New upper bounds for nonbinary codes based on quadruples

Bart Litjens and Sven Polak

Based on joint work with Lex Schrijver

Korteweg-de Vries Institute for Mathematics
Faculty of Science
University of Amsterdam

June 30th, 2016

Outline of the talk

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(ii) Interesting parameter in cryptography: a code $C \subseteq [q]^n$ with $d_{\min}(C) = 2e + 1$ is *e-error correcting*.

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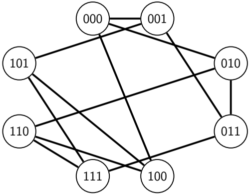

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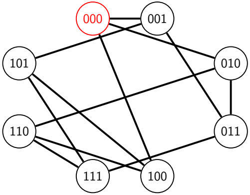

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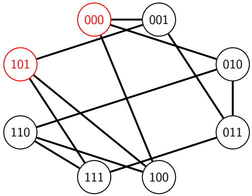

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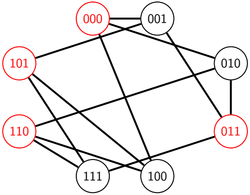

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Schrijver (starting in 2005): hierarchy of semidefinite programming upper bounds via k-tuples of codewords ($k \geq 2$).

k	Studied by
2	Delsarte (1973)
3	Schrijver (2005) for $q = 2$ and Gijswijt, Schrijver and Tanaka (2006) for $q \in \{3, 4, 5\}$
4	Gijswijt, Mittelmann and Schrijver (2012) for $q = 2$

Delsarte bound

$$\theta^*(q,n,d) := \max\Big\{ \sum_{u,v\in[q]^n} X_{u,v} \;\Big|\; X \in \mathbb{R}_{\geq 0}^{[q]^n\times[q]^n} \text{ with:}$$

$$\begin{aligned} &\text{(i)} \quad \text{trace}(X) = 1, \\ &\text{(ii)} \quad X_{u,v} = 0 \text{ if } \{u,v\} \in E, \\ &\text{(iii)} \; X \text{ is positive semidefinite}\Big\}. \end{aligned}$$

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- $(1/|G|)\sum_{\pi\in G} X^\pi$ is a G-invariant optimum solution. Hence the SDP has at most $n+1$ variables.

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Given $x : \mathcal{C}_2 \to \mathbb{R}_{\geq 0}$, define the $\mathcal{C}_1 \times \mathcal{C}_1$-matrix M_x by

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It can be proven that the Delsarte bound equals

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$$B_q(n,d) := \max\Big\{ \sum_{v\in[q]^n} x(\{v\}) \mid x : \mathcal{C}_4 \to \mathbb{R}_{\geq 0} \text{ with:}$$
$$\text{(i)} \quad x(\emptyset) = 1,$$
$$\text{(ii)} \quad x(C) = 0 \text{ if } d_{\min}(C) < d,$$
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Proposition. $A_q(n,d) \leq B_q(n,d)$

Proof. Let $C \subseteq [q]^n$ be a code of minimum distance at least d and maximum size. Define x by $x(S) = 1$ if $S \subseteq C$ and $x(S) = 0$ else. Then x is feasible. Moreover,

$$\sum_{u\in[q]^n} x(\{u\}) = |C| = A_q(n,d),$$

which yields the proposition. □

Intermezzo: constant weight codes

Suppose that $q = 2$ and let $n, d, w \in \mathbb{N}$.

Constant weight codes

The weight $\text{wt}(u)$ of a codeword $u \in \{0,1\}^n$ is the number of nonzero entries in u.

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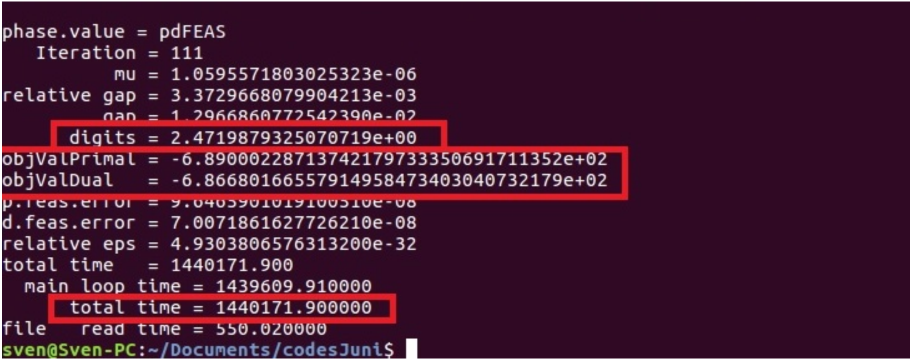

Intermezzo: $k = 5$ for binary codes

Suppose that $q = 2$ and let $n, d \in \mathbb{N}$.

SDP-bound on $A_2(n,d)$ based on quintuples, $k = 5$

Let $\mathbf{0} := 0\dots0$ and let $\mathcal{C}'_k$ be the collection of codes $C \subseteq [q]^n$ with $|C| \leq k$ and $\mathbf{0} \in C$. Then $A_2(n,d) \leq Q(n,d)$, where

$$Q(n,d) := \max\Big\{ \sum_{v\in[q]^n} x(\{\mathbf{0}, v\}) \mid x : \mathcal{C}'_5 \to \mathbb{R}_{\geq 0} \text{ with: }$$

$$\begin{aligned} &\text{(i)} \quad x(\{\mathbf{0}\}) = 1, \\ &\text{(ii)} \quad x(C) = 0 \text{ if } d_{\min}(C) < d, \\ &\text{(iii)} \quad M_x \text{ is positive semidefinite}\Big\}, \end{aligned}$$

where $(M_x)_{C,C'} = x(C \cup C')$ for all $x \in \mathcal{C}'_3$.

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```
sven@Sven-PC:~/Documents/codesJuni$ sdpa_dd Q17_8.dat-s DDversieQ17_8.result
SDPA-DD start at    Sat Jun 18 17:16:27 2016
data      is Q17_8.dat-s : sparse
parameter is ./param.sdpa
out       is DDversieQ17_8.result

DENSE computations
   mu      thetaP  thetaD  objP      objD      alphaP  alphaD  beta
 0 1.0e+10 1.0e+00 1.0e+00 -0.00e+00 -1.00e+05 2.6e-01 2.1e-01 4.00e-01
 1 8.7e+09 7.4e-01 7.9e-01 -8.30e+04 -1.47e+05 1.4e-01 1.9e-01 4.00e-01
 2 8.6e+09 6.4e-01 6.4e-01 -1.60e+05 -3.58e+05 1.6e-01 1.9e-01 4.00e-01
 3 8.4e+09 5.4e-01 5.2e-01 -1.87e+05 -1.13e+06 1.7e-01 2.4e-01 4.00e-01
 4 8.2e+09 4.5e-01 3.9e-01 -1.80e+05 -2.78e+06 2.0e-01 2.2e-01 4.00e-01
 5 7.7e+09 3.6e-01 3.1e-01 -1.62e+05 -4.84e+06 2.2e-01 2.2e-01 4.00e-01
 6 7.1e+09 2.8e-01 2.4e-01 -1.45e+05 -7.73e+06 2.2e-01 2.5e-01 4.00e-01
 7 6.7e+09 2.2e-01 1.8e-01 -1.34e+05 -1.24e+07 2.4e-01 2.5e-01 4.00e-01
 8 6.1e+09 1.7e-01 1.4e-01 -1.22e+05 -1.90e+07 2.5e-01 2.5e-01 4.00e-01
 9 5.5e+09 1.2e-01 1.0e-01 -1.09e+05 -2.82e+07 2.6e-01 2.6e-01 4.00e-01
10 4.9e+09 9.2e-02 7.5e-02 -9.53e+04 -4.12e+07 2.7e-01 2.6e-01 4.00e-01


36 4.4e+07 4.0e-07 1.3e-28 -3.51e+01 -3.59e+10 4.3e-01 6.3e-01 4.00e-01
37 3.7e+07 2.3e-07 1.8e-28 -3.44e+01 -3.50e+10 4.4e-01 7.0e-01 4.00e-01
38 3.1e+07 1.3e-07 5.1e-28 -3.39e+01 -3.38e+10 4.5e-01 8.2e-01 4.00e-01
39 2.7e+07 7.0e-08 1.5e-27 -3.36e+01 -3.26e+10 4.7e-01 7.5e-01 4.00e-01
40 2.2e+07 3.7e-08 1.0e-26 -3.34e+01 -3.20e+10 4.9e-01 7.9e-01 4.00e-01
41 1.8e+07 1.9e-08 8.7e-27 -3.33e+01 -3.00e+10 5.0e-01 9.9e-01 4.00e-01
42 1.5e+07 9.5e-09 6.2e-26 -3.31e+01 -2.64e+10 5.0e-01 1.0e+00 4.00e-01
43 1.3e+07 4.8e-09 2.0e-25 -3.30e+01 -2.43e+10 5.0e-01 1.0e+00 4.00e-01
44 1.0e+07 2.4e-09 2.6e-25 -3.30e+01 -2.21e+10 5.0e-01 1.0e+00 4.00e-01
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- For example, if we assume that $q \geq 3$, then

$$\{\{1, 3\}, \{2\}, \{4\}\} \mapsto S_q \cdot (0, 1, 0, 2).$$

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- For example, writing $\{\{1,2\},\{3\},\{4\}\}$ as $12,3,4$, letting $n = 4$ and $q \geq 3$ then

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$\Longrightarrow |\Omega|$ bounded by a polynomial in n.

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- Replace variable $x(C)$ in the matrix M_x, with $C \in \mathcal{C}_4$, by $y(w)$, with $w \in \Omega$ the orbit containing C.

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Theorem (Maschke's theorem + Schur's lemma)

$End_G(\mathbb{R}^{\mathcal{C}_2}) \xrightarrow{\sim} \bigoplus_i \mathbb{R}^{m_i \times m_i}$ (as linear spaces), via $A \mapsto U^t A U$. Moreover, A is positive semidefinite if and only if each of the blocks of $U^t A U$ is.

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- The coefficients can be computed in time polynomial in n.

Results [L., P. and Schrijver, 2016]

Table: New upper bounds on $A_q(n, d)$

q	n	d	Best lower bound known	**New upper bound**	Best upper bound previously known
4	6	3	164	**176**	179
4	7	3	512	**596**	614
4	7	4	128	**155**	169
5	7	4	250	**489**	545
5	7	5	53	**87**	108

Mixed binary/ternary codes

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Definition

$N(n_2, n_3, d) := \max\{|C| \mid C \subseteq [2]^{n_2}[3]^{n_3},\ d_{\min}(C) \geq d\}$.

Motivation: football pools

Source: http://www.uefa.com/uefaeuro/draws/

Motivation: the (extended) football pool problem

Fix $0 \leq e \leq n_2 + n_3$. Suppose n_3 games are played with possible outcome win/draw/loss and n_2 games with possible outcome win/loss.

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$\Longrightarrow$ amounts to determining $N(n_2, n_3, d)$ with $d = 2e + 1$.

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$\Longrightarrow$ Linear programming bound with $\leq \frac{(n_2+n_3+1)(n_2+n_3+2)}{2}$ constraints (Brouwer, Hämäläinen, Östergård and Sloane, 1998).

Semidefinite programming upper bound

SDP-bound on $N(n_2, n_3, d)$ based on triples, $k = 3$

Let $\mathbf{0} := 0 \dots 0$ and let $\mathcal{C}_3'$ be the collection of codes $C \subseteq [2]^{n_2}[3]^{n_3}$ with $|C| \leq 3$ and $\mathbf{0} \in C$. Then $N(n_2, n_3, d) \leq N_3(n_2, n_3, d)$, where

$$N_3(n_2, n_3, d) := \max\Big\{ \sum_{v \in [2]^{n_2}[3]^{n_3}} x(\{\mathbf{0}, v\}) \mid x : \mathcal{C}_3' \to \mathbb{R}_{\geq 0} \text{ with:}$$

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Reductions and results

- Symmetry reduction using the group $(S_2^{n_2} \rtimes S_{n_2}) \times (S_3^{n_3} \rtimes S_{n_3})$.

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Results (L., 2016)

In total 135 improved upper bounds were found: 131 from the SDP with $k = 3$, one new bound from the SDP with $k = 4$ and three implicit improvements.

A selection of the results

Table: A part of the table with best known bounds on $N(n_2, n_3, 4)$. The improved bounds are boldface.

$n_2 \backslash n_3$	2	3	4	5	6
2	2	3	8	22	51-**61**
3	3	6	15	36-**43**	92-**117**
4	6	11	28-30	62-**83**	158-**228**
5	8	20	50-**59**	114-**160**	288-**436**
6	16	34-40	96-**114**	216-**308**	576-**825**
7	36-30	64-80	192-**220**	408-**585**	1152-**1576**
8	50-**59**	128-**153**	384-**407**	768-**1103**	2304-**3027**
9	96-**108**	256-**288**	548-**771**	1536-**2105**	
10	192-**212**	420-**548**	1050-**1480**		
11	384	784-**1032**			