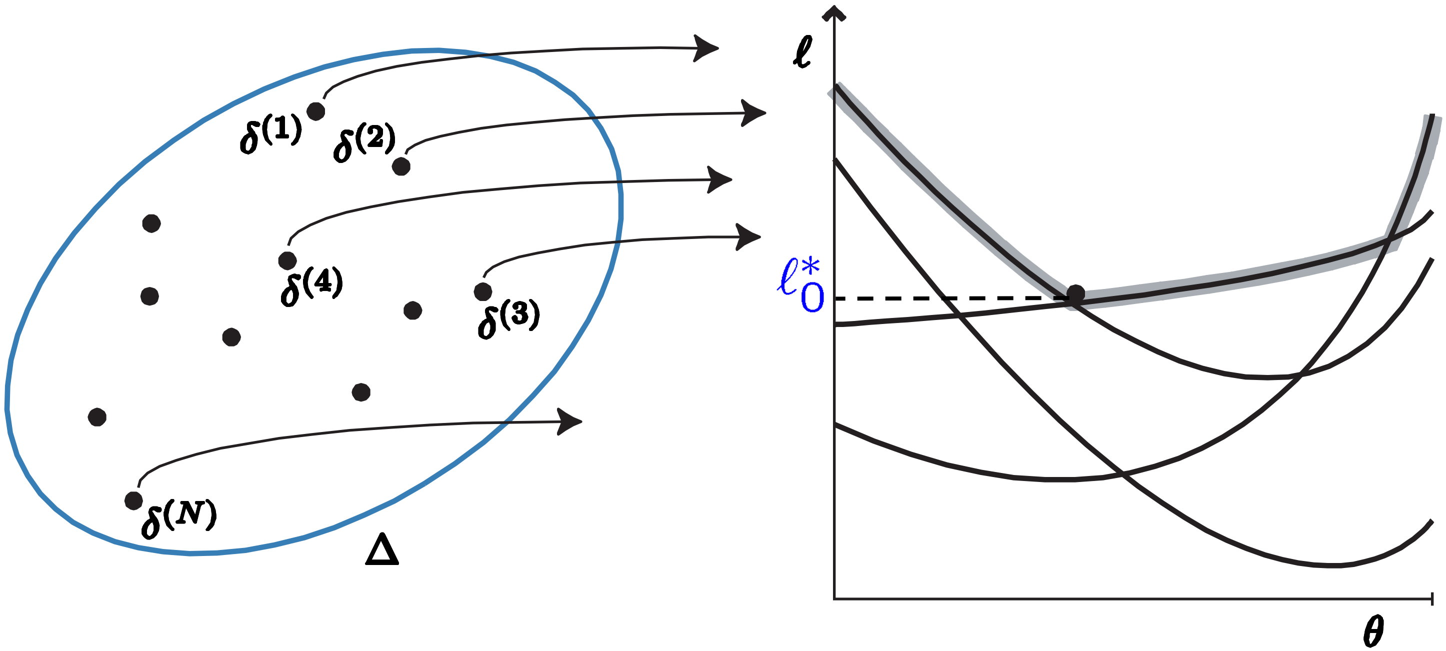

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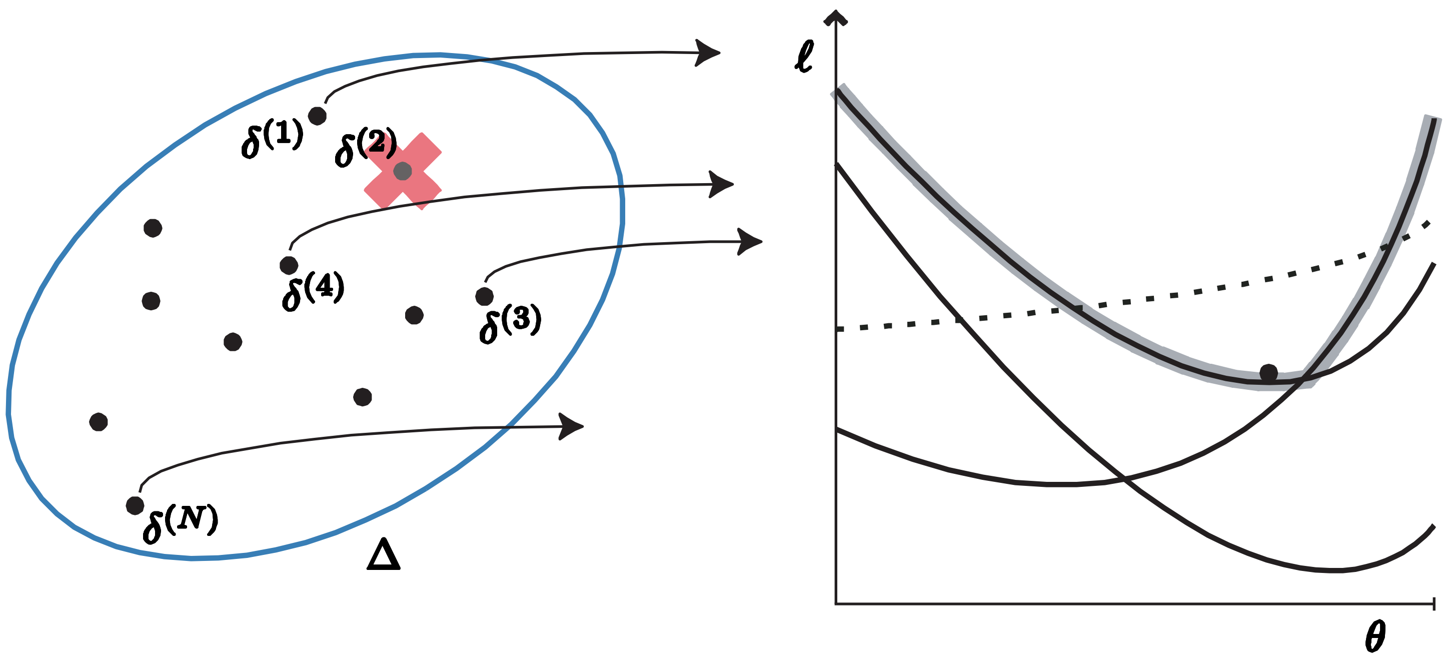

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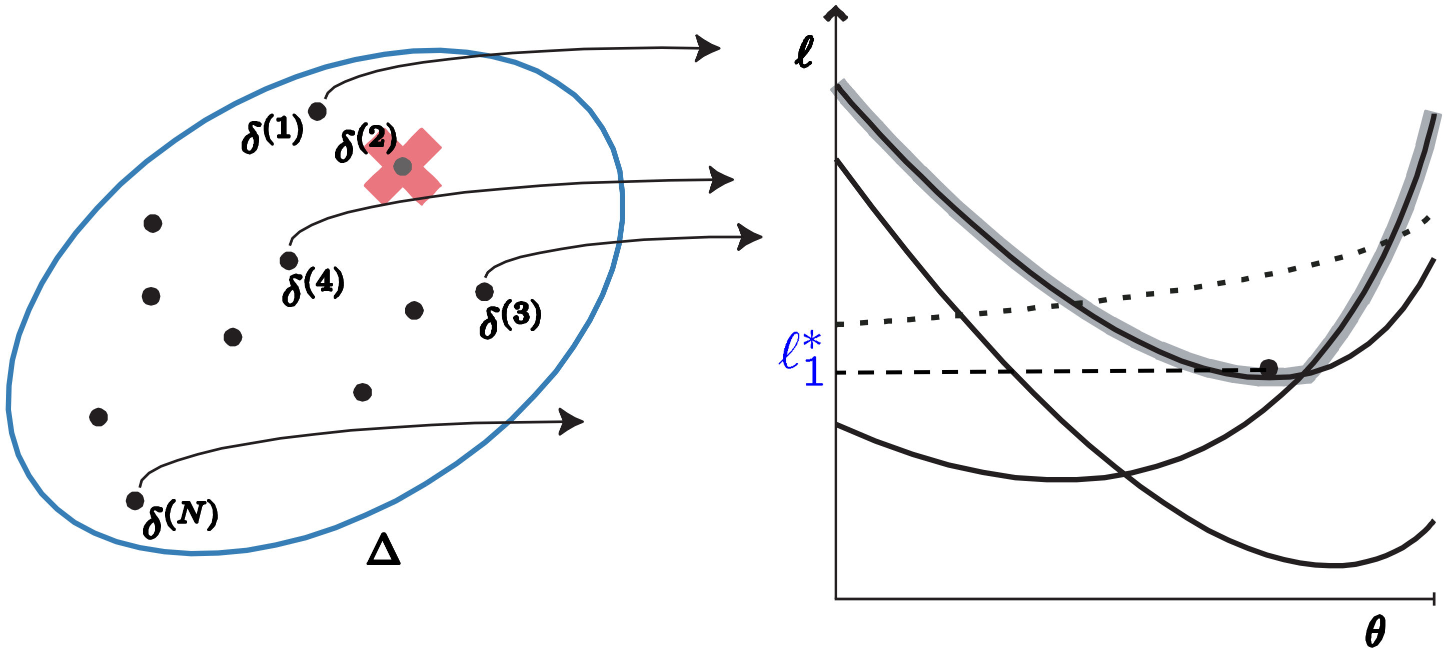

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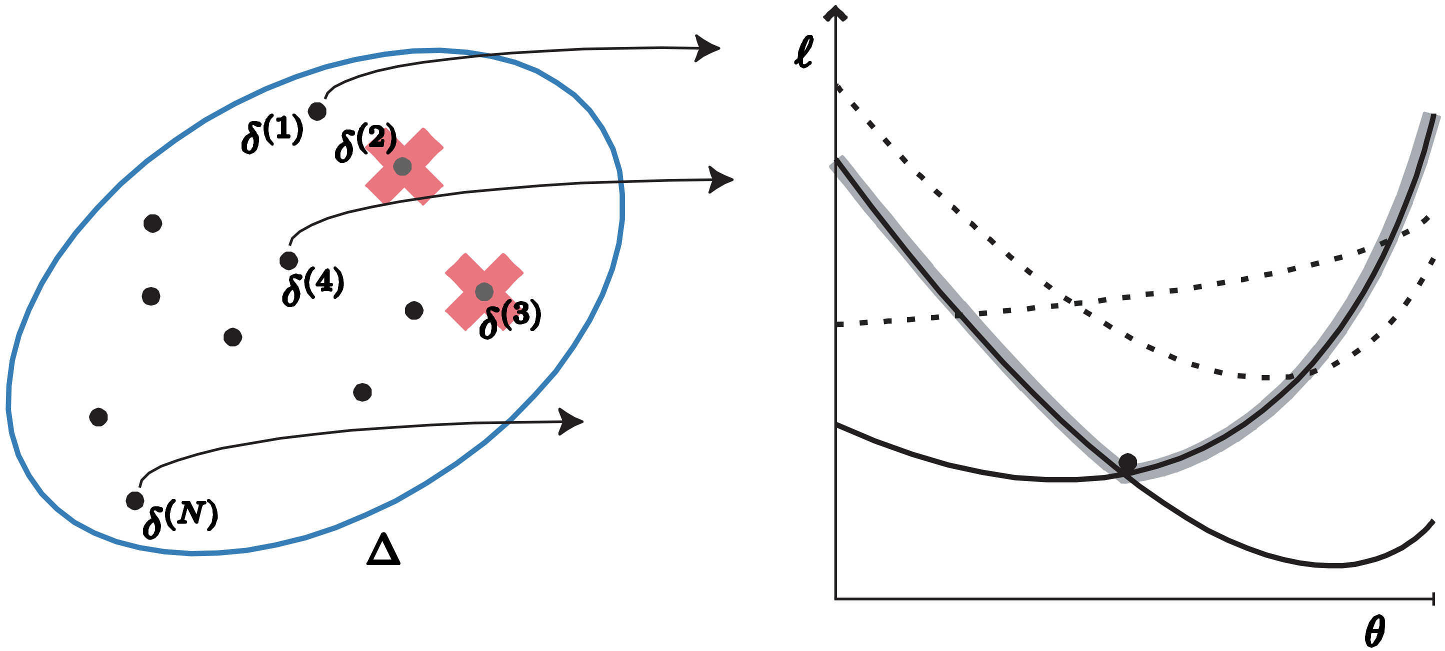

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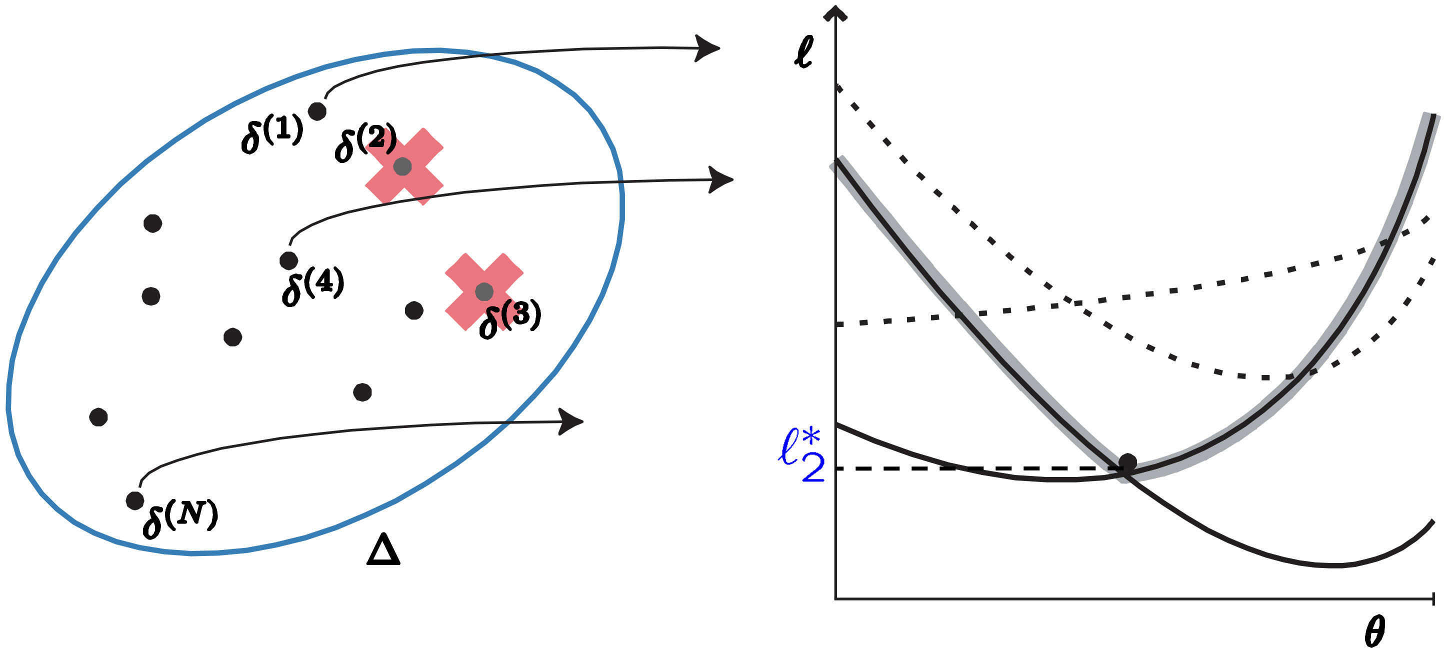

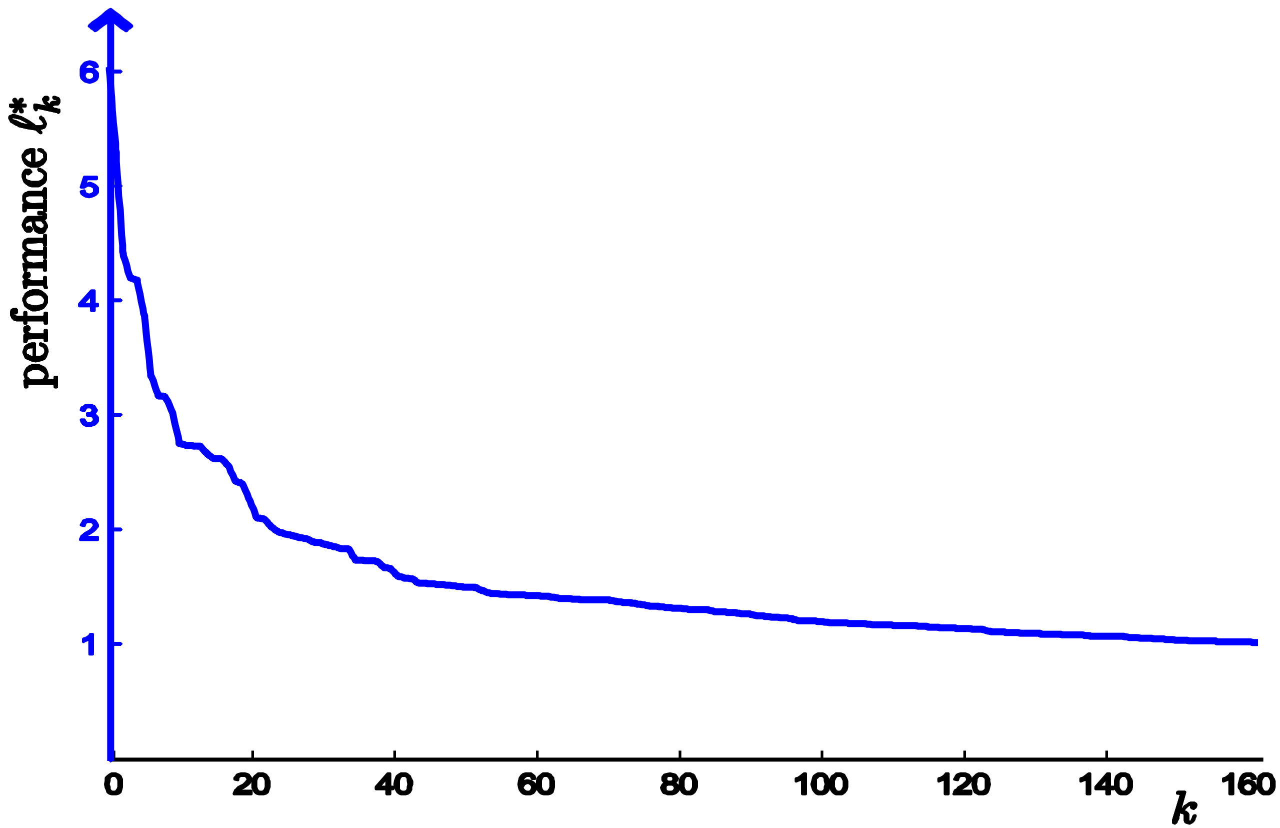

performance ℓ_k^*
6
5
4
3
2
1
0
20
40
60
80
100
120
140
160
k

Theorem (with S. Garatti)

$$N \geq N(\epsilon, \beta) \doteq \frac{2}{\epsilon}\left(\ln\frac{1}{\beta} + n_\theta\right).$$

Then, ℓ_k^* is ϵ_k-level guaranteed where:

$$\epsilon_k = \frac{k}{N} + O\left(\frac{1}{\sqrt{N}}\right)$$

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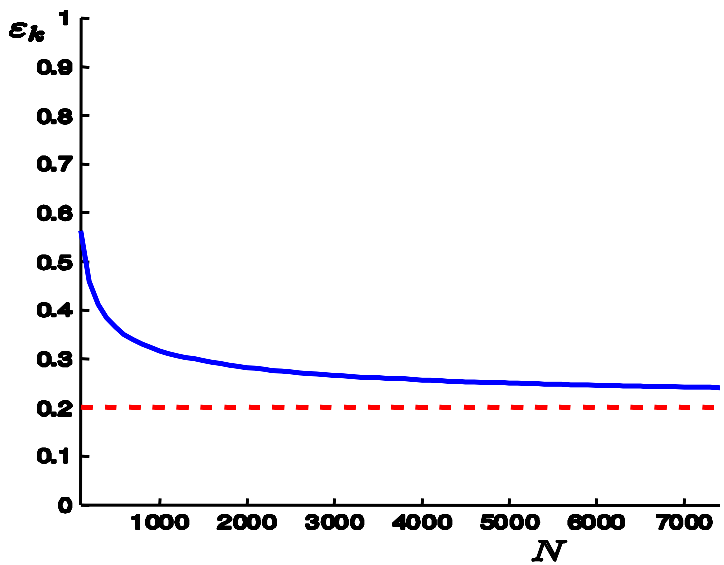

Comments

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… do it greedy

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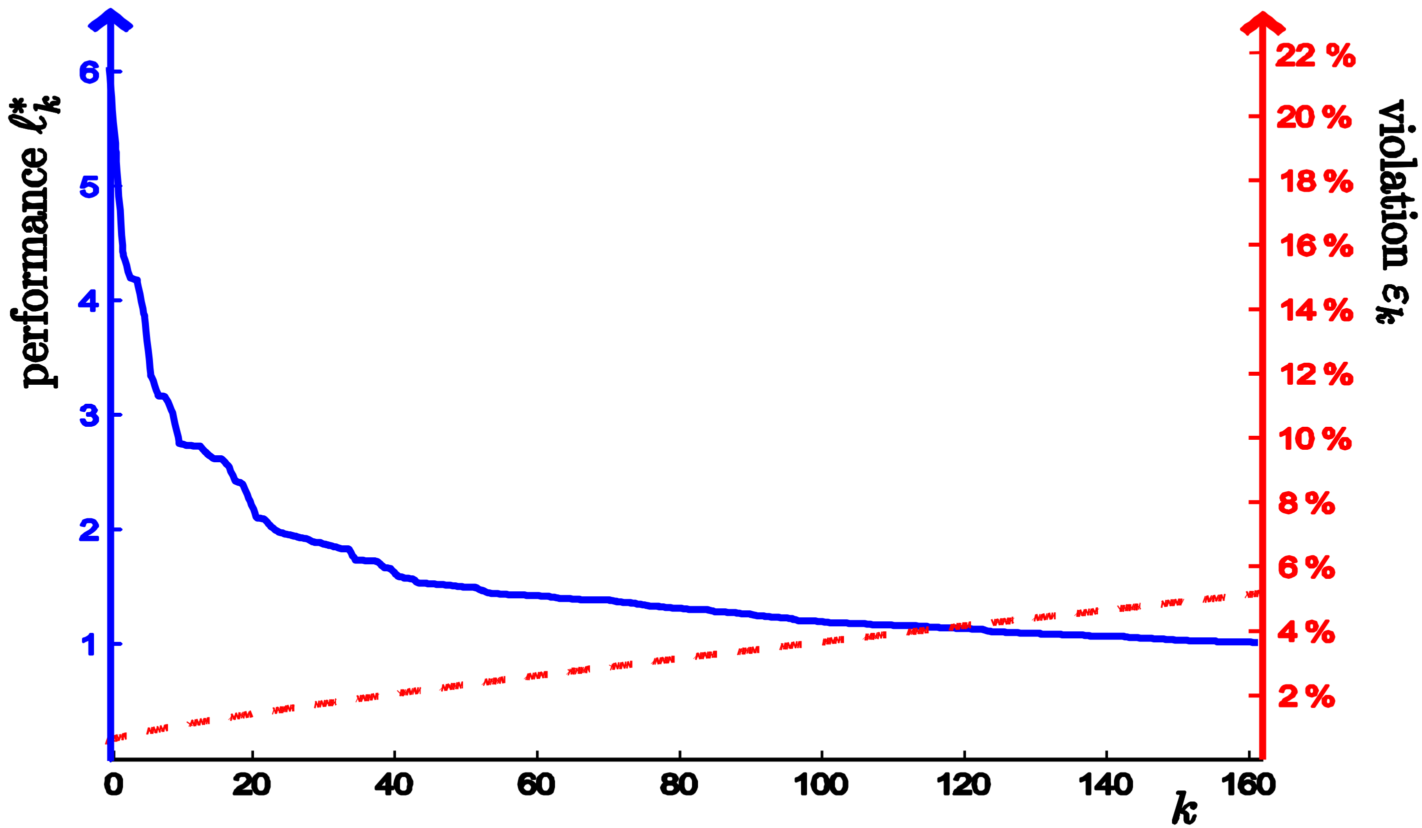
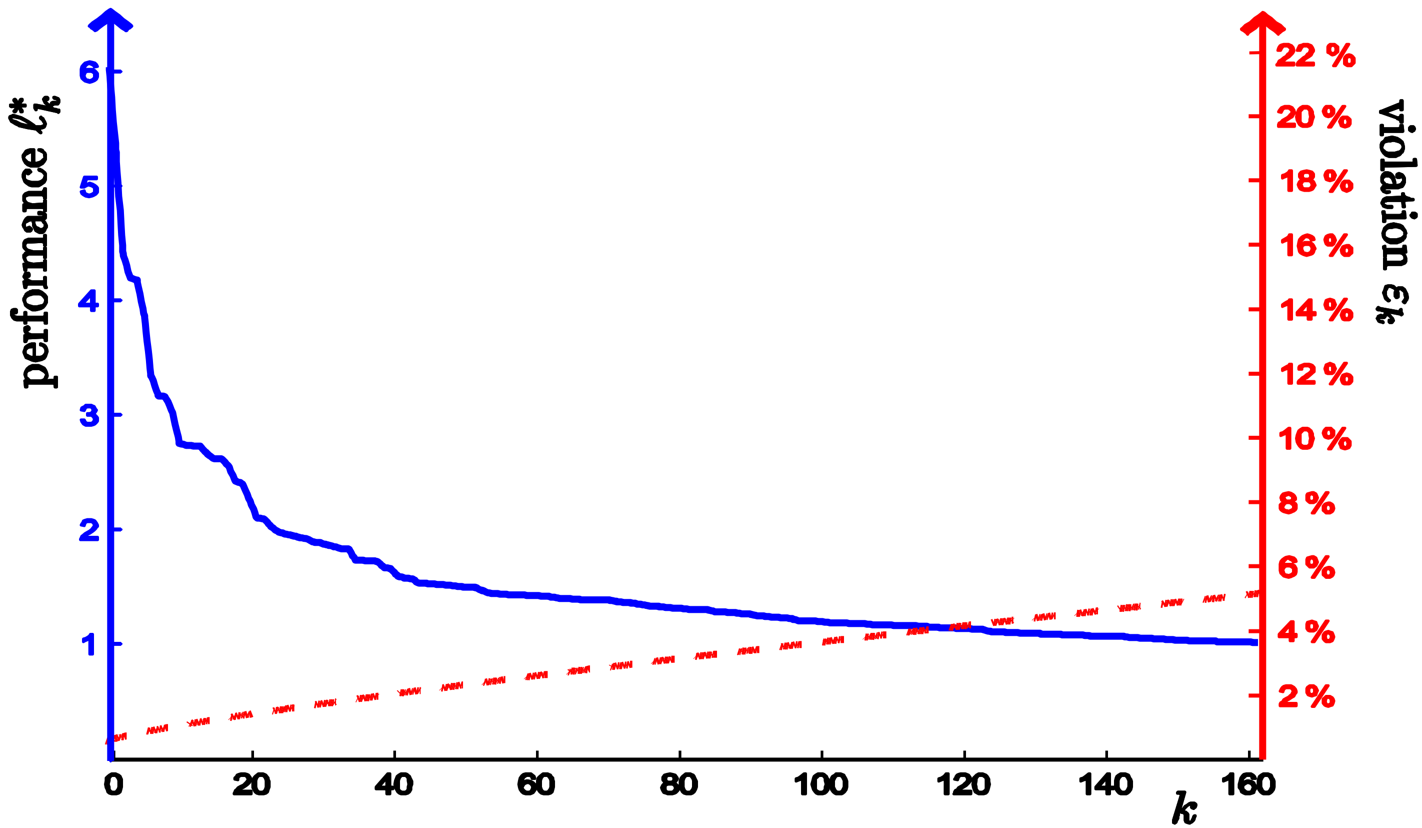
- value can be inspected

  violation probability is guaranteed by the theorem

performance - violation plot



Example: feedforward noise compensation



Example: feedforward noise compensation

sample: $a_i, b_i, c_i, d_i \in \Delta, \quad i = 1, 2, \ldots, 5427;$

solve:

$$\min_{k_1,k_2} \left[\max_i \frac{(c_i + b_i k_1)^2 + (d_i + b_i k_2)^2 + 2a_i(c_i + b_i k_1)(d_i + b_i k_2)}{1 - a_i^2} \right]$$

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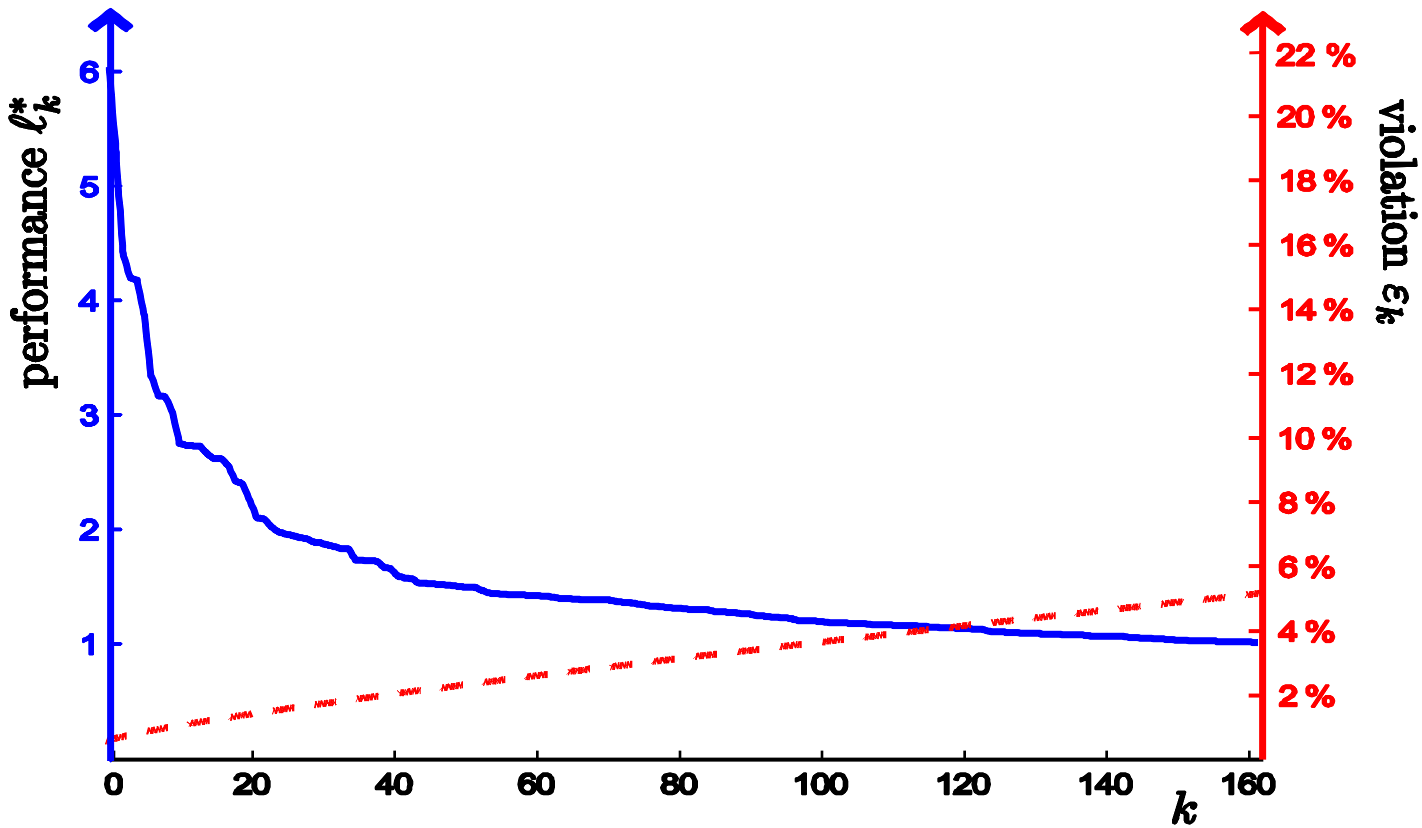
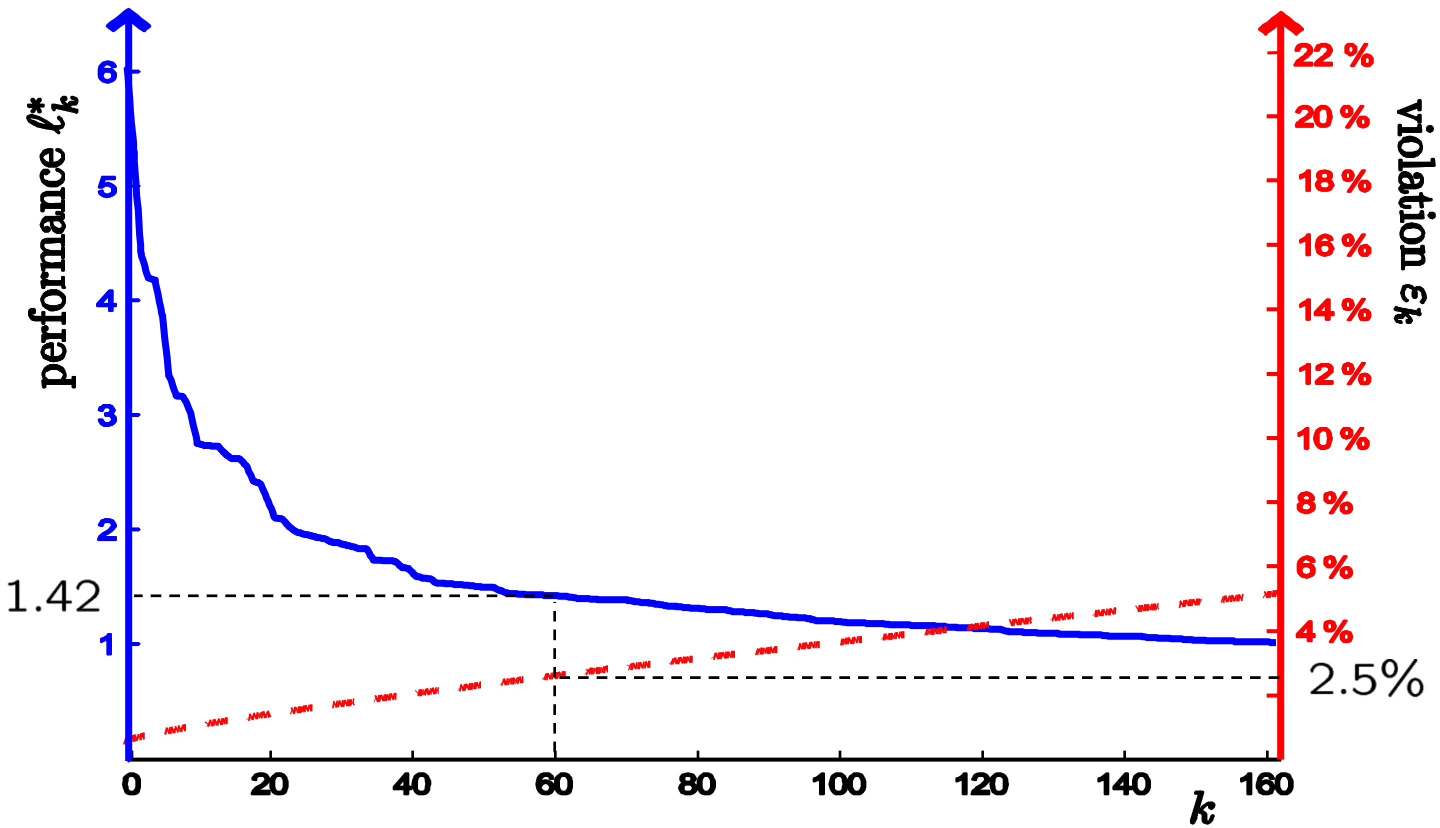
⇒ eliminate $k = 1, 2, \ldots$ constraints

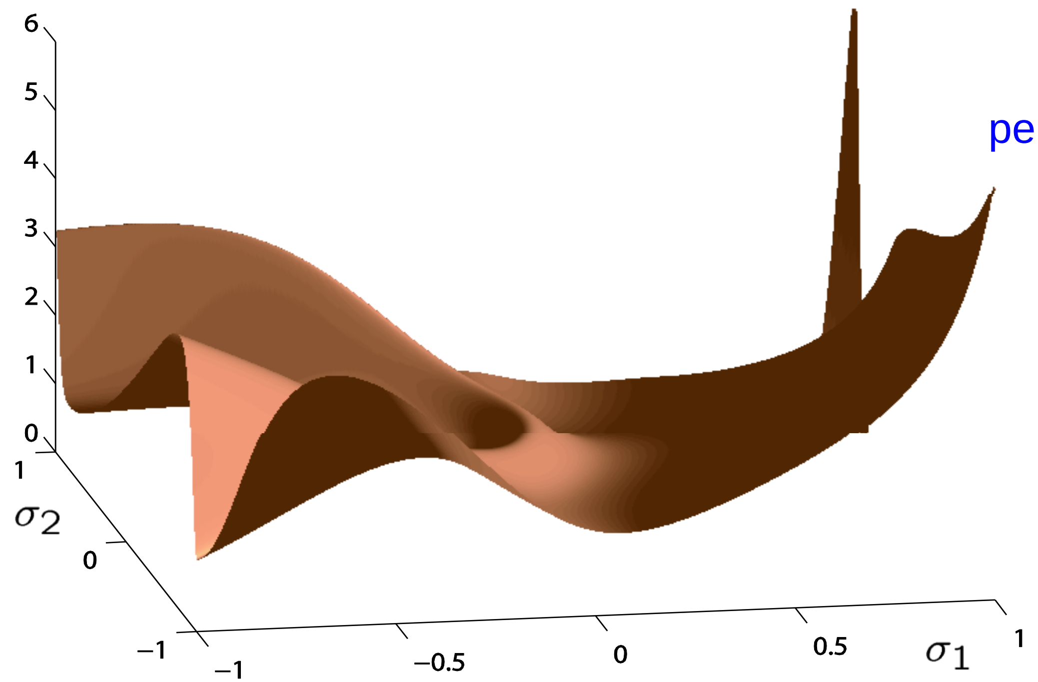
Example: feedforward noise compensation



Example: feedforward noise compensation

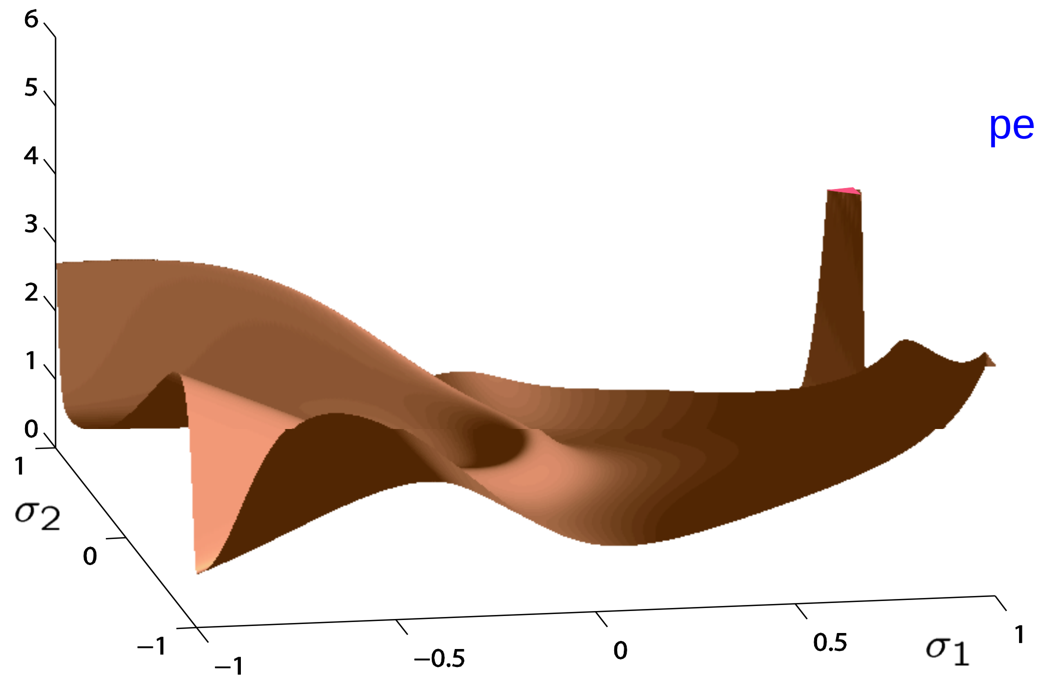


Example: feedforward noise compensation



performance profile

Example: feedforward noise compensation



performance profile

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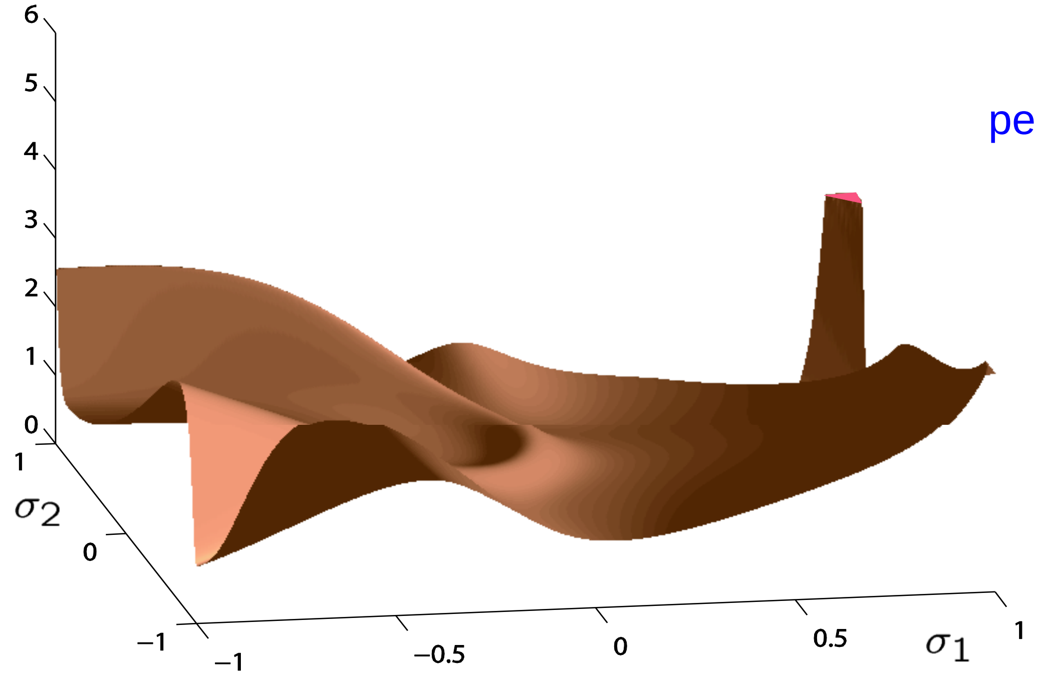

performance profile

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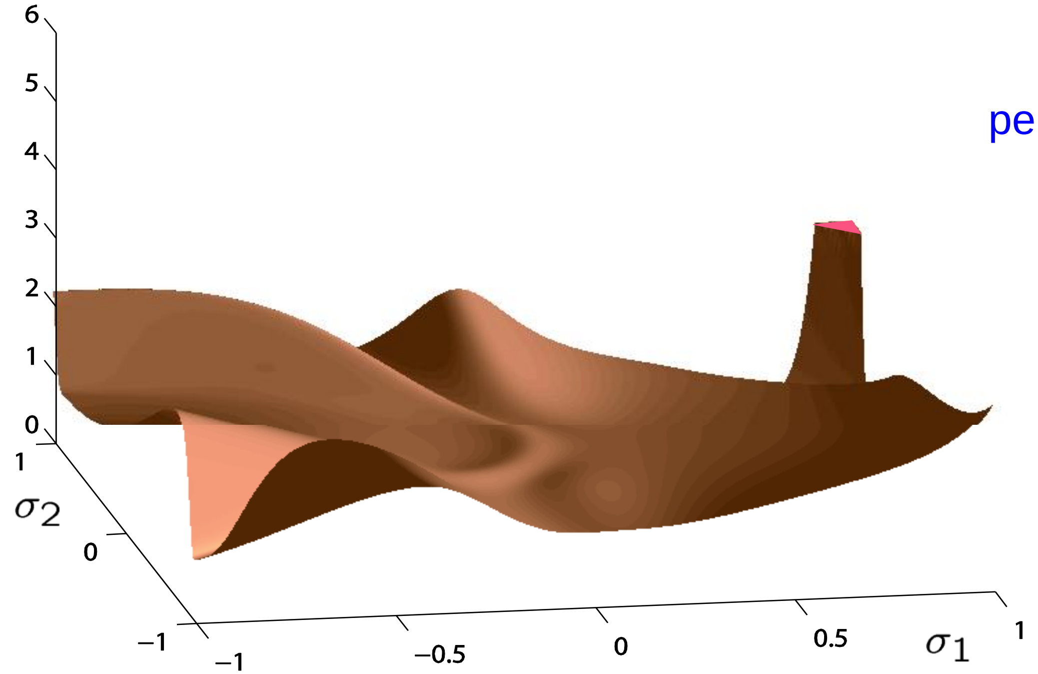

performance profile

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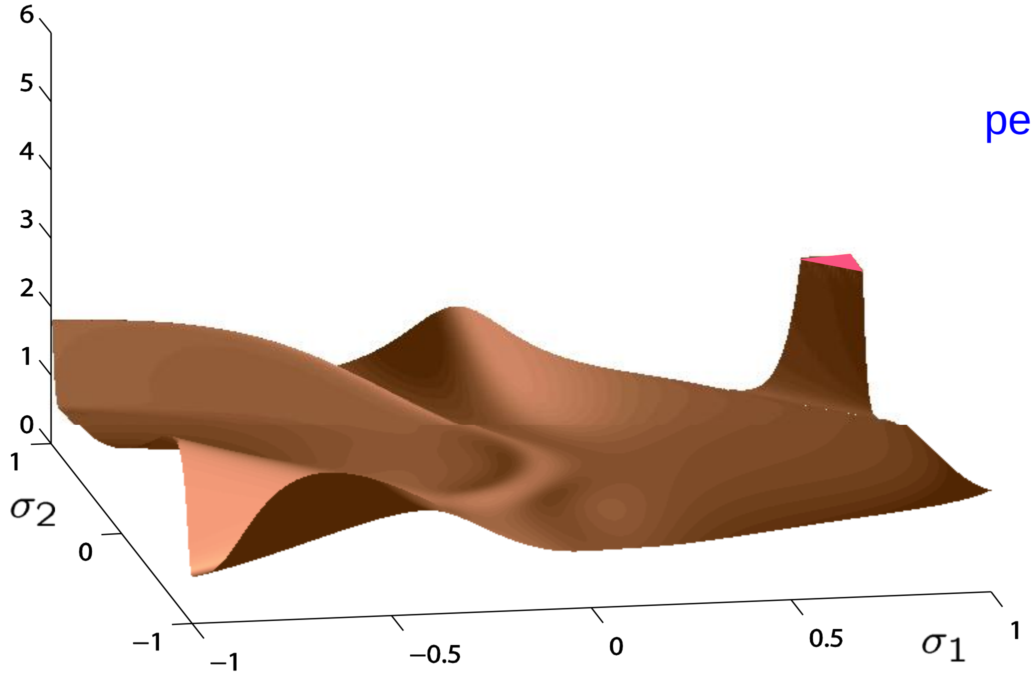

performance profile

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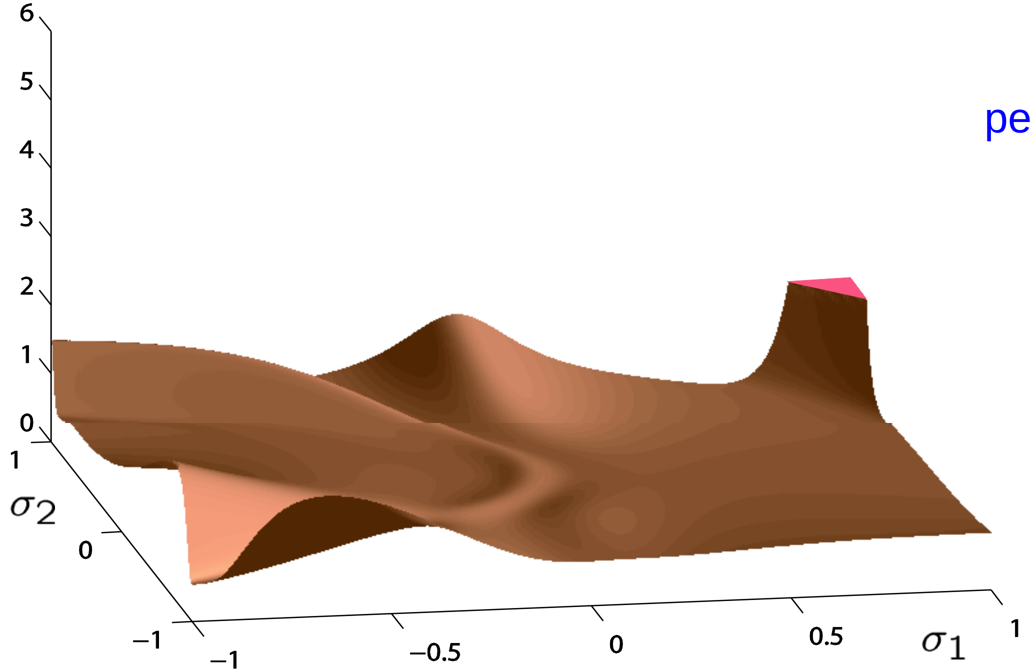

performance profile

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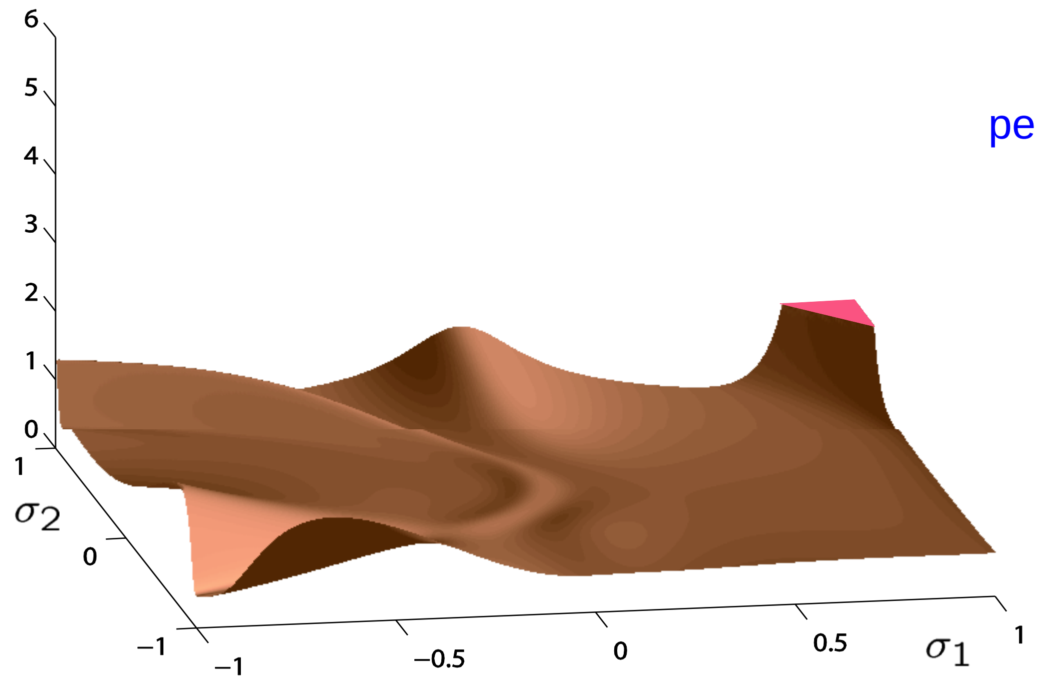

performance profile

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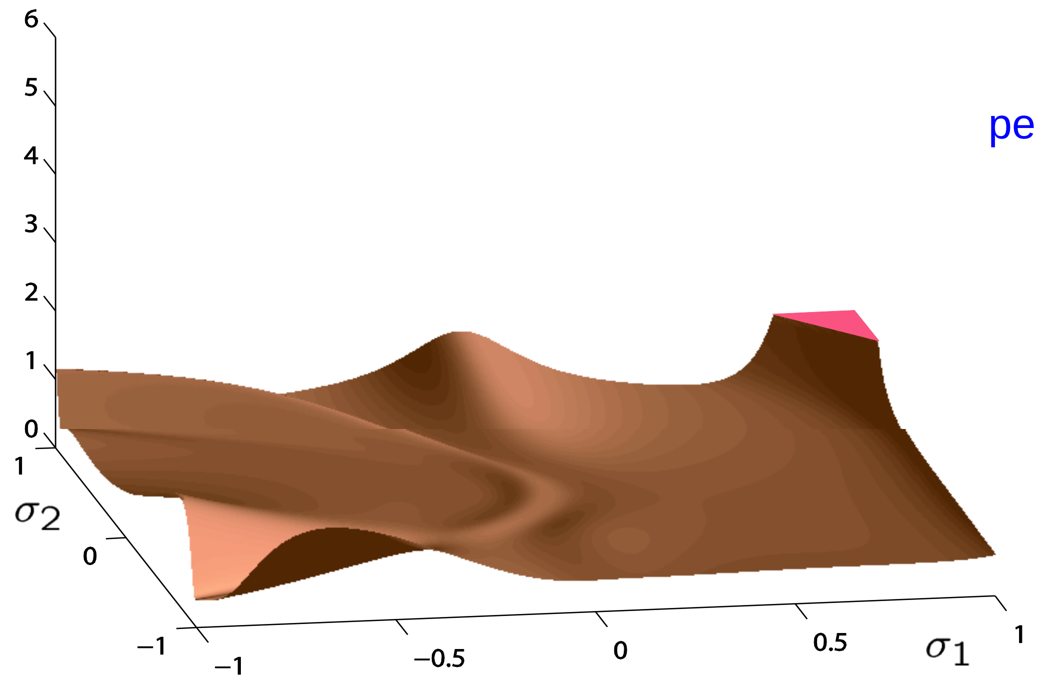
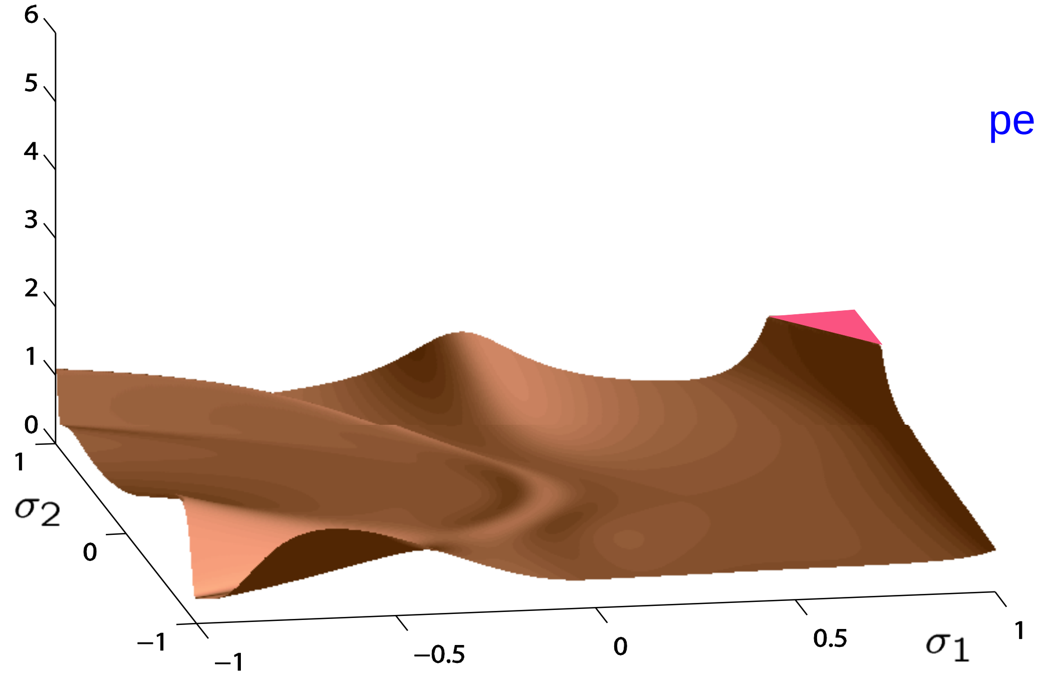


performance profile

Example: feedforward noise compensation



performance profile

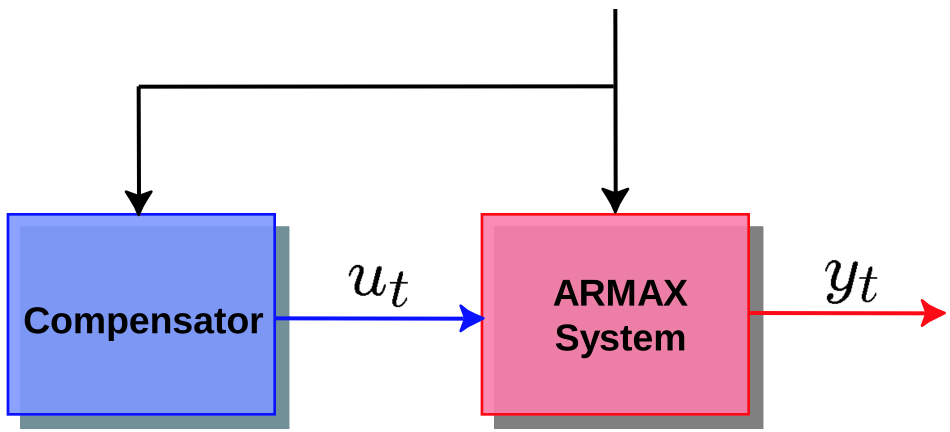
$k = 60$

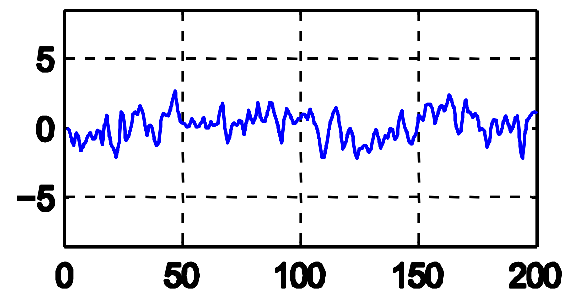
$l^*_{60} = 1.42$

$\epsilon_{60} = 2.5\%$

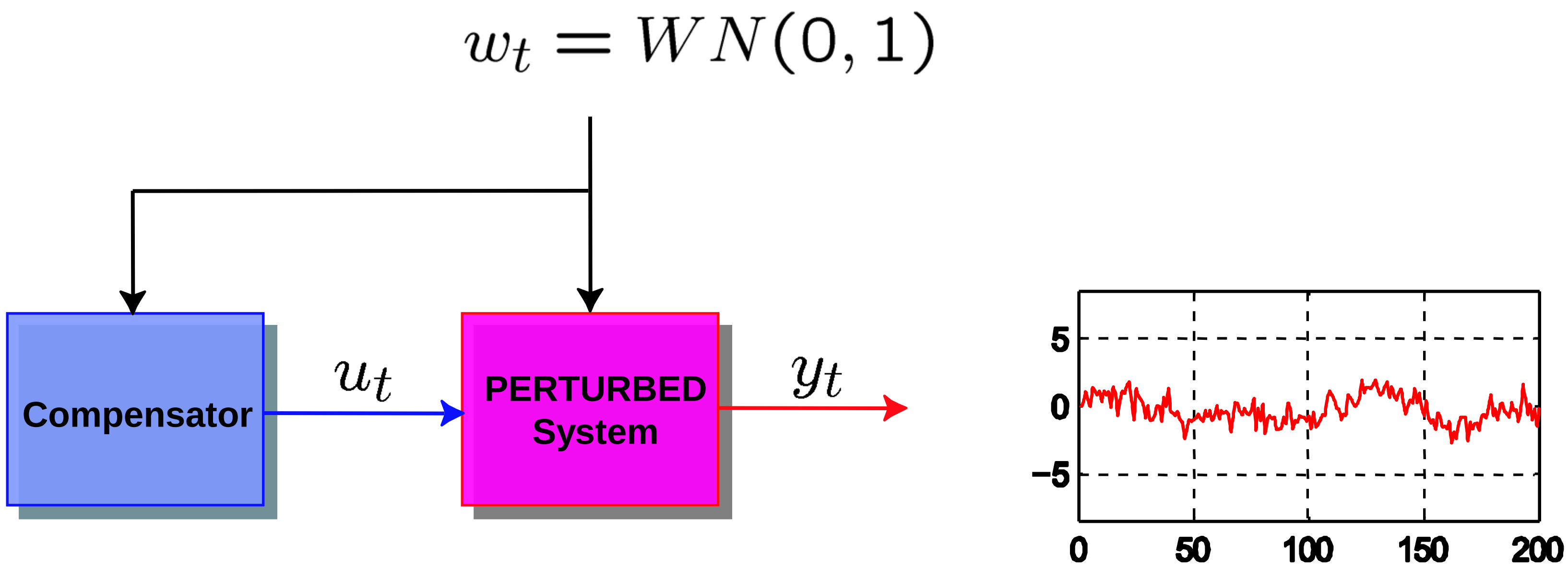
Example: feedforward noise compensation

$$w_t = WN(0, 1)$$





Example: feedforward noise compensation



Conclusions

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Applications in:
- management
- finance
- prediction
- control

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